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Multiple Zeta Values over Global Function Fields

Riad Masri

Abstract. Let K be a global function field with finite constant field $\mathbb{F}_q$ of order q. In this paper we develop the analytic theory of a multiple zeta function $Z_d(K; s_1, \ldots, s_d)$ in d independent complex variables defined over K. This is the function field analog of the Euler-Zagier multiple zeta function $\zeta_d(s_1, \ldots, s_d)$ of depth d (see [**Z1**]). Our main result is that $Z_d(K; s_1, \ldots, s_d)$ has a meromorphic continuation to all $(s_1, \ldots, s_d)$ in $\mathbb{C}^d$ and is a rational function in each of $q^{-s_1}, \ldots, q^{-s_d}$ with a specified denominator.

1. Introduction

1. In [**Z1**] Zagier defined the multiple zeta function of depth d by

$$\zeta_d(s_1, \ldots, s_d) = \sum_{0<n_1<\cdots<n_d} n_1^{-s_1} \cdots n_d^{-s_d}, \quad s = (s_1, \ldots, s_d) \in \mathbb{C}^d,$$

which is absolutely convergent and analytic in the region

$$\mathrm{Re}(s_k + \cdots + s_d) > d - k + 1, \quad k = 1, \ldots, d. \tag{1.1}$$

He then defined the multiple zeta values of depth d by

$$\zeta_d(a_1, \ldots, a_d) = \sum_{0<n_1<\cdots<n_d} n_1^{-a_1} \cdots n_d^{-a_d}, \quad a_k \in \mathbb{Z}_{\geq 1}, \quad a_d > 1,$$

and described in part the fundamental role these numbers play in geometry, number theory, and physics. This has continued to be revealed in the work of Drinfeld, Goncharov, Kontsevich, Manin, and Zagier, among many others (see the discussion below).

In this paper we initiate the study of multiple zeta values over global function fields. Our objective is to develop the analytic theory of two new multiple zeta functions analogous to ζ_d. The first of these is associated to the polynomial ring $\mathbb{F}_q[T]$ over the finite field $\mathbb{F}_q$ of order q, and the second of these is associated to a global function field K with finite constant field $\mathbb{F}_q$. We will prove that each of these functions has a meromorphic continuation to all s in $\mathbb{C}^d$ and is a rational function in each of $q^{-s_1}, \ldots, q^{-s_d}$ with a specified denominator.

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We now describe the main result of this paper. Let K be a global function field with finite constant field $\mathbb{F}_q$. For a divisor D of K let $\deg(D)$ be its degree and $|D| = q^{\deg(D)}$ be its norm. Let $\mathcal{D}_K^+$ be the sub semi-group of effective divisors of K. We define the multiple zeta function of depth d over K by

$$Z_d\left(K; s_1, \ldots, s_d\right) = \sum_{\substack{(D_1,\ldots,D_d) \in \mathcal{D}_K^+ \times \cdots \times \mathcal{D}_K^+ \\ 0 \leq \deg(D_1) \leq \cdots \leq \deg(D_d)}} \prod_{k=1}^{d} |D_k|^{-s_k},$$

which is absolutely convergent and analytic in the region (1.1). Our main result is the following

Main Theorem. *The multiple zeta function $Z_d\left(K; s_1, \ldots, s_d\right)$ has a meromorphic continuation to all s in $\mathbb{C}^d$ and is a rational function in each of $q^{-s_1}, \ldots, q^{-s_d}$ with a specified denominator. Further, $Z_d\left(K; s_1, \ldots, s_d\right)$ has possible simple poles on the linear subvarieties*

$$s_k + \cdots + s_d = 0, 1, \ldots, d-k+1, \quad k = 1, \ldots, d.$$

The proofs of the Main Theorem and other results in this paper are straightforward. In each case we use either properties of the ring $\mathbb{F}_q[T]$ or the Riemann-Roch theorem for global function fields to reduce to an analysis involving sums of geometric progressions. Nonetheless, we believe that the multiple zeta function $Z_d\left(K; s_1, \ldots, s_d\right)$ provides an interesting example of a multiple Dirichlet series with many possibilities for further research (see the discussion at the end of the introduction). Our ultimate hope is that the study of multiple zeta values over global function fields will lead to a better understanding of multiple zeta values over $\mathbb{Q}$.

In the remaining part of the introduction we will provide some motivating background on the multiple zeta values over $\mathbb{Q}$ and describe the results of this paper in more detail. The multiple zeta values and their generalizations, the multiple polylogarithms at N-th roots of unity,

$$\mathrm{Li}_{a_1,\ldots,a_d}\left(\zeta_N^{\alpha_1}, \ldots, \zeta_N^{\alpha_d}\right) = \sum_{0 < n_1 < \cdots < n_d} \frac{(\zeta_N^{\alpha_1})^{n_1} \cdots (\zeta_N^{\alpha_d})^{n_d}}{n_1^{a_1} \cdots n_d^{a_d}}, \quad \zeta_N = e^{2\pi i/N},$$

have been the focus of much attention in the past 15 years. A particularly important development was Kontsevich's discovery that the multiple zeta values can be expressed as an iterated integral of Chen type. This led to A. Goncharov's interpretation of the multiple zeta values as periods of mixed motives and his remarkable work [**G1, G2, G3, G4, G5**] on mixed Tate motives over $\mathrm{Spec}(\mathbb{Z})$ and proof of the upper bound in Zagier's dimension conjecture.

Zagier's dimension conjecture is perhaps the central problem in the subject. This is a statement about the non-trivial $\mathbb{Q}$-linear relations which arise between multiple zeta values of the same weight (see [**Z1**]). Let Z_w be the $\mathbb{Q}$-algebra generated by all multiple zeta values of weight $w = a_1 + \cdots + a_d$. The dimension conjecture states that $\dim_{\mathbb{Q}} Z_w = d_w$, where $d_0 = 1$, $d_1 = 0$, $d_2 = 1$, and $d_w = d_{w-2} + d_{w-3}$ for $w \geq 3$. For $w \geq 2$, this formula implies that d_w is less than or equal to 2^{w-2}, the number of multiple zeta values of weight w, and hence that there are many non-trivial $\mathbb{Q}$-linear relations between multiple zeta values of the same weight. Goncharov proved the inequality $\dim_{\mathbb{Q}} Z_w \leq d_w$.

For some examples of how the multiple zeta values appear in geometry, number theory, and physics we refer the reader to the work of Drinfeld [**D**], Goncharov and

Manin [**GM**], Kontsevich [**K1, K2**], Manin [**Ma1, Ma2**], and Zagier [**Z2**]. In the excellent survey article [**KZ**] Kontsevich and Zagier discuss the multiple zeta values in the context of periods and special values of L-functions.

Also of interest are the analytic properties of ζ_d. The meromorphic continuation of ζ_d to all s in $\mathbb{C}^d$ was first established by Goncharov and Kontsevich in [**G4**]. The existence of such a meromorphic continuation was initially obscured by the presence of points of indeterminacy, which are special types of singularities arising in several complex variables (see [**M**]). Goncharov and Kontsevich established the meromorphic continuation by using a d-dimensional Mellin transform to express ζ_d as the pairing of a meromorphic distribution and a test function in a certain modified Schwartz class. These methods were subsequently extended by J. Kelliher and the author in [**KM**] to give a sufficient condition for the meromorphic continuation of a more general class of multiple Dirichlet series of Euler-Zagier type.

2. Let $\mathbb{F}_q[T]$ be the ring of polynomials with coefficients in the finite field $\mathbb{F}_q$ of order q. For f in $\mathbb{F}_q[T]$ let $\deg(f)$ be its degree and $|f| = q^{\deg(f)}$ be its norm. Recall that the zeta function of $\mathbb{F}_q[T]$ is defined by

$$Z(\mathbb{F}_q[T], s) = \sum_{\substack{f \in \mathbb{F}_q[T] \\ f \text{ monic}}} |f|^{-s}, \quad \mathrm{Re}(s) > 1.$$

We formally define the multiple zeta function of depth d over $\mathbb{F}_q[T]$ by

$$Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right) = \sum_{\substack{(f_1, \ldots, f_d) \in \mathbb{F}_q[T] \times \cdots \times \mathbb{F}_q[T] \\ f_1, \ldots, f_d \text{ monic} \\ 0 \le \deg(f_1) \le \cdots \le \deg(f_d)}} \prod_{k=1}^{d} |f_k|^{-s_k}. \tag{1.2}$$

In the following theorem we establish the main analytic properties of (1.2).

Theorem 1.1. *(1) $Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right)$ is absolutely convergent and analytic in the region (1.1).*

(2) $Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right)$ has a meromorphic continuation to all s in $\mathbb{C}^d$ and is a rational function in each of $q^{-s_1}, \ldots, q^{-s_d}$. In fact,

$$\prod_{k=1}^{d} \left(1 - q^{d-k+1-(s_k + \cdots + s_d)}\right) Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right) = 1.$$

(3) $Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right)$ has simple poles on the linear subvarieties

$$s_k + \cdots + s_d = d - k + 1, \quad k = 1, \ldots, d.$$

(4) The function

$$\xi_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right) := \prod_{k=1}^{d} \frac{q^{-(s_k + \cdots + s_d - (d-k))}}{1 - q^{-(s_k + \cdots + s_d - (d-k))}} Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right)$$

is invariant under the involution

$$s_1 \mapsto 2d - 1 - 2(s_2 + \cdots + s_d) - s_1.$$

(5) $Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right)$ has the Euler product

$$Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right) = \prod_{\substack{P \in \mathbb{F}_q[T] \\ P \text{ monic} \\ P \text{ irreducible}}} \prod_{k=1}^{d} \left(1 - \frac{1}{|P|^{s_k + \cdots + s_d - (d-k)}}\right)^{-1}.$$

From part (2) of Theorem 1.1 and the identity

$$Z(\mathbb{F}_q[T], s) = \frac{1}{1 - q^{1-s}}$$

we immediately obtain the following

Corollary 1.2. *$Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right)$ has the factorization*

$$Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right) = \prod_{k=1}^{d} Z\left(\mathbb{F}_q[T], s_k + \cdots + s_d - (d-k)\right).$$

Remarks. (i) $Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right)$ is the first example of a Zagier type multiple zeta function possessing a functional equation and Euler product. In fact, it is not difficult to see that ζ_d cannot satisfy a functional equation in any of the independent variables $s_1, \ldots, s_d$. This is because of a shifting in the variables $n_1, \ldots, n_d$ which occurs if one expresses ζ_d as a sum over $\mathbb{Z}^d_{\geq 1}$.

(ii) Using L' Hopital's rule one can compute the residues of $Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right)$ at the simple poles on the linear subvarieties

$$s_k + \cdots + s_d = d - k + 1, \quad k = 1, \ldots, d.$$

For example, suppose $d = 2$. Then the double zeta function $Z_2\left(\mathbb{F}_q[T]; s, w\right)$ has simple poles on the linear subvarieties $s + w = 2$ and $w = 1$ with the following residues:

(1) For $s \neq 1$,

$$\lim_{w \to 1} (w-1) Z_2\left(\mathbb{F}_q[T]; s, w\right) = \frac{\left(1 - q^{1-s}\right)^{-1}}{\log(q)}.$$

(2) For $w \neq 1$,

$$\lim_{s \to 2-w} (s - (2-w)) Z_2\left(\mathbb{F}_q[T]; s, w\right) = \frac{\left(1 - q^{1-w}\right)^{-1}}{\log(q)}.$$

(3) For $s \neq 1$,

$$\lim_{w \to 2-s} (w - (2-s)) Z_2\left(\mathbb{F}_q[T]; s, w\right) = \frac{\left(1 - q^{s-1}\right)^{-1}}{\log(q)}.$$

(iii) Upon examining the proof of part (4) of Theorem 1.1 one sees that the functional equation is actually satisfied by

$$\frac{q^{-(s_1 + \cdots + s_d - (d-1))}}{1 - q^{-(s_1 + \cdots + s_d - (d-1))}} Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right).$$

However, as suggested by Corollary 1.2, by including the factors

$$\frac{q^{-(s_k + \cdots + s_d - (d-k))}}{1 - q^{-(s_k + \cdots + s_d - (d-k))}}, \quad k = 2, \ldots, d,$$

we can obtain additional functional relations. These are no longer involutions, but instead involve a mixing of the variables. For example, suppose $d = 2$. Let

$$w \mapsto 1 - w$$

in

$$\xi_2\left(\mathbb{F}_q[T]; s, w\right).$$

Then arguing as in the proof of part (4) of Theorem 1.1 we obtain the functional relation

$$\xi_2\left(\mathbb{F}_q[T]; s, 1-w\right) = \xi_2\left(\mathbb{F}_q[T]; s - 2w + 1, w\right).$$

3. We now briefly review some background on function fields (see [**R**]). A function field in one variable over a constant field F is a field K containing F and at least one element x transcendental over F such that $K/F(x)$ is a finite algebraic extension. A function field in one variable over a finite constant field $F = \mathbb{F}_q$ is called a global function field. Throughout this paper we assume that K is a global function field.

A prime in K is a discrete valuation ring R with maximal ideal P such that $\mathbb{F}_q \subset P$ and the quotient field of R equals K. The degree $\deg(P)$ of P is the dimension of R/P over $\mathbb{F}_q$, which is finite. The group $\mathcal{D}_K$ of divisors of K is the free abelian group generated by the primes in K. A typical divisor D is written additively by $D = \sum_P a(P)P$. The degree of D is defined by $\deg(D) = \sum_P a(P)\deg(P)$.

Given $a \in K^*$, the divisor (a) of a is defined by $(a) = \sum_P \mathrm{ord}_P(a)P$. The map $K^* \to \mathcal{D}_K$ defined by $a \mapsto (a)$ is a homomorphism whose image $\mathcal{P}_K$ is the group of principal divisors. Two divisors D_1 and D_2 are linearly equivalent $D_1 \sim D_2$ if their difference is principal; that is, $D_1 - D_2 = (a)$ for some $a \in K^*$. Define the divisor class group by $Cl_K = \mathcal{D}_K/\mathcal{P}_K$.

It can be shown that the degree of a principal divisor is zero (see [**R**], Proposition 5.1). Thus, the degree map $\deg : Cl_K \to \mathbb{Z}$ is a homomorphism. Let $\ker(\deg) = Cl_K^0$ be the group of divisor classes of degree zero. It can also be shown that the number $\left|Cl_K^0\right|$ of divisor classes of degree zero is finite (see [**R**], Lemma 5.6). Define the class number of K to be $h_K = \left|Cl_K^0\right|$. Because K has divisors of degree one (see [**S**]), one obtains the exact sequence

$$0 \to Cl_K^0 \to Cl_K \to \mathbb{Z} \to 0.$$

A divisor D is effective if $a(P) \geq 0$ for all P. This is denoted by $D \geq 0$. Given a divisor D, let

$$L(D) = \{x \in K^* : \ (x) + D \geq 0\} \cup \{0\}.$$

It can be shown that $L(D)$ is a finite dimensional vector space over $\mathbb{F}_q$. Let $l(D)$ the dimension of $L(D)$ over $\mathbb{F}_q$.

We are now in a position to state the following form of the Riemann-Roch theorem for global function fields.

Theorem 1.3 (Riemann-Roch). *There is an integer $g \geq 0$ and a divisor class $\mathcal{C}$ such that for $C \in \mathcal{C}$ and $A \in \mathcal{D}_K$ we have*

$$l(A) = \deg(A) - g + 1 + l(C - A).$$

The integer g, which is uniquely determined by K, is called the genus of K.

4. For a divisor D of K let $|D| = q^{\deg(D)}$ be its norm. Then $|D|$ is a positive integer, and for any two divisors D_1 and D_2, $|D_1 + D_2| = |D_1|\,|D_2|$. Let $\mathcal{D}_K^+$ be

the sub semi-group of effective divisors of K. Recall that the zeta function of K is defined by

$$Z(K,s)=\sum_{D\in\mathcal{D}_K^+}|D|^{-s}, \quad \mathrm{Re}(s)>1.$$

We formally define the multiple zeta function of depth d over K by

$$Z_d\left(K;s_1,\ldots,s_d\right)=\sum_{\substack{(D_1,\ldots,D_d)\in\mathcal{D}_K^+\times\cdots\times\mathcal{D}_K^+\\ 0\le\deg(D_1)\le\cdots\le\deg(D_d)}}\prod_{k=1}^{d}|D_k|^{-s_k}. \tag{1.3}$$

If K is a global function field of genus $g=0$, then $K=\mathbb{F}_q(T)$ is the rational function field. In the following theorem we establish the main analytic properties of (1.3) for $K=\mathbb{F}_q(T)$.

Theorem 1.4. *(1) $Z_d\left(\mathbb{F}_q(T);s_1,\ldots,s_d\right)$ is absolutely convergent and analytic in the region (1.1).*

(2) $Z_d\left(\mathbb{F}_q(T);s_1,\ldots,s_d\right)$ has a meromorphic continuation to all s in $\mathbb{C}^d$ and is a rational function in each of $q^{-s_1},\ldots,q^{-s_d}$. In fact,

$$Q\left(q^{-s_1},\ldots,q^{-s_d}\right)Z_d\left(\mathbb{F}_q(T);s_1,\ldots,s_d\right)$$

is a polynomial of degree $\le 2d-1$ in each of $q^{-s_1},\ldots,q^{-s_d}$, where

$$Q\left(q^{-s_1},\ldots,q^{-s_d}\right)=(q-1)^d\left(1-q^{-s_d}\right)\left(1-q^{1-s_d}\right)$$
$$\times\prod_{k=1}^{d-1}\left(1-q^{-(s_k+\cdots+s_d)}\right)\left(1-q^{1-(s_k+\cdots+s_d)}\right)\left(1-q^{2-(s_k+\cdots+s_d)}\right).$$

(3) $Z_d\left(\mathbb{F}_q(T);s_1,\ldots,s_d\right)$ has possible simple poles on the linear subvarieties

$$s_k+\cdots+s_d=0,1,\ldots,d-k+1, \quad k=1,\ldots,d.$$

A multiple zeta value is said to be *reducible* (*completely reducible*) if it can be written as a rational linear combination of products of lower depth (depth one) multiple zeta values. In the following corollary we establish that $Z_d\left(\mathbb{F}_q(T);s_1,\ldots,s_d\right)$ is always a rational linear combination of products of 1-dimensional zeta functions over $\mathbb{F}_q[T]$.

Corollary 1.5. *$Z_d\left(\mathbb{F}_q(T);s_1,\ldots,s_d\right)$ is a rational linear combination of products of zeta functions from the set*

$$\left\{Z\left(\mathbb{F}_q[T],s_k+\cdots+s_d+l\right):\ k=1,\ldots,d,\ l=-1,0,1\right\}.$$

Example. The double zeta function $Z_2\left(\mathbb{F}_q(T);s,w\right)$ has the following decomposition as a rational linear combination of products of 1-dimensional zeta functions over $\mathbb{F}_q[T]$:

$$\begin{aligned}Z_2\left(\mathbb{F}_q(T);s,w\right)=&\frac{q^2}{(q-1)^2}Z\left(\mathbb{F}_q[T],s+w-1\right)Z\left(\mathbb{F}_q[T],w\right)\\&-\frac{q}{(q-1)^2}Z\left(\mathbb{F}_q[T],s+w\right)Z\left(\mathbb{F}_q[T],w+1\right)\\&-\frac{q}{(q-1)^2}Z\left(\mathbb{F}_q[T],s+w\right)Z\left(\mathbb{F}_q[T],w\right)\\&+\frac{1}{(q-1)^2}Z\left(\mathbb{F}_q[T],s+w+1\right)Z\left(\mathbb{F}_q[T],w+1\right).\end{aligned}$$

Corollary 1.5 is in stark contrast to what occurs for the multiple zeta values, where one must place restrictions on the depth d and weight w to guarantee reducibility. For example, when the depth d and weight w of a multiple zeta value have different parity, the multiple zeta value is reducible (see [**T**]). As a first instance of this there is the following fact due to Euler and Zagier: *Every double zeta value $\zeta_2(a,b)$ of odd weight $k=a+b$ is a rational linear combination of the numbers $\zeta(k)$ and $\zeta(r)\zeta(k-r)$ where $2\le r\le k/2$.*

5. If K is a global function field of genus $g\ge 1$, $Z_d\left(K;s_1,\dots,s_d\right)$ no longer decomposes as a rational linear combination of products of 1-dimensional zeta functions. This indicates a more complicated arithmetic structure. In the following theorem we establish the main analytic properties of (1.3) for K of genus $g\ge 1$.

THEOREM 1.6. *Let K be a global function field of genus $g\ge 1$.*

(1) $Z_d\left(K;s_1,\dots,s_d\right)$ is absolutely convergent and analytic in the region (1.1).

(2) $Z_d\left(K;s_1,\dots,s_d\right)$ has a meromorphic continuation to all s in $\mathbb{C}^d$ and is a rational function in each of $q^{-s_1},\dots,q^{-s_d}$ with a specified denominator.

(3) $Z_d\left(K;s_1,\dots,s_d\right)$ has possible simple poles on the linear subvarieties

$$s_k+\cdots+s_d=0,1,\dots,d-k+1,\quad k=1,\dots,d.$$

As indicated in part (2) of Theorem 1.6 the denominator of the rational function $Z_d\left(K;s_1,\dots,s_d\right)$ can always be specified. For example, in 2-dimensions we obtain the following

COROLLARY 1.7. *Let K be a global function field of genus $g\ge 1$. Further, let $u=q^{-(s+w)}$ and $v=q^{-w}$. Then*

$$Z_2\left(K;s,w\right)=\frac{P(u,v)}{Q(u,v)},$$

where

$$P(u,v)\in\mathbb{Q}[u,v]$$

is a polynomial of degree $\le 2g+1$ in u and degree $\le (1+2+\cdots+2g)+2g-2$ in v, and

$$Q(u,v)=(1-u)(1-qu)(1-q^2u)(1-v)(1-qv)\prod_{n=0}^{2g-2}v^n\in\mathbb{Z}[u,v].$$

Remarks. (i) The polynomial $P(u,v)$ can be given explicitly as follows. Let b_n be the number of effective divisors of K of degree n. Then

$$P(u,v)=P_1(u,v)+P_2(u,v)+P_3(u,v),$$

where

$$P_1(u,v)=Q(u,v)\sum_{n=0}^{2g-2}\sum_{m=0}^{2g-2-n}b_nb_{m+n}u^nv^m,$$

$$P_2(u,v)=$$

$$\frac{h_K}{q-1}(1-u)(1-qu)(1-q^2u)\left[q^g(1-v)-(1-qv)\right]v^{2g-1}\sum_{k=0}^{2g-2}b_ku^k\prod_{\substack{n=0\\ n\neq k}}^{2g-2}v^n,$$

and

$$P_3(u,v) = \frac{h_K^2}{(q-1)^2} \prod_{n=0}^{2g-2} v^n \quad \times$$
$$\Big[(1-qu)(1-v)(1-u)\left(q^2\right)^g - \left(1-q^2u\right)(1-v)(1-u)q^g$$
$$-(1-qv)\left(1-q^2u\right)(1-u)q^g + (1-qv)\left(1-q^2u\right)(1-qu)\Big]\, u^{2g-1}.$$

(ii) The rational function $P(u,v)/Q(u,v)$ depends in a complicated way on the function field K. For example, if K has genus $g=1$ then

$$Z_2(K;s,w) = Z(K,w) + \frac{h_K^2}{(q-1)^2} \frac{\prod_{n=0}^{2g-2} v^n}{Q(u,v)} \quad \times$$
$$\left[(1-qu)(1-v)(1-u)q^2 - \left(1-q^2u\right)(1-v)(1-u)q^g\right.$$
$$\left.-(1-qv)\left(1-q^2u\right)(1-u)q^g + (1-qv)\left(1-q^2u\right)(1-qu)\right] u^{2g-1}.$$

6. To conclude we want to emphasize that $Z_d(K;s_1,\dots,s_d)$ provides ample opportunity for further research. Three potentially interesting questions are the following. First, do the special values $Z_d(K;a_1,\dots a_d)$, $a_k \in \mathbb{Z}_{\geq 1}$, $a_d > 1$, have an integral representation analogous to $\zeta_d(a_1,\dots,a_d)$, and if so, does this lead to a cohomological interpretation of these special values? Second, in 1-dimension the polynomial appearing in the numerator of the rational function $Z(K,s)$ is the characteristic polynomial of the action of the Frobenius automorphism on the Tate module (see [**R**], pg. 275). Is there a similar interpretation of the polynomial $P(u,v)$? Third, for K of genus $g \geq 1$ can anything in general can be said about the dependence of the rational function $Z_d(K;s_1,\dots,s_d)$ on the function field K?

This paper is organized as follows. In section 2 we prove Theorem 1.1. In section 3 we prove Theorem 1.4 and Corollary 1.5. Finally, in section 4 we prove Theorem 1.6 and Corollary 1.7.

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2. Proof of Theorem 1.1

The following analytic properties of $Z(\mathbb{F}_q[T],s)$ will be used repeatedly (see [**R**]). We refer to these as properties 1-4.

(1) $Z(\mathbb{F}_q[T],s)$ has a meromorphic continuation to all s in $\mathbb{C}$ and is a rational function in q^{-s}. In fact, $Z(\mathbb{F}_q[T],s) = 1/(1-q^{1-s})$.
(2) $Z(\mathbb{F}_q[T],s)$ has a simple pole at $s=1$ with residue $1/\log(q)$.
(3) The function

$$\xi\left(\mathbb{F}_q[T],s\right) := \frac{q^{-s}}{1-q^{-s}} Z(\mathbb{F}_q[T],s)$$

satisfies the functional equation

$$\xi\left(\mathbb{F}_q[T],1-s\right) = \xi\left(\mathbb{F}_q[T],s\right).$$

(4) $Z(\mathbb{F}_q[T], s)$ has the Euler product

$$Z(\mathbb{F}_q[T], s) = \prod_{\substack{P \in \mathbb{F}_q[T] \\ P \text{ monic} \\ P \text{ irreducible}}} \left(1 - \frac{1}{|P|^s}\right)^{-1}.$$

Proof of Theorem 1.1. Given a finite set X let $|X|$ denote the number of elements in X. Define the nonnegative integers

$$a_{n_1,\ldots,n_d} = \left|\left\{(f_1, \ldots, f_d) \in \mathbb{F}_q[T]^d : \ f_k \text{ monic}, \ \deg(f_k) = n_k, \ k = 1, \ldots, d\right\}\right|$$

and

$$b_{n_k} = |\{f_k \in \mathbb{F}_q[T] : \ f_k \text{ monic}, \ \deg(f_k) = n_k\}|.$$

Then

$$a_{n_1,\ldots,n_d} = \prod_{k=1}^{n} b_{n_k},$$

so that formally,

$$\begin{aligned} Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right) &= \sum_{0 \le n_1 \le \cdots \le n_d} a_{n_1,\ldots,n_d} \prod_{k=1}^{d} \left(q^{n_k}\right)^{-s_k} \\ &= \sum_{0 \le n_1 \le \cdots \le n_d} \prod_{k=1}^{d} b_{n_k} \left(q^{n_k}\right)^{-s_k}. \end{aligned}$$

The last sum can be expressed as

$$\sum_{0 \le n_1 \le \cdots \le n_d} \prod_{k=1}^{d} b_{n_k} \left(q^{n_k}\right)^{-s_k} = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \prod_{k=1}^{d} b_{n_1+\cdots+n_k} \left(q^{-s_k}\right)^{n_1+\cdots+n_k}.$$

Therefore,

$$Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \prod_{k=1}^{d} b_{n_1+\cdots+n_k} \left(q^{-s_k}\right)^{n_1+\cdots+n_k}. \tag{2.1}$$

Because the number of monic polynomials of degree n in $\mathbb{F}_q[T]$ is q^n,

$$b_{n_1+\cdots+n_k} = q^{n_1+\cdots+n_k},$$

so that

$$\begin{aligned} \prod_{k=1}^{d} b_{n_1+\cdots+n_k} \left(q^{-s_k}\right)^{n_1+\cdots+n_k} &= \prod_{k=1}^{d} q^{n_1+\cdots+n_k} \left(q^{-s_k}\right)^{n_1+\cdots+n_k} \\ &= \prod_{k=1}^{d} \left(q^{d-k+1}\right)^{n_k} \left(q^{-(s_k+\cdots+s_d)}\right)^{n_k} \\ &= \prod_{k=1}^{d} \left(q^{d-k+1-(s_k+\cdots+s_d)}\right)^{n_k}. \end{aligned}$$

Thus, if

$$\operatorname{Re}(s_k + \cdots + s_d) > d - k + 1, \quad k = 1, \ldots, d,$$

substituting in (2.1) and summing geometric series yields

$$Z_d\left(\mathbb{F}_q[T];s_1,\ldots,s_d\right)=\prod_{k=1}^{d}\sum_{n_k=0}^{\infty}\left(q^{d-k+1-(s_k+\cdots+s_d)}\right)^{n_k}$$

$$=\prod_{k=1}^{d}\left(1-q^{d-k+1-(s_k+\cdots+s_d)}\right)^{-1}. \tag{2.2}$$

This proves (1).

It follows from (2.2) that $Z_d\left(\mathbb{F}_q[T];s_1,\ldots,s_d\right)$ has a meromorphic continuation to all s in $\mathbb{C}^d$ and is a rational function in $q^{-s_1},\ldots,q^{-s_d}$ with

$$\prod_{k=1}^{d}\left(1-q^{d-k+1-(s_k+\cdots+s_d)}\right)Z_d\left(\mathbb{F}_q[T];s_1,\ldots,s_d\right)=1.$$

This proves (2).

It also follows from (2.2) that $Z_d\left(\mathbb{F}_q[T];s_1,\ldots,s_d\right)$ has simple poles on the linear subvarieties

$$s_k+\cdots+s_d=d-k+1,\quad k=1,\ldots,d.$$

This proves (3).

Write

$$d-k+1-(s_k+\cdots+s_d)=1-(s_k+\cdots+s_d-(d-k)).$$

Then (2.2) and property 1 yield the factorization

$$Z_d\left(\mathbb{F}_q[T];s_1,\ldots,s_d\right)=\prod_{k=1}^{d}Z\left(\mathbb{F}_q[T],s_k+\cdots+s_d-(d-k)\right). \tag{2.3}$$

From the definition of $\xi(\mathbb{F}_q[T],s)$ and (2.3) we obtain

$$\xi_d\left(\mathbb{F}_q[T];s_1,\ldots,s_d\right)=\prod_{k=1}^{d}\xi\left(\mathbb{F}_q[T],s_k+\cdots+s_d-(d-k)\right). \tag{2.4}$$

From property 3 we obtain the functional equation

$$\xi\left(\mathbb{F}_q[T],1-s_1+\cdots+s_d-(d-1)\right)=\xi\left(\mathbb{F}_q[T],s_1+\cdots+s_d-(d-1)\right),$$

or equivalently,

$$\xi\left(\mathbb{F}_q[T],d-(s_1+\cdots+s_d)\right)=\xi\left(\mathbb{F}_q[T],s_1+\cdots+s_d-(d-1)\right). \tag{2.5}$$

Let

$$s_1\mapsto -s_1-2(s_2+\cdots+s_d)+2d-1$$

in

$$\xi_d\left(\mathbb{F}_q[T];s_1,\ldots,s_d\right).$$

It follows that

$$\begin{aligned}
&\xi_d\left(\mathbb{F}_q[T]; -s_1 - 2(s_2 + \cdots + s_d) + 2d - 1, \ldots, s_d\right) \\
&= \xi\left(\mathbb{F}_q[T], (-s_1 - 2(s_2 + \cdots + s_d) + 2d - 1) + s_2 + \cdots + s_d - (d-1)\right) \\
&\quad \times \prod_{k=2}^{d} \xi\left(\mathbb{F}_q[T], s_k + \cdots + s_d - (d-k)\right) \quad (\text{by } (2.4)) \\
&= \xi\left(\mathbb{F}_q[T], d - (s_1 + \cdots + s_d)\right) \prod_{k=2}^{d} \xi\left(\mathbb{F}_q[T], s_k + \cdots + s_d - (d-k)\right) \\
&= \xi\left(\mathbb{F}_q[T], s_1 + \cdots + s_d - (d-1)\right) \prod_{k=2}^{d} \xi\left(\mathbb{F}_q[T], s_k + \cdots + s_d - (d-k)\right) \quad (\text{by } (2.5)) \\
&= \prod_{k=1}^{d} \xi\left(\mathbb{F}_q[T], s_k + \cdots + s_d - (d-k)\right) \\
&= \xi_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right) \quad (\text{by } (2.4)).
\end{aligned}$$

This proves (4).

Finally, from the factorization (2.3) and property 4 we obtain the Euler product

$$Z_d\left(\mathbb{F}_q[T]; s_1, \ldots, s_d\right) = \prod_{\substack{P \in \mathbb{F}_q[T] \\ P \text{ monic} \\ P \text{ irreducible}}} \prod_{k=1}^{d} \left(1 - \frac{1}{|P|^{s_k + \cdots + s_d - (d-k)}}\right)^{-1}.$$

This proves (5). □

3. Proofs of Theorem 1.4 and Corollary 1.5

It is known that for all integers $n \geq 0$ the number b_n of effective divisors of degree n is finite (see [**R**], Lemma 5.5). Further, it is known that if n is sufficiently large compared to the genus of K this number takes an explicit form.

Proposition 3.1. *For all nonnegative integers $n > 2g - 2$,*

$$b_n = h_K \frac{q^{n-g+1} - 1}{q - 1}.$$

We include a proof for convenience.

Proof. Let A be a divisor and $\overline{A}$ be its divisor class. We will need the following two facts.

(1) The number of effective divisors in $\overline{A}$ is

$$\frac{q^{l(A)} - 1}{q - 1}.$$

(2) If $\deg(A) > 2g - 2$, then $l(A) = \deg(A) - g + 1$.

Fact 1 can be found in [**R**], Lemma 5.7.

To prove Fact 2, first observe that by Theorem 1.3, Fact 2 is equivalent to $l(C-A) = 0$. Now, by Theorem 1.3 with $A = C$, $\deg(C) = l(C)+g-2$ (here we used that $l(0) = 1$), and by Theorem 1.3 with $A = 0$, $l(C) = g$. Thus, $\deg(C) = 2g - 2$. Since we have assumed that $\deg(A) > 2g - 2$, it follows that

$$\deg(C - A) = \deg(C) - \deg(A) < 2g - 2 + 2 - 2g = 0.$$

Finally, by [**R**], Lemma 5.3, $\deg(C - A) < 0$ implies that $l(C - A) = 0$. This proves Fact 2.

We now prove the proposition. From the exact sequence

$$0 \to Cl_K^0 \to Cl_K \to \mathbb{Z} \to 0$$

we conclude that for each nonnegative integer n there are $h_K = \left|Cl_K^0\right|$ divisor classes of degree n. List these as $\left\{\overline{A}_1, \ldots, \overline{A}_{h_K}\right\}$. By Fact 1, the number of effective divisors in $\overline{A_i}$ is

$$\frac{q^{l(A_i)} - 1}{q - 1}.$$

Therefore,

$$b_n = \sum_{i=1}^{h_K} \frac{q^{l(A_i)} - 1}{q - 1}.$$

Assume that $n = \deg(A_i) > 2g - 2$. By Fact 2, $l(A_i) = n - g + 1$. We conclude that

$$b_n = h_K \frac{q^{n-g+1} - 1}{q - 1}.$$

□

Proof of Theorem 1.4. Define the nonnegative integers

$$a_{n_1,\ldots,n_d} = \left|\left\{(D_1, \ldots, D_d) \in D_K^+ \times \cdots \times D_K^+ : \ \deg(D_k) = n_k, \ k = 1, \ldots, d\right\}\right|$$

and

$$b_{n_k} = \left|\left\{D_k \in D_K^+ : \ \deg(D_k) = n_k\right\}\right|.$$

Then

$$a_{n_1,\ldots,n_d} = \prod_{k=1}^{n} b_{n_k},$$

so that formally,

$$Z_d\left(\mathbb{F}_q(T); s_1, \ldots, s_d\right) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \prod_{k=1}^{d} b_{n_1+\cdots+n_k} \left(q^{-s_k}\right)^{n_1+\cdots+n_k}. \tag{3.1}$$

By Proposition 3.1,

$$|b_{n_1+\cdots+n_k}| \leq C_k q^{n_1+\cdots+n_k}$$

for some constants $C_k > 0$, $k = 1, \ldots, d$. This yields the estimate

$$\begin{aligned}\prod_{k=1}^{d} \left|b_{n_1+\cdots+n_k} \left(q^{-s_k}\right)^{n_1+\cdots+n_k}\right| &\leq \prod_{k=1}^{d} C_k \left|q^{n_1+\cdots n_k} \left(q^{-s_k}\right)^{n_1+\cdots+n_k}\right| \\ &= \prod_{k=1}^{d} C_k \left(q^{d-k+1-\mathrm{Re}(s_k+\cdots+s_d)}\right)^{n_k}.\end{aligned}$$

Thus, if
$$\mathrm{Re}(s_k+\cdots+s_d)>d-k+1,\quad k=1,\ldots,d,$$
substituting in (3.1) and summing geometric series yields
$$\begin{aligned}\sum_{n_1=0}^{\infty}\cdots\sum_{n_d=0}^{\infty}\prod_{k=1}^{d}&\left|b_{n_1+\cdots+n_k}\left(q^{-s_k}\right)^{n_1+\cdots+n_k}\right|\\ &\le\prod_{k=1}^{d}C_k\sum_{n_k=0}^{\infty}q^{d-k+1-\mathrm{Re}(s_k+\cdots+s_d)}\\ &=\prod_{k=1}^{d}C_k\left(1-q^{d-k+1-\mathrm{Re}(s_k+\cdots+s_d)}\right)^{-1}.\end{aligned}$$
This proves (1) for *any* global function field.

Suppose first that $d=2$. Because $\mathbb{F}_q(T)$ has genus $g=0$ and class number $h_{\mathbb{F}_q(T)}=1$, Proposition 3.1 implies that
$$b_n=\frac{q^{n+1}-1}{q-1}$$
for all integers $n\ge 0$. Then
$$\begin{aligned}&b_nb_{n+m}\left(q^{-s}\right)^n\left(q^{-w}\right)^{n+m}\\ &=\frac{1}{(q-1)^2}\left(q^{n+1}-1\right)\left(q^{n+m+1}-1\right)\left(q^{-s}\right)^n\left(q^{-w}\right)^{n+m}\\ &=\frac{1}{(q-1)^2}\left[q^{n+1}q^{n+m+1}-q^{n+1}-q^{n+m+1}+1\right]\left(q^{-(s+w)}\right)^n\left(q^{-w}\right)^m\end{aligned}$$
for all integers $n,m\ge 0$. Substitute in (3.1) and expand to obtain
$$\begin{aligned}&(q-1)^2Z_2\left(\mathbb{F}_q(T);s,w\right)\\ &=q^2\sum_{n=0}^{\infty}\left(q^{2-(s+w)}\right)^n\sum_{m=0}^{\infty}\left(q^{1-w}\right)^m-q\sum_{n=0}^{\infty}\left(q^{1-(s+w)}\right)^n\sum_{m=0}^{\infty}\left(q^{-w}\right)^m\\ &\quad-q\sum_{n=0}^{\infty}\left(q^{1-(s+w)}\right)^n\sum_{m=0}^{\infty}\left(q^{1-w}\right)^m+\sum_{n=0}^{\infty}\left(q^{-(s+w)}\right)^n\sum_{m=0}^{\infty}\left(q^{-w}\right)^m.\end{aligned}$$
Summing geometric series in the preceding expression yields
$$\begin{aligned}&Z_2\left(\mathbb{F}_q(T);s,w\right)=\\ &\frac{q^2}{(q-1)^2}\frac{1}{1-q^{2-(s+w)}}\frac{1}{1-q^{1-w}}-\frac{q}{(q-1)^2}\frac{1}{1-q^{1-(s+w)}}\frac{1}{1-q^{-w}}\\ &-\frac{q}{(q-1)^2}\frac{1}{1-q^{1-(s+w)}}\frac{1}{1-q^{1-w}}+\frac{1}{(q-1)^2}\frac{1}{1-q^{-(s+w)}}\frac{1}{1-q^{-w}}.\end{aligned}\tag{3.2}$$
It follows from (3.2) that $Z_2\left(\mathbb{F}_q(T);s,w\right)$ has a meromorphic continuation to all (s,w) in $\mathbb{C}^2$ and is a rational function in q^{-s} and q^{-w}.

Let
$$\begin{aligned}&Q\left(q^{-s},q^{-w}\right)=\\ &(q-1)^2\left(1-q^{-(s+w)}\right)\left(1-q^{1-(s+w)}\right)\left(2-q^{-(s+w)}\right)\left(1-q^{-w}\right)\left(1-q^{1-w}\right).\end{aligned}$$

Multiply both sides of (3.2) by $Q\left(q^{-s}, q^{-w}\right)$ to obtain

$$\begin{aligned}
&Q\left(q^{-s}, q^{-w}\right) Z_2\left(\mathbb{F}_q(T); s, w\right) = \\
&\quad q^2\left(1-q^{-(s+w)}\right)\left(1-q^{1-(s+w)}\right)\left(1-q^{-w}\right) \\
&\quad - q\left(1-q^{-(s+w)}\right)\left(1-q^{2-(s+w)}\right)\left(1-q^{1-w}\right) \\
&\quad - q\left(1-q^{-(s+w)}\right)\left(1-q^{2-(s+w)}\right)\left(1-q^{-w}\right) \\
&\quad + \left(1-q^{1-(s+w)}\right)\left(1-q^{2-(s+w)}\right)\left(1-q^{1-w}\right).
\end{aligned}$$

Thus,

$$Q\left(q^{-s}, q^{-w}\right) Z_2\left(\mathbb{F}_q(T); s, w\right)$$

is polynomial of degree ≤ 3 in each of q^{-s} and q^{-w}. This proves (2).

It also follows from (3.2) that $Z_2\left(\mathbb{F}_q(T); s, w\right)$ has possible simple poles on the linear subvarieties $s+w=0,1,2$ and $w=0,1$. This proves (3).

The proof for $d \geq 3$ is analogous. Substitute the product

$$\prod_{k=1}^{d} b_{n_1+\cdots+n_k} = \frac{1}{(q-1)^d} \prod_{k=1}^{d}\left(q^{n_1+\cdots+n_k+1}-1\right)$$

into (3.1), expand, and sum geometric series to obtain

$$Z_d\left(\mathbb{F}_q(T); s_1, \ldots, s_d\right) = \sum_{i=1}^{2^d} R_i\left(q^{-s_1}, \ldots, q^{-s_d}\right), \tag{3.3}$$

where $R_i\left(q^{-s_1}, \ldots, q^{-s_d}\right)$ is a rational function which is a product of one function from each of the sets

$$\begin{aligned}
&\left\{(q-1)^{-3}, \ldots, (q-1)^{-d}\right\}, \\
&\left\{q, \ldots, q^d\right\}, \\
&\left\{\left(1-q^{-s_d}\right)^{-1}, \left(1-q^{1-s_d}\right)^{-1}\right\},
\end{aligned}$$

and $d-1$ functions from the set

$$\left\{\left(1-q^{l-(s_k+\cdots+s_d)}\right)^{-1} : l=0,1,2 \text{ and } k=1,\ldots,d-1.\right\}$$

It follows from (3.3) that $Z_d\left(\mathbb{F}_q(T); s_1, \ldots, s_d\right)$ has a meromorphic continuation to all s in $\mathbb{C}^d$ and is a rational function in $q^{-s_1}, \ldots, q^{-s_d}$.

Let

$$\begin{aligned}
Q\left(q^{-s_1}, \ldots, q^{-s_d}\right) &= (q-1)^d\left(1-q^{-s_d}\right)\left(1-q^{1-s_d}\right) \\
&\times \prod_{k=1}^{d-1}\left(1-q^{-(s_k+\cdots+s_d)}\right)\left(1-q^{1-(s_k+\cdots+s_d)}\right)\left(1-q^{2-(s_k+\cdots+s_d)}\right).
\end{aligned}$$

Then it is not difficult to show that

$$Q\left(q^{-s_1}, \ldots, q^{-s_d}\right) R_i\left(q^{-s_1}, \ldots, q^{-s_d}\right)$$

is a polynomial of degree $\leq 2d-1$ in each of $q^{-s_1}, \ldots, q^{-s_d}$.

Finally, it follows from the explicit form of the functions $R_i\left(q^{-s_1},\ldots,q^{-s_d}\right)$ that

$$\sum_{i=1}^{2^d} R_i\left(q^{-s_1},\ldots,q^{-s_d}\right)$$

has possible simple poles on the linear subvarieties

$$s_k+\cdots+s_d=0,1,\ldots,d-k+1,\quad k=1,\ldots,d.$$

□

Proof of Corollary 1.5. It follows from property 1 that each function $R_i\left(q^{-s_1},\ldots,q^{-s_d}\right)$ is the product of a rational number and products of zeta functions from the set

$$\left\{Z\left(\mathbb{F}_q[T],s_k+\cdots+s_d+l\right):\ k=1,\ldots,d,\ l=-1,0,1\right\}.$$

The corollary now follows from (3.3). □

4. Proofs of Theorem 1.6 and Corollary 1.7

Proof of Theorem 1.6. Part (1) was established in the proof of Theorem 1.4.

We will prove parts (2) and (3) for $d=2$, the proofs for $d\geq 3$ being a more complicated elaboration on the same idea.

Write

$$b_n b_{n+m}\left(q^{-s}\right)^n\left(q^{-w}\right)^{n+m}=b_n b_{n+m}\left(q^{-(s+w)}\right)^n\left(q^{-w}\right)^m$$

and substitute in (3.1) to obtain

$$Z_2\left(K;s,w\right)=\sum_{n=0}^{\infty}b_n\left(q^{-(s+w)}\right)^n\sum_{m=0}^{\infty}b_{n+m}\left(q^{-w}\right)^m. \tag{4.1}$$

Let $u=q^{-(s+w)}$ and $v=q^{-w}$. Decompose the sum (4.1) as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}b_nu^n\sum_{m=0}^{\infty}b_{n+m}v^m&=\sum_{n=0}^{2g-2}b_nu^n\sum_{m=0}^{\infty}b_{n+m}v^m+\sum_{n=2g-1}^{\infty}b_nu^n\sum_{m=0}^{\infty}b_{n+m}v^m\\&=A(u,v)+B(u,v)+C(u,v),\end{aligned}$$

where

$$\begin{aligned}A(u,v)&=\sum_{n=0}^{2g-2}b_nu^n\sum_{m=0}^{2g-2-n}b_{m+n}v^m,\\B(u,v)&=\sum_{n=0}^{2g-2}b_nu^n\sum_{m=2g-1-n}^{\infty}b_{m+n}v^m,\end{aligned}$$

and

$$C(u,v)=\sum_{n=2g-1}^{\infty}b_nu^n\sum_{m=0}^{\infty}b_{n+m}v^m.$$

It is immediate that

$$A(u,v)=\sum_{n=0}^{2g-2}\sum_{m=0}^{2g-2-n}b_nb_{m+n}u^nv^m$$

is analytic for all (s,w) in $\mathbb{C}^2$ and is a rational function in q^{-s} and q^{-w}.

To analyze $B(u,v)$, first observe that if $m \geq 2g-1-n$, then $m+n > 2g-2$. Apply Proposition 3.1 and sum geometric series to obtain

$$\begin{aligned}
\sum_{m=2g-1-n}^{\infty} b_{m+n} v^m &= \frac{h_K}{q-1} \sum_{m=2g-1-n}^{\infty} \left(q^{m+n-g+1} - 1\right) v^m \\
&= \frac{h_K}{q-1} \left[q^{n-g+1} \sum_{m=2g-1-n}^{\infty} (qv)^m - \sum_{m=2g-1-n}^{\infty} v^m \right] \\
&= \frac{h_K}{q-1} \left[q^{n-g+1} \frac{(qv)^{2g-1-n}}{1-qv} - \frac{v^{2g-1-n}}{1-v} \right] \\
&= \frac{h_K}{q-1} \left[\frac{q^g}{1-qv} - \frac{1}{1-v} \right] v^{2g-1-n}. \qquad (4.2)
\end{aligned}$$

Substitute (4.2) into $B(u,v)$ to obtain

$$B(u,v) = \frac{h_K}{q-1} \left[\frac{q^g}{1-qv} - \frac{1}{1-v} \right] v^{2g-1} \sum_{n=0}^{2g-2} b_n \left(uv^{-1}\right)^n .$$

This expression shows that $B(u,v)$ has a meromorphic continuation to all (s,w) in $\mathbb{C}^2$ and is a rational function in q^{-s} and q^{-w}. Further, this expression shows that $B(u,v)$ has simple poles at $v = 1$ and $v = q^{-1}$, which correspond to the linear subvarieties $w = 0$ and $w = 1$, respectively.

To analyze $C(u,v)$, first observe that if $n \geq 2g-1$, then $m+n > 2g-2$ for all $m \geq 0$. Apply Proposition 3.1 and sum geometric series to obtain

$$\begin{aligned}
\sum_{m=0}^{\infty} b_{m+n} v^m &= \frac{h_K}{q-1} \sum_{m=0}^{\infty} \left(q^{m+n-g+1} - 1\right) v^m \\
&= \frac{h_K}{q-1} \left[q^{n-g+1} \sum_{m=0}^{\infty} (qv)^m - \sum_{m=0}^{\infty} v^m \right] \\
&= \frac{h_K}{q-1} \left[q^{n-g+1} \frac{1}{1-qv} - \frac{1}{1-v} \right]. \qquad (4.3)
\end{aligned}$$

Substitute (4.3) into $C(u,v)$, apply Proposition 3.1, and sum geometric series to obtain

$$C(u,v) = \frac{h_K}{q-1} \left[\frac{q^{-g+1}}{1-qv} \sum_{n=2g-1}^{\infty} b_n (qu)^n - \frac{1}{1-v} \sum_{n=2g-1}^{\infty} b_n u^n \right]$$

$$
\begin{aligned}
&= \frac{h_K}{q-1}\left[\frac{q^{-g+1}}{1-qv}\sum_{n=2g-1}^{\infty}\left[\frac{h_K}{q-1}\left(q^{n-g+1}-1\right)\right](qu)^n\right.\\
&\qquad\left.-\frac{1}{1-v}\sum_{n=2g-1}^{\infty}\left[\frac{h_K}{q-1}\left(q^{n-g+1}-1\right)\right]u^n\right]\\
&= \left(\frac{h_K}{q-1}\right)^2\left[\frac{q^{-g+1}}{1-qv}\left[q^{-g+1}\sum_{n=2g-1}^{\infty}(q^2u)^n-\sum_{n=2g-1}^{\infty}(qu)^n\right]\right.\\
&\qquad\left.-\frac{1}{1-v}\left[q^{-g+1}\sum_{n=2g-1}^{\infty}(qu)^n-\sum_{n=2g-1}^{\infty}u^n\right]\right]\\
&= \left(\frac{h_K}{q-1}\right)^2\left[\frac{q^{-g+1}}{1-qv}\left[q^{-g+1}\frac{(q^2u)^{2g-1}}{1-q^2u}-\frac{(qu)^{2g-1}}{1-qu}\right]\right.\\
&\qquad\left.-\frac{1}{1-v}\left[q^{-g+1}\frac{(qu)^{2g-1}}{1-qu}-\frac{u^{2g-1}}{1-u}\right]\right]\\
&= \left(\frac{h_K}{q-1}\right)^2\left[\frac{(q^2)^g}{(1-qv)(1-q^2u)}-\frac{q^g}{(1-qv)(1-qu)}\right.\\
&\qquad\left.-\frac{q^g}{(1-v)(1-qu)}+\frac{1}{(1-v)(1-u)}\right]u^{2g-1}.
\end{aligned}
$$

This expression shows that $C(u,v)$ has a meromorphic continuation to all (s,w) in $\mathbb{C}^2$ and is a rational function in q^{-s} and q^{-w}. Further, this expression shows that $C(u,v)$ has possible simple poles at $v=1$, $v=q^{-1}$, $u=1$, $u=q^{-1}$, and $u=q^{-2}$, which correspond to the linear subvarieties $w=0$, $w=1$, $s+w=0$, $s+w=1$, and $s+w=2$, respectively. □

Proof of Corollary 1.7. First, observe that the polynomial

$$Q(u,v) = (1-qv)\left(1-q^2u\right)(1-qu)(1-v)(1-u)\prod_{n=0}^{2g-2}v^n$$

has degree 3 in u and degree $1+2+\cdots+2g$ in v.

Because

$$A(u,v) = \sum_{n=0}^{2g-2}\sum_{m=0}^{2g-2-n}b_n b_{m+n}u^n v^m$$

has degree $\leq 2g-2$ in u and degree $\leq 2g-2$ in v, $Q(u,v)A(u,v)$ has degree $\leq 2g+1$ in u and degree $\leq (1+2+\cdots+2g)+2g-2$ in v.

Write

$$\sum_{n=0}^{2g-2}b_n\left(uv^{-1}\right)^n = \frac{\sum_{k=0}^{2g-2}b_k u^k\prod_{\substack{n=0\\ n\neq k}}^{2g-2}v^n}{\prod_{n=0}^{2g-2}v^n},$$

so that

$$B(u,v) = \frac{h_K}{q-1}\left[\frac{q^g(1-v)-(1-qv)}{(1-qv)(1-v)}\right]v^{2g-1}\frac{\sum_{k=0}^{2g-2}b_k u^k\prod_{\substack{n=0\\ n\neq k}}^{2g-2}v^n}{\prod_{n=0}^{2g-2}v^n}.$$

Then

$$Q(u,v)B(u,v)=\frac{h_K}{q-1}\quad\times$$
$$\left(1-q^2u\right)(1-qu)(1-u)\left[q^g(1-v)-(1-qv)\right]v^{2g-1}\sum_{k=0}^{2g-2}b_ku^k\prod_{\substack{n=0\\n\neq k}}^{2g-2}v^n$$

has degree $\leq 2g+1$ in u and degree $(1+2+\cdots+2g-2)+2g$ in v.

Similarly, write

$$\begin{aligned}C(u,v)=&\frac{h_K^2}{(q-1)^2}\frac{1}{(1-qv)\left(1-q^2u\right)(1-qu)(1-v)(1-u)}\\ &\times\left[(1-qu)(1-v)(1-u)q^2-\left(1-q^2u\right)(1-v)(1-u)q^g\right.\\ &\quad\left.-(1-qv)\left(1-q^2u\right)(1-u)q^g+(1-qv)\left(1-q^2u\right)(1-qu)\right]u^{2g-1}.\end{aligned}$$

Then

$$\begin{aligned}Q(u,v)C(u,v)=&\frac{h_K^2}{(q-1)^2}\prod_{n=0}^{2g-2}v^n\\ &\times\Big[(1-qu)(1-v)(1-u)\left(q^2\right)^g-\left(1-q^2u\right)(1-v)(1-u)q^g\\ &\quad-(1-qv)\left(1-q^2u\right)(1-u)q^g+(1-qv)\left(1-q^2u\right)(1-qu)\Big]u^{2g-1}\end{aligned}$$

has degree $2g+1$ in u and degree $(1+2+\cdots+2g-2)+1$ in v.

The corollary follows by comparing the degrees of the polynomials on the right hand side of the expression

$$Q(u,v)Z_2(K;s,w)=Q(u,v)A(u,v)+Q(u,v)B(u,v)+Q(u,v)C(u,v).$$

□

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Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: masrirm@mpim-bonn.mpg.de