6.5 Fundamental Theorem of Calculus

1) $\int_1^3 (4 - 32x^{-3})\, dx = \left(4x - \frac{32}{-2}x^{-2}\right)\Big|_1^3$

$= \left(4x + \frac{16}{x^2}\right)\Big|_1^3$

$= 12 + \frac{16}{9} - (4+16) = \boxed{-6.2}$

2) $\int_0^9 x\sqrt{9-x}\, dx = -\int_9^0 (9-u)u^{1/2}\, du = \int_9^0 (-9u^{1/2} + u^{3/2})\, du$

$u = 9-x$

$\frac{du}{dx} = -1$

$du = -1\, dx$

$-1\, du = dx$

$u = 9 - x$

$x = 9 - u$

and

when

$x=0,\ u=9$

$x=9,\ u=0$

$= \left(-9 \cdot \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right)\Big|_9^0$

$= \left(-6u^{3/2} + \frac{2}{5}u^{5/2}\right)\Big|_9^0$

$= (0+0) - \left(-6 \cdot 9^{3/2} + \frac{2}{5} \cdot 9^{5/2}\right)$

$= +6 \cdot 3^3 - \frac{2}{5} \cdot 3^5$

$= 6 \cdot 27 - \frac{2}{5}(243)$

$= 162 - \frac{486}{5}$

$= \boxed{64.8}$

3) Find the exact value of $\int_0^{\sqrt{3}} \frac{x}{4-x^2}\, dx$

$u = 4 - x^2$

$\frac{du}{dx} = -2x$

$du = -2x\, dx$

$-\frac{1}{2}du = x\, dx$

if $x = 0$

$u = 4$

if $x = \sqrt{3}$

$u = 1$

$\int_0^{\sqrt{3}} \frac{1}{4-x^2} \cdot x\, dx = \int_4^1 \frac{1}{u}\left(-\frac{1}{2}\right) du = -\frac{1}{2}\int_4^1 \frac{1}{u}\, du$

$= \left(-\frac{1}{2}\ln|u|\right)\Big|_4^1$

$= -\frac{1}{2}\ln 1 - \left(-\frac{1}{2}\ln|4|\right)$

$= 0 + \frac{1}{2}\ln 4$

$\boxed{\frac{1}{2}\ln 4}$

4) The Ringin' Ringtone Company finds their marginal cost is defined by $C'(x) = 200 - 2x$ where x is the number of ringtones produced each month. Compute the increase in cost to change the production level from 50 ringtones each month to 100 ringtones per month.

$$C'(x) = 200 - 2x$$

$$\int_{50}^{100} C'(x)\,dx = \int_{50}^{100} (200-2x)\,dx = \left(200x - x^2\right)\Big|_{50}^{100}$$

$$= (20{,}000 - 10{,}000) - (10{,}000 - 2500)$$

$$= +2500 \qquad \boxed{\$2500}$$

Check: $\int_{50}^{100} (200-2x)\,dx$

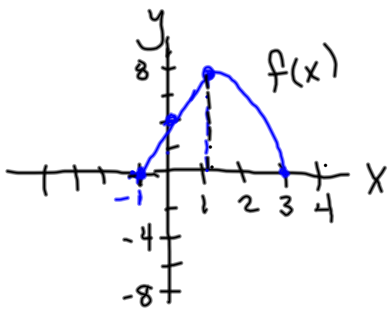
Math 9 fnInt $(200-2x, x, 50, 100)$

$= 2500$

5) a. Write the integral to represent the area below $f(x)$ and above the x-axis when

$f(x) = \begin{cases} 4x+4, & x < 1 \\ 9 - x^2, & x \geq 1 \end{cases}$ $\quad A = \int_{-1}^{1} (4x+4)\,dx + \int_{1}^{3} (9 - x^2)\,dx$

b. Sketch the graph.



c. Find the area.

$$A = \left(2x^2 + 4x\right)\Big|_{-1}^{1} + \left(9x - \tfrac{1}{3}x^3\right)\Big|_{1}^{3}$$

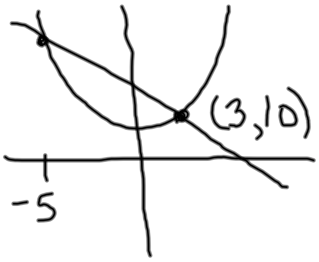
$$A = (2+4) - (2-4) + (27-9) - \left(9 - \tfrac{1}{3}\right)$$

$$A = 6 - (-2) + 18 - 8\tfrac{2}{3}$$

$$A = \boxed{17.\overline{3}}$$

7.1 Area Between Curves

6) Find the area bounded by the curves $f(x) = x^2 + 1$ and $g(x) = -2x + 16$.



$x^2+1=-2x+16$

$x^2+2x-15=0$

$(x+5)(x-3)=0$

$x=-5,\ x=3$

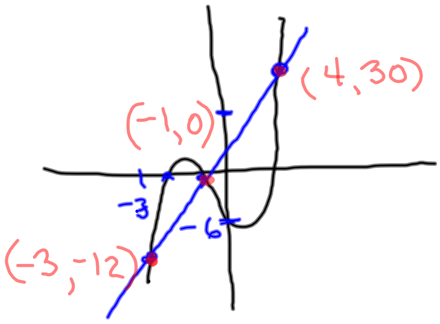
$$A=\int_{-5}^{3} (-2x+16)-(x^2+1)\,dx$$

$$A=\left(-x^2+16x-\tfrac{1}{3}x^3-x\right)\Big|_{-5}^{3}$$

$$A=-9+48-9-3-\left(-25-80+\tfrac{125}{3}+5\right)$$

$$A=\boxed{85.\overline{3}}$$

7) Find the area bounded by the curves $f(x) = x^3 - 7x - 6$ and $g(x) = 6x + 6$.



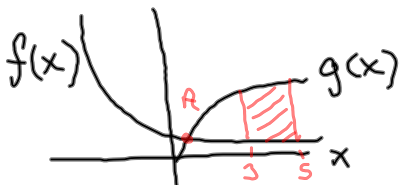
$$A=\int_{-3}^{-1}(x^3-7x-6)-(6x+6)\,dx + \int_{-1}^{4}(6x+6)-(x^3-7x-6)\,dx$$

$y_1=x^3-7x-6$

$y_2=6x+6$

Area = fnInt$(y_1-y_2, x, -3, -1)+(y_2-y_1, x, -1, 4)$

8) Find the area bounded by the curves $f(x) = e^{-x}$ and $g(x) = \sqrt{x}$ over the interval $3 \le x \le 5$.

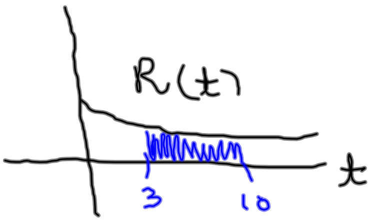


$$\text{Area}=\int_3^5 \left(\sqrt{x}-e^{-x}\right)dx$$

fnInt$(\sqrt{x}-e^{-x}, x, 3, 5)$

$A=(0.4263, 0.6529)$

9) The rate of production of oil from a producing field is described by $R(t) = \frac{50}{t+20} + 15$, on the interval from $0 \le t \le 12$ where $R(t)$ is the rate of production in thousands of barrels per year t years after pumping begins. Find the area between the graph of R and the t axis over the interval $[3, 10]$, and interpret the results.



$R(t) = P'(t)$ R = rate of production

$\int_3^{10} R(t)\,dt = P(t)$ production of oil 4–10 year

fnInt(50/(t+20)+15, t, 3, 10)

= 118.2851583 thous. barrels

$\boxed{\text{118,285 barrels}}$

produced $4^{th} \to 10^{th}$ yr.

note: $\int_0^3 R(t)\,dt$ will find oil produced 1st 3 years

7.2 Applications

10) Find the consumers' surplus at a price level of \$12 for the price-demand function $p = D(x) = 40 - .5x$.

$D(x) = 40 - \frac{1}{2}x$

$y = 12$

$12 = 40 - \frac{1}{2}x$

$\frac{1}{2}x = 40 - 12$

$(56, 12)$

$\frac{1}{2}x = 28$

$x = 56$

* $C.S. = \int_0^{56} (40 - \frac{1}{2}x - 12)\,dx$

$C.S. = \int_0^{56} (28 - \frac{1}{2}x)\,dx$

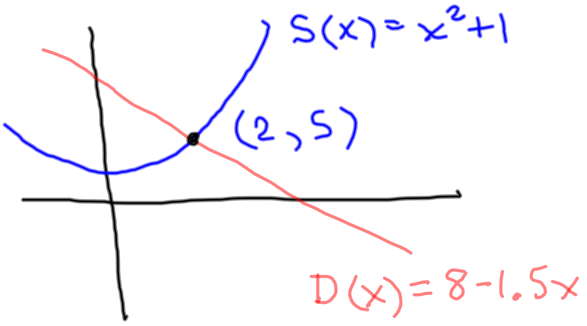
$C.S. = \left(28x - \frac{1}{4}x^2\right)\Big|_0^{56}$

$C.S. = 28(56) - \frac{1}{4}(56)^2 - (0)$

$C.S. = \$784$

OR * fnInt$(40 - \frac{1}{2}x - 12, x, 0, 56)$ = 784 dollars

11) Find the point of equilibrium for the Company XYZ which has a price-demand function of $p = D(x) = 8 - 1.5x$ and a price-supply function of $p = x^2 + 1$.



Method 1

$Y_1 = x^2 + 1$

$Y_2 = 8 - 1.5x$

2nd Trace 5: intersect

(2, 5)

Method 2: Solve

$x^2 + 1 = 8 - 1.5x$

$x^2 + 1.5x - 7 = 0$

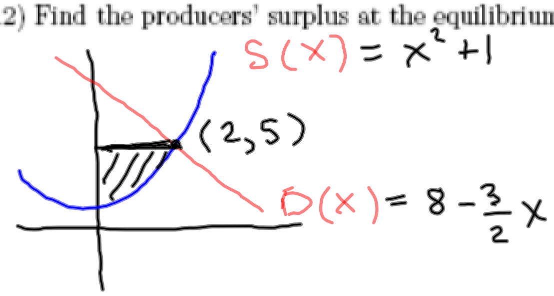
$2x^2 + 3x - 14 = 0$

$(2x + 7)(x - 2) = 0$

$x = -7/2 \quad x = 2$

three possible Questions
- (2, 5) point of Equilibrium
- 2 = Equilibrium Quantity
- 5 = Equilibrium price

12) Find the producers' surplus at the equilibrium point for the previous problem.



$$P.S. = \int_0^2 (5 - [x^2 + 1])\, dx$$

$$P.S. = \int_0^2 (4 - x^2)\, dx$$

fnInt$(4 - x^2, x, 0, 2)$ dx =

$5.\overline{3} \approx \$5.33$

13) Find the consumers' surplus at the equilibrium point for the previous problem.

$$C.S. = \int_0^2 (8 - \tfrac{3}{2}x) - (5)\, dx$$

fnInt$(8 - 1.5x - 5, x, 0, 2) = 3$ $\boxed{\$3.00}$

14) The rate of change of the income provided by the vending machines on the first floor of Blocker is given by $f(t) = 2500e^{0.03t}$, where t is time in years since the installation of the machine. Find the total income produced the third year.

$$I = \int_2^3 2500 e^{0.03t} \, dt$$

$$I = \frac{2500}{.03} \int_2^3 e^{0.03t} (.03) \, dt$$

$$I = \left(\frac{2500}{.03} e^{0.03t} \right) \Bigg|_2^3 = \boxed{\$2,694.81}$$

Miscellaneous Topics

15) $\int \left(ax^3 - bx^2 + \frac{c}{x^2} + \frac{\pi}{x} \right) dx$ $= \int \left(ax^3 - bx^2 + cx^{-2} + \pi \cdot \frac{1}{x} \right) dx$

$$= \frac{ax^4}{4} - \frac{b}{3}x^3 - cx^{-1} + \pi \ln|x| + C$$

16) $\int 4t \cdot e^{t^2} \, dt$ $= \int e^{t^2} (4t) \, dt = \int e^u \cdot 2 \, du$

$$u = t^2$$

$$\frac{du}{dt} = 2t$$

$$du = 2t \, dt$$

$$2 \, du = 4t \, dt$$

$$= 2e^u + C$$

$$= 2e^{t^2} + C$$

17) $\int \left(\frac{8x^3 - 12x + 6}{x^2} \right) dx$ $= \int \left(8x - 12 \cdot \frac{1}{x} + 6x^{-2} \right) dx$

$$= 4x^2 - 12 \ln|x| - 6x^{-1} + C$$

$$= 4x^2 - 12 \ln|x| - \frac{6}{x} + C$$

18) $\int (9x+27)\sqrt{3x^2+18x-6}\,dx \quad = \int (u)^{1/2}\cdot \frac{3}{2}du =$

$u = 3x^2+18x-6$

$du = (6x+18)\,dx$

$\frac{3}{2}du = \frac{3}{2}(6x+18)\,dx$

$\frac{3}{2}du = (9x+27)\,dx$

$\frac{3}{2}\cdot\frac{2}{3}u^{3/2} + C$

$u^{3/2} + C$

$(3x^2+18x-6)^{3/2} + C$

19) $\int (x-2)\cdot(x+5)^2\,dx \quad = \int u^2(u-5-2)\,du$

$u = x+5$

$du = dx$

$u-5 = x$

$= \int u^2(u-7)\,du$

$= \int (u^3-7u^2)\,du$

$= \frac{1}{4}u^4 - \frac{7}{3}u^3 + C =$

$\frac{1}{4}(x+5)^4 - \frac{7}{3}(x+5)^3 + C$

20) $\int (x^3-2x+1)\cdot(x^2+5x-2)\,dx$

$\int (x^5+5x^4-2x^3-2x^3-10x^2+4x+x^2+5x-2)\,dx$

$= \int (x^5+5x^4-4x^3-9x^2+9x-2)\,dx$

$= \frac{1}{6}x^6 + x^5 - x^4 - 3x^3 + \frac{9}{2}x^2 - 2x + C$

21) $\int_a^b (5-16x^{-3})\,dx \quad = \left(5x - \frac{16}{-2}x^{-2}\right)\Big|_a^b = \left(5x + \frac{8}{x^2}\right)\Big|_a^b$

$= 5b + \frac{8}{b^2} - 5a - \frac{8}{a^2}$

22) $\int_0^a x\sqrt{9-x}\;dx \quad = \int_9^{9-a} (9-u)\sqrt{u}\,du = \int_9^{9-a} (9u^{1/2} - u^{3/2})\,du$

$u = 9-x$

$x = 9-u$

$du = dx$

if $x=0$, $u=9$

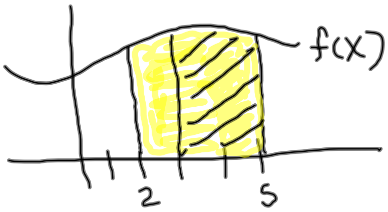
if $x=a$, $u=9-a$

$= \left(6u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_9^{9-a}$

$$= \boxed{6(9-a)^{3/2} - \frac{2}{5}(9-a)^{5/2} - 6(9)^{3/2} + \frac{2}{5}(9)^{5/2}}$$

23) $\int \pi \cdot e^2 dt$ $= \boxed{\pi e^2 t + C}$

24) Given the $\int_2^5 f(x)\,dx = 50$, and the $\int_3^5 f(x)\,dx = 30$, find the area under the curve $f(x)$ over the interval [2,3].



Yellow region = 50

Striped region = 30

Yellow and not striped

$\boxed{20}$

25) $\int x e^{-3x^2}\,dx$ $= \int e^{-3x^2} \cdot x\,dx = \int e^{u} \cdot \left(-\frac{1}{6}\right) du =$

$u = -3x^2$

$\frac{du}{dx} = -6x$

$du = -6x\,dx$

$-\frac{1}{6}du = x\,dx$

$-\frac{1}{6}e^{u} + C$

$\boxed{-\frac{1}{6}e^{-3x^2} + C}$