QUANTITATIVE NONVANISHING OF L-SERIES ASSOCIATED TO CANONICAL HECKE CHARACTERS

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Abstract. We prove quantitative nonvanishing theorems for central values and central derivatives of L–series associated to canonical Hecke characters of imaginary quadratic fields. These results have applications to the study of Chow groups of Kuga-Sato varieties. Some key ingredients in the proofs are bounds for ℓ-torsion in class groups obtained recently by Ellenberg and Venkatesh [EV], and subconvexity bounds for automorphic L–functions due to Duke, Friedlander, and Iwaniec [DFI].

1. Introduction and statement of results

1. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of discriminant $-D$ with $D > 3$ and $D \equiv 3 \bmod 4$. Let $\mathcal{O}_D$ be the ring of integers of K, ε be the quadratic character of K of conductor $\sqrt{-D}\mathcal{O}_D$, and ψ_k be a Hecke character of K of conductor $\sqrt{-D}\mathcal{O}_D$ satisfying

$$\psi_k(\alpha\mathcal{O}_D) = \varepsilon(\alpha)\alpha^{2k-1}, \quad \text{for} \quad (\alpha\mathcal{O}_D, \sqrt{-D}\mathcal{O}_D) = 1, \quad k \in \mathbb{Z}_{\geq 1}.$$

Assume in addition that ψ_k satisfies

$$\overline{\psi_k(\mathfrak{a})} = \psi_k(\overline{\mathfrak{a}}) \quad \text{for ideals } \mathfrak{a} \text{ prime to } \sqrt{-D}\mathcal{O}_D. \tag{1.1}$$

Let $d \equiv 1 \bmod 4$ be a squarefree integer relatively prime to D, and $\psi_{d,k} = (d/N(\cdot))\psi_k$ be the Hecke character of K of conductor $d\sqrt{-D}\mathcal{O}_D$ given by the quadratic twist of ψ_k by $(d/N(\cdot))$. Clearly, $\psi_{d,k}$ also satisfies (1.1). Let $\Psi_{d,k}$ be the set of Hecke characters of the form $\psi_{d,k}$. This set consists of $h(-D)$ characters, where $h(-D)$ is the class number of K. Finally, let $L(\psi_{d,k}, s)$ be the L–series of $\psi_{d,k}$. Its central value is $L(\psi_{d,k}, k)$.

The Hecke characters $\psi_{d,k}$ are examples of *canonical* Hecke characters of K in the sense of Rohrlich [R2]. These characters are arithmetic in nature, and are associated to CM motives. For example, when $k = 1$, they are associated to the so-called $\mathbb{Q}$-curves $A(D)$ first studied by B. Gross in [G]. In particular, the nonvanishing of the central values $L(\psi_{d,1}, 1)$ has applications to the study of the Mordell-Weil rank of these curves (see e.g. [MY]). More generally, there is a natural generalization of the Birch and Swinnerton-Dyer conjecture to L–functions of CM motives, where the nonvanishing of the central values $L(\psi_{d,k}, k)$ for $k \geq 2$ is related via the Bloch-Beilinson conjecture to the dimension of the image of an étale Abel-Jacobi map on the Chow group of a Kuga-Sato variety (see e.g. [M]).

The nonvanishing of the central values $L(\psi_{d,k}, k)$ has been studied extensively by Rohrlich [R1, R2], Montgomery and Rohrlich [MR], Rodriguez-Villegas [RV1, RV2], Rodriguez-Villegas and Zagier [RVZ], Rodriguez-Villegas and Yang [RVY], Yang [Y], Miller and Yang [MY], and Liu and Xu [LX]. In these papers it is assumed that there is *one* orbit of the Galois action on $\Psi_{d,k}$. This allows these authors to prove there exists a $\psi_{d,k} \in \Psi_{d,k}$ for which $L(\psi_{d,k}, k) \neq 0$, and to use a well-known theorem of Shimura [Sh] to conclude that $L(\psi_{d,k}, k) \neq 0$ for *all* $\psi_{d,k} \in \Psi_{d,k}$.

If there is *more than one* orbit of the Galois action on $\Psi_{d,k}$, the existence of one nonvanishing central value no longer implies that all of the central values are nonvanishing. It then becomes of interest to understand how the number of nonvanishing central values in the family $\{L(\psi_{d,k}, k) : \psi_{d,k} \in \Psi_{d,k}\}$ grows with the discriminant $-D$. In this paper we will address this problem by obtaining lower bounds of the form

$$|\{\psi_{d,k} \in \Psi_{d,k} : \ L(\psi_{d,k}, k) \neq 0\}| \gg D^{\delta} \tag{1.2}$$

for some absolute constant(s) $\delta > 0$.

To obtain bounds of the form (1.2) one must resort to methods different from those in the above cited papers. We will develop two methods for obtaining such bounds. Briefly, the first of these combines results on Galois conjugacy of unramified twists of Hecke characters due to Rohrlich [R3] with bounds for ℓ-torsion in class groups obtained recently by Ellenberg and Venkatesh [EV]. The second of these uses subconvexity bounds for automorphic L–functions due to Duke, Friedlander, and Iwaniec [DFI]. A more detailed description of our methods is given in subsections 1.3 and 1.4. Our main results are stated in subsection 1.4.

2. The setup for this paper is as follows. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of discriminant $-D$ with $D > 3$ and $D \equiv 3 \bmod 4$. Let $\mathcal{O}_D$ be the ring of integers of K. Let $\varepsilon(n) = (-D/n) = (n/D)$ be the Kronecker symbol associated to K (that the Kronecker symbol equals the Legendre symbol follows from the quadratic reciprocity law). We also view ε as a quadratic character of $(\mathcal{O}_D/\sqrt{-D}\mathcal{O}_D)^{\times}$ via the isomorphism

$$\mathbb{Z}/D\mathbb{Z} \to \mathcal{O}_D/\sqrt{-D}\mathcal{O}_D.$$

Let ψ_k be a Hecke character of K of conductor $\sqrt{-D}\mathcal{O}_D$ satisfying

$$\psi_k(\alpha\mathcal{O}_D) = \varepsilon(\alpha)\alpha^{2k-1}, \quad \text{for} \quad (\alpha\mathcal{O}_D, \sqrt{-D}\mathcal{O}_D) = 1, \quad k \in \mathbb{Z}_{\geq 1}.$$

Throughout this paper we assume that ψ_k satisfies (1.1).

Let $d \equiv 1 \bmod 4$ be a squarefree integer relatively prime to D. Then $(d/N(\cdot))$ is a primitive Hecke character of K of conductor $d\mathcal{O}_D$. Let

$$\psi_{d,k} = (d/N(\cdot))\psi_k$$

be the Hecke character of K of conductor $d\sqrt{-D}\mathcal{O}_D$ given by the quadratic twist of ψ_k by $(d/N(\cdot))$. Clearly, $\psi_{d,k}$ also satisfies (1.1).

Let $\mathrm{CL}(K)$ be the ideal class group of K, $h(-D)$ be the class number of K, and $\mathrm{CL}(K)^\wedge$ be the group of ideal class group characters of K,

$$\xi : \mathrm{CL}(K) \to \overline{\mathbb{Q}}^{\times}.$$

Let $\Psi_{d,k}$ be the set of Hecke characters of the form $\psi_{d,k}$. This set consists of $h(-D)$ characters, and if $\psi_{d,k}$ is any one of them,

$$\Psi_{d,k} = \{\psi_{d,k}\xi : \ \xi \in \mathrm{CL}(K)^\wedge\}.$$

The L–series of $\psi_{d,k}$ is defined by

$$L(\psi_{d,k}, s) = \sum_{\mathfrak{a}} \psi_{d,k}(\mathfrak{a})N(\mathfrak{a})^{-s}, \quad \text{for} \quad \mathrm{Re}(s) > k + \frac{1}{2},$$

where the sum is over nonzero ideals $\mathfrak{a}$. The L–series $L(\psi_{d,k}, s)$ has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 2k - s$ with central value $L(\psi_{d,k}, k)$ and root number $w(\psi_{d,k}) = \pm 1$. By a slight generalization of the calculation in [G, Theorem 9.1.1], one can show that

$$w(\psi_{d,k}) = (-1)^{k-1}\mathrm{sign}(d)(-1)^{\frac{D+1}{4}}. \tag{1.3}$$

The root number formula (1.3) is crucial to the proofs of our main results (see the second paragraph of subsection 1.4). Therefore, for the convenience of the reader, we will give a proof of (1.3) in the Appendix.

3. Throughout this subsection we assume that either $D = p$ is prime or $\gcd(D, 2k-1) = 1$.

Let $N = K(\zeta_{2k-1})$ be the proper cyclotomic extension of K obtained by adjoining a primitive $(2k-1)$-st root of unity ζ_{2k-1} to K. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/N)$ acts on $\Psi_{d,k}$ by

$$\mathfrak{a} \mapsto \psi_{d,k}(\mathfrak{a})^\sigma, \quad \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/N).$$

For $\psi_{d,k} \in \Psi_{d,k}$, let $\mathcal{O}_{\psi_{d,k}}$ be the Galois orbit

$$\mathcal{O}_{\psi_{d,k}} = \{\psi_{d,k}^\sigma : \ \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/N)\}.$$

Let $\mathrm{CL}_{2k-1}(K)$ be the $(2k-1)$-torsion subgroup of $\mathrm{CL}(K)$. We will make use of results of Rohrlich [R3] to prove the following proposition.

Proposition 1.1. *Assume that either $D = p$ is prime or* $\gcd(D, 2k-1) = 1$. *Then for all $\psi_{d,k} \in \Psi_{d,k}$ we have*

$$\mathcal{O}_{\psi_{d,k}} = \{\psi_{d,k}\xi : \ \xi \in \mathrm{CL}(K)^\wedge, \ \xi \text{ is trivial on } \mathrm{CL}_{2k-1}(K)\}. \tag{1.4}$$

By results of Shimura [Sh],

$$\text{if } L(\psi_{d,k}, k) \neq 0 \text{ for some } \psi_{d,k} \in \Psi_{d,k}, \tag{$*$}$$

then

$$L(\psi_{d,k}^\sigma, k) \neq 0 \text{ for all } \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/N).$$

Thus, assuming condition $(*)$ is satisfied we have

$$h(-D) \geq |\{\psi_{d,k} \in \Psi_{d,k} : \ L(\psi_{d,k}, k) \neq 0\}| \geq \left|\mathcal{O}_{\psi_{d,k}}\right|. \tag{1.5}$$

The study of the nonvanishing of the central values $L(\psi_{d,k}, k)$ can now be divided into two cases.

Case 1: Assume that $\gcd(2k-1, h(-D)) = 1$. Then $\mathrm{CL}_{2k-1}(K) = \{1\}$, so by (1.4) there is one orbit $\mathcal{O}_{\psi_{d,k}}$ of size $\left|\mathcal{O}_{\psi_{d,k}}\right| = h(-D)$. Thus, if condition $(*)$ is satisfied, (1.5) implies that

$$|\{\psi_{d,k} \in \Psi_{d,k} : \ L(\psi_{d,k}, k) \neq 0\}| = h(-D). \tag{1.6}$$

As remarked, this case has been studied extensively in [R1, R2, MR, RV1, RV2, RVZ, RVY, Y, MY, LX]. The strategy in these papers is to impose the hypothesis $\gcd(2k-1, h(-D)) = 1$, prove by some method that condition $(*)$ is satisfied for some (possibly ineffective) range of discriminants $-D$, and conclude that (1.6) holds for $-D$ in this range. The best result in this case is due to Tonghai Yang [Y], who proved that (1.6) holds for $D \equiv 7 \bmod 8$ and $d \ll D^{\frac{1}{4}-\epsilon}$ (with $d > 0$ and $d \equiv 1 \bmod 4$).

Case 2: Assume that $\gcd(2k-1, h(-D)) > 1$, which is the primary case of interest in this paper. Then by (1.4) there are $|\mathrm{CL}_{2k-1}(K)|$ orbits $\mathcal{O}_{\psi_{d,k}}$, each of size

$$\left|\mathcal{O}_{\psi_{d,k}}\right| = \frac{h(-D)}{|\mathrm{CL}_{2k-1}(K)|}. \tag{1.7}$$

In this case it is of interest to obtain *quantitative* information about the growth of the number of nonvanishing central values in the family $\{L(\psi_{d,k}, k) : \ \psi_{d,k} \in \Psi_{d,k}\}$ as $D \to \infty$. That is, we seek a lower bound of the form

$$|\{\psi_{d,k} \in \Psi_{d,k} : \ L(\psi_{d,k}, k) \neq 0\}| \gg D^{\delta}$$

for some absolute constant $\delta > 0$.

4. We now state our main results.

First, observe that by (1.3) we have $w(\psi_{d,k}) = w(\psi_{d,k}\xi)$ for all $\xi \in \mathrm{CL}(K)^{\wedge}$. Then proceeding exactly as in [LX], one finds that for $|d| \ll D^{\frac{1}{12}-\epsilon}$ there exists a Hecke character $\psi_{d,k,j} \in \Psi_{d,k}$ such that $L^{(j)}(\psi_{d,k,j}, k) \neq 0$ for $j = 0, 1$. Thus, condition $(*)$ is satisfied by the central values and central derivatives of $L(\psi_{d,k}, s)$.

We will use (1.5), (1.7), Siegel's theorem, and bounds for ℓ-torsion in class groups due to Ellenberg and Venkatesh [EV] to prove the following quantitative nonvanishing theorem for the central values of $L(\psi_{d,k}, s)$.

Theorem 1.2. *Assume that* $\mathrm{sign}(d) = (-1)^{k+\frac{D-3}{4}}$ *(i.e. the root number of* $\psi_{d,k}$ *is 1). Furthermore, assume that either* $D = p$ *is prime or* $\gcd(D, 2k-1) = 1$.

(a) *For* $|d| \ll D^{\frac{1}{12}-\epsilon}$ *we have*

$$|\{\psi_{d,2} \in \Psi_{d,2} : \ L(\psi_{d,2}, 2) \neq 0\}| \gg_{\delta} D^{\delta}$$

for all $\delta < 1/6$.

(b) *Let* $k \geq 3$. *Then assuming GRH, for* $|d| \ll D^{\frac{1}{12}-\epsilon}$ *we have*

$$|\{\psi_{d,k} \in \Psi_{d,k} : \ L(\psi_{d,k}, k) \neq 0\}| \gg_{\delta} D^{\delta}$$

for all $\delta < 1/2(2k-1)$.

We will replace Shimura's theorem with a result of Zhang [Z] to prove an analogue of (1.5) for the central derivatives of $L(\psi_{d,k}, s)$. We then proceed as in Theorem 1.2 to prove the following quantitative nonvanishing theorem for the central derivatives of $L(\psi_{d,k}, s)$.

Theorem 1.3. *Assume that* $\mathrm{sign}(d) = (-1)^{k+\frac{D+1}{4}}$ *(i.e. the root number of* $\psi_{d,k}$ *is* -1*). Furthermore, assume that either* $D = p$ *is prime or* $\gcd(D, 2k-1) = 1$.

(a) *For* $|d| \ll D^{\frac{1}{12}-\epsilon}$ *we have*

$$|\{\psi_{d,2} \in \Psi_{d,2} : \ L'(\psi_{d,2}, 2) \neq 0\}| \gg_{\delta} D^{\delta}$$

for all $\delta < 1/6$.

(b) *Let* $k \geq 3$. *Then assuming GRH, for* $|d| \ll D^{\frac{1}{12}-\epsilon}$ *we have*

$$|\{\psi_{d,k} \in \Psi_{d,k} : \ L'(\psi_{d,k}, k) \neq 0\}| \gg_{\delta} D^{\delta}$$

for all $\delta < 1/2(2k-1)$.

There is an alternate approach to quantitative nonvanishing. We will combine lower bounds for the averages

$$\sum_{\psi_{d,k}\in\Psi_{d,k}} |L^{(j)}(\psi_{d,k},k)|, \quad \text{for} \quad j=0,1,$$

with subconvexity bounds for $L^{(j)}(\psi_{d,k},k)$ due to Duke, Friedlander, and Iwaniec [DFI] to prove the following quantitative nonvanishing theorems for the central values and central derivatives of $L(\psi_{d,k},s)$.

Theorem 1.4. *Assume that* $\operatorname{sign}(d)=(-1)^{k+\frac{D-3}{4}}$. *Then for all* $\delta<1/60$ *we have*

$$|\{\psi_{d,k}\in\Psi_{d,k}:\ L(\psi_{d,k},k)\neq 0\}| \gg_{d,k,\delta} g(D)$$

where

$$g(D)\sim c(k)D^{\delta} \quad as \quad D\to\infty$$

for some ineffective constant $c(k)>0$.

Theorem 1.5. *Assume that* $\operatorname{sign}(d)=(-1)^{k+\frac{D+1}{4}}$. *Then for all* $\delta<1/60$ *we have*

$$|\{\psi_{d,k}\in\Psi_{d,k}:\ L'(\psi_{d,k},k)\neq 0\}| \gg_{d,k,\delta} D^{\delta}$$

as $D\to\infty$.

Remark 1.6. It is important to emphasize that the lower bounds in Theorems 1.2 and 1.3 hold for *all* d in the range $|d|\ll D^{\frac{1}{12}-\epsilon}$, whereas the lower bounds in Theorems 1.4 and 1.5 depend on a fixed d. The implied constants in all of these lower bounds are ineffective.

Remark 1.7. More general quantitative nonvanishing theorems have recently been obtained by the author and Tonghai Yang [M, MaY]. These theorems are corollaries of asymptotic formulas for averages of the central values $L(\psi_{d,k},k)$, which are proved using explicit formulas for the central values $L(\psi_{d,k},k)$ and the equidistribution of Galois (sub)orbits of Heegner points on modular curves. See also the recent work of Michel and Venkatesh [MV].

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2. The size of the Galois orbit $\mathcal{O}_{\psi_{d,k}}$

In this section we prove Proposition 1.1.

Let $I(d\sqrt{-D}\mathcal{O}_D)$ be the group of nonzero fractional ideals $\mathfrak{a}$ of K prime to $d\sqrt{-D}\mathcal{O}_D$. Then

$$\psi_{d,k}: I(d\sqrt{-D}\mathcal{O}_D)\to \overline{\mathbb{Q}}^{\times}$$

is a homomorphism satisfying

$$\psi_{d,k}(\alpha\mathcal{O}_D)=\varepsilon(\alpha)\alpha^{2k-1}, \quad \text{for} \quad (\alpha\mathcal{O}_D, d\sqrt{-D}\mathcal{O}_D)=1.$$

Define a map
$$\mathrm{CL}(K) \to \mathrm{CL}(K)$$
by
$$[\mathfrak{a}] \to [\mathfrak{a}^{2k-1}].$$
Let $\mathrm{CL}_{2k-1}(K)$ denote the kernel of this map and G denote the inverse image of $\mathrm{CL}_{2k-1}(K)$ in $I(d\sqrt{-D}\mathcal{O}_D)$. Thus $G < I(d\sqrt{-D}\mathcal{O}_D)$ is the subgroup of all $\mathfrak{a}$ such that $\mathfrak{a}^{2k-1}$ is principal. We view a class group character of K as a homomorphism
$$\xi : I(d\sqrt{-D}\mathcal{O}_D) \to \overline{\mathbb{Q}}^{\times}$$
trivial on principal ideals prime to $d\sqrt{-D}\mathcal{O}_D$. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, let $\psi_{d,k}^{\sigma}$ denote the Hecke character of K given by
$$\mathfrak{a} \mapsto \psi_{d,k}(\mathfrak{a})^{\sigma}.$$

Let $N = K(\zeta_{2k-1})$ be the proper cyclotomic extension of K obtained by adjoining a primitive $(2k-1)$-st root of unity ζ_{2k-1} to K. We claim that the values of $\psi_{d,k}$ on G lie in N. To see this, let $\mathfrak{a} \in G$. Then $\mathfrak{a}^{2k-1} = \alpha\mathcal{O}_D$ for some $\alpha \in K^{\times}$, so that
$$\psi_{d,k}(\mathfrak{a})^{2k-1} = \psi_{d,k}(\mathfrak{a}^{2k-1}) = \psi_{d,k}(\alpha\mathcal{O}_D) = \varepsilon(\alpha)\alpha^{2k-1}.$$
Clearly, this implies
$$\psi_{d,k}(\mathfrak{a}) = \pm\zeta\alpha$$
for some $\zeta \in \mu_{2k-1}$, i.e. $\psi_{d,k}(\mathfrak{a}) \in N$.

The following proposition is now a consequence of Rohrlich [R3, Theorem 2].

Proposition 2.1. *Assume that the following condition holds for all $\mathfrak{a}$ in $I(d\sqrt{-D}\mathcal{O}_D)$:*
$$\textit{if } \mathfrak{a}^2 \in G \textit{ and } \psi_{d,k}(\mathfrak{a}) \in N, \textit{ then } \mathfrak{a} \in G. \tag{2.1}$$
Then the characters $\psi_{d,k}^{\sigma}$ with $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/N)$ are precisely the characters $\psi_{d,k}\xi$ with ξ an ideal class character of K trivial on $\mathrm{CL}_{2k-1}(K)$.

It follows from Proposition 2.1 that to prove Proposition 1.1, it suffices to prove that condition (2.1) is satisfied by all $\psi_{d,k} \in \Psi_{d,k}$.

First, assume that $D = p$ is prime. Then $h(-p)$ is odd, so $\mathrm{CL}(K)$ contains no elements of order 2, and condition (2.1) is satisfied by all $\psi_{d,k} \in \Psi_{d,k}$.

Next, assume that $\gcd(D, 2k-1) = 1$. We will prove that
$$\text{if } \mathfrak{a}^2 \text{ is principal and } \psi_{d,k}(\mathfrak{a}) \in N, \text{ then } \mathfrak{a} \text{ is principal,}$$
and hence that condition (2.1) is satisfied by all $\psi_{d,k} \in \Psi_{d,k}$.

Assume that
$$\mathfrak{a}^2 = (\alpha), \quad \text{for some} \quad \alpha \in K^{\times}. \tag{2.2}$$
Then
$$\psi_{d,k}(\mathfrak{a})^2 = \psi_{d,k}(\mathfrak{a}^2) = \psi_{d,k}((\alpha)) = \varepsilon(\alpha)\alpha^{2k-1} \in K, \tag{2.3}$$
so that
$$(\psi_{d,k}(\mathfrak{a}))^2 = (\psi_{d,k}(\mathfrak{a})^2) = (\alpha^{2k-1}) = (\alpha)^{2k-1} = \mathfrak{a}^{2(2k-1)}.$$

Taking square roots, we obtain

$$\mathfrak{a}^{2k-1} = (\psi_{d,k}(\mathfrak{a})).$$

We will use the hypotheses $\gcd(D, 2k-1) = 1$ and $\psi_{d,k}(\mathfrak{a}) \in N$ to prove that $\psi_{d,k}(\mathfrak{a}) \in K$. Thus

$$\mathfrak{a}^2 = (\alpha) \quad \text{and} \quad \mathfrak{a}^{2k-1} = (\psi_{d,k}(\mathfrak{a})), \quad \text{for} \quad \alpha, \psi_{d,k}(\mathfrak{a}) \in K^\times,$$

from which it follows that $\mathfrak{a}$ is principal.

It remains to prove that $\psi_{d,k}(\mathfrak{a}) \in K$. Assume that $\psi_{d,k}(\mathfrak{a})$ is *not* in K. We will derive a contradiction in two steps. In step one we prove that the quadratic extension $K(\psi_{d,k}(\mathfrak{a}))/K$ is unramified at all primes not dividing 2. In step two we use the hypotheses $\gcd(D, 2k-1) = 1$ and $\psi_{d,k}(\mathfrak{a}) \in N$ to prove that $K(\psi_{d,k}(\mathfrak{a}))/K$ is ramified at some prime not dividing 2.

Step 1: Assume that $\psi_{d,k}(\mathfrak{a})$ is not in K. We know by (2.3) that $\psi_{d,k}(\mathfrak{a})^2 \in K$, thus $K(\psi_{d,k}(\mathfrak{a}))/K$ is a quadratic extension. Therefore, if $\mathfrak{p}$ does not divide 2,

$$K(\psi_{d,k}(\mathfrak{a}))/K \text{ is unramified at } \mathfrak{p} \text{ if and only if } 2 | val_{\mathfrak{p}}(\psi_{d,k}(\mathfrak{a})^2).$$

Using equations (2.3) and (2.2) we compute

$$\begin{aligned} val_{\mathfrak{p}}(\psi_{d,k}(\mathfrak{a})^2) &= val_{\mathfrak{p}}(\varepsilon(\alpha)\alpha^{2k-1}) \\ &= (2k-1) val_{\mathfrak{p}}(\alpha) \\ &= (2k-1) val_{\mathfrak{p}}(\mathfrak{a}^2). \end{aligned}$$

Clearly, $2 | val_{\mathfrak{p}}(\mathfrak{a}^2)$. Thus $K(\psi_{d,k}(\mathfrak{a}))/K$ is unramified at primes $\mathfrak{p}$ not dividing 2.

Step 2: By assumption $\psi_{d,k}(\mathfrak{a}) \in N$, thus $K(\psi_{d,k}(\mathfrak{a})) \subset N$. This yields the following field diagram:

$$\begin{tikzcd}
N \arrow[dd, dash] \arrow[dr, dash] & \\
& \mathbb{Q}(\zeta_{2k-1}) \arrow[dd, dash] \\
K(\psi_{d,k}(\mathfrak{a})) \arrow[dd, dash] \arrow[dr, dash] & \\
& L \arrow[dd, dash] \\
K \arrow[dr, dash] & \\
& \mathbb{Q}
\end{tikzcd}$$

In the diagram $L = K(\psi_{d,k}(\mathfrak{a})) \cap \mathbb{Q}(\zeta_{2k-1})$. We claim that L has degree 2 over $\mathbb{Q}$. Clearly, $[L : \mathbb{Q}] = 2$ if and only if K is not contained in $\mathbb{Q}(\zeta_{2k-1})$. We know that $K/\mathbb{Q}$ is ramified at some prime, and $\mathbb{Q}(\zeta_{2k-1})/\mathbb{Q}$ is ramified at precisely those primes $p | 2k-1$. Therefore, if $K \subset \mathbb{Q}(\zeta_{2k-1})$, $K/\mathbb{Q}$ is ramified at some prime $p | 2k-1$. This contradicts the assumption $\gcd(D, 2k-1) = 1$.

Since $L \subset \mathbb{Q}(\zeta_{2k-1})$, $L/\mathbb{Q}$ is ramified at some prime $p|2k-1$ ($p \neq 2$). Because $\gcd(D, 2k-1) = 1$, we have $\gcd(p, D) = 1$. Thus $K/\mathbb{Q}$ is unramified at p. It follows from the diagram

$$\begin{tikzcd}
K(\psi_{d,k}(\mathfrak{a})) \arrow[dd, dash] \arrow[dr, dash] & \\
& L \arrow[dd, dash] \\
K \arrow[dr, dash] & \\
& \mathbb{Q}
\end{tikzcd}$$

that $K(\psi_{d,k}(\mathfrak{a}))/K$ is ramified at some prime $\mathfrak{p}|p$ which does not divide 2.

3. Quantitative nonvanishing via bounds for ℓ-torsion

In this section we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Assume that $w(\psi_{d,k}) = 1$. We observed in subsection 1.4 that for $|d| \ll D^{\frac{1}{12}-\epsilon}$, there exists at least one Hecke character $\psi_{d,k}$ such that $L(\psi_{d,k}, k) \neq 0$. By Weil's converse theorem, $\psi_{d,k}$ is associated to a normalized, primitive, CM newform f of weight $2k$ and level d^2D^2 (see section 4). It follows from Shimura [Sh, Theorem 2 and Proposition 4] that

$$L(\psi_{d,k}^{\sigma}, k) \neq 0 \quad \text{for all} \quad \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/N).$$

We now see that for $|d| \ll D^{\frac{1}{12}-\epsilon}$,

$$|\{\psi_{d,k} \in \Psi_{d,k} : \ L(\psi_{d,k}, k) \neq 0\}| \geq \left|\mathcal{O}_{\psi_{d,k}}\right|. \tag{3.1}$$

Assume that either $D = p$ is prime or $\gcd(D, 2k-1) = 1$. Then by Proposition 1.1,

$$\mathcal{O}_{\psi_{d,k}} = \{\psi_{d,k}\xi : \ \xi \in \mathrm{CL}(K)^{\wedge}, \ \xi_{|\mathrm{CL}_{2k-1}(K)} = 1\}.$$

There is a (non-canonical) isomorphism

$$\{\xi \in \mathrm{CL}(K)^{\wedge} : \ \xi_{|\mathrm{CL}_{2k-1}(K)} = 1\} = (\mathrm{CL}(K)/\mathrm{CL}_{2k-1}(K))^{\wedge} \cong \mathrm{CL}(K)/\mathrm{CL}_{2k-1}(K).$$

Thus

$$\left|\mathcal{O}_{\psi_{d,k}}\right| = |\mathrm{CL}(K)/\mathrm{CL}_{2k-1}(K)| = \frac{h(-D)}{|\mathrm{CL}_{2k-1}(K)|}. \tag{3.2}$$

From (3.1) and (3.2) we find that for $|d| \ll D^{\frac{1}{12}-\epsilon}$,

$$|\{\psi_{d,k} \in \Psi_{d,k} : \ L(\psi_{d,k}, k) \neq 0\}| \geq \frac{h(-D)}{|\mathrm{CL}_{2k-1}(K)|}. \tag{3.3}$$

In beautiful recent work, Ellenberg and Venkatesh [EV] established unconditional and conditional bounds for ℓ-torsion in class groups. The following theorem is a consequence of special cases of [EV, Propositions 1 and 2].

Theorem 3.1. *(a)* $|\mathrm{CL}_3(K)| \ll_{\epsilon} D^{\frac{1}{3}+\epsilon}$.

(b) Let $k \geq 3$. Then assuming GRH, $|\mathrm{CL}_{2k-1}(K)| \ll_{\epsilon} D^{\frac{1}{2}-\frac{1}{2(2k-1)}+\epsilon}$.

By combining (3.3) with Siegel's theorem

$$h(-D) \gg_\epsilon D^{\frac{1}{2}-\epsilon}$$

and the upper bounds for $|\mathrm{CL}_{2k-1}(K)|$ in Theorem 3.1, we obtain Theorem 1.2.

Proof of Theorem 1.3. Assume that $w(\psi_{d,k}) = -1$. We observed in subsection 1.4 that for $|d| \ll D^{\frac{1}{12}-\epsilon}$, there exists at least one Hecke character $\psi_{d,k}$ such that $L'(\psi_{d,k}, k) \neq 0$. It follows from Zhang [Z, Corollary 0.3.5] that

$$L'(\psi_{d,k}^{\sigma}, k) \neq 0 \quad \text{for all} \quad \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/N).$$

Here we have again used Weil's converse theorem to associate $\psi_{d,k}$ to a primitive, weight $2k$ newform f. Thus, for $|d| \ll D^{\frac{1}{12}-\epsilon}$ we have

$$|\{\psi_{d,k} \in \Psi_{d,k} : \ L'(\psi_{d,k}, k) \neq 0\}| \geq \left|\mathcal{O}_{\psi_{d,k}}\right|.$$

Assume that either $D = p$ is prime or $\gcd(D, 2k-1) = 1$. Then arguing as in the proof of Theorem 1.2, we obtain Theorem 1.3.

4. Quantitative nonvanishing via subconvexity bounds

In this section we prove Theorems 1.4 and 1.5.

Using $\Psi_{d,k} = \{\psi_{d,k}\xi : \ \xi \in \mathrm{CL}(K)^\wedge\}$ and orthogonality one obtains the identity

$$\sum_{\psi_{d,k} \in \Psi_{d,k}} L(\psi_{d,k}, s) = h(-D) L(\psi_{d,k}, s, [\mathcal{O}_D]), \tag{4.1}$$

where

$$L(\psi_{d,k}, s, [\mathcal{O}_D]) = \sum_{\mathfrak{a} \in [\mathcal{O}_D]} \psi_{d,k}(\mathfrak{a}) N(\mathfrak{a})^{-s}, \quad \text{for} \quad \mathrm{Re}(s) > k + \frac{1}{2},$$

is the partial Hecke L–series associated to the ideal class $[\mathcal{O}_D]$.

Define

$$a_D^{(j)}(d,k) := L^{(j)}(\psi_{d,k}, k, [\mathcal{O}_D]), \quad \text{for} \quad j = 0, 1.$$

Let $\psi_{d,k,u}$ denote the unitarization of $\psi_{d,k}$. By combining (4.1), the identity

$$L(\psi_{d,k}, s + k - \frac{1}{2}) = L(\psi_{d,k,u}, s),$$

and Siegel's theorem

$$h(-D) \gg_\epsilon D^{\frac{1}{2}-\epsilon},$$

we obtain the estimate

$$\sum_{\psi_{d,k,u} \in \Psi_{d,k,u}} |L^{(j)}(\psi_{d,k,u}, 1/2)| \gg_\epsilon |a_D^{(j)}(d,k)| D^{\frac{1}{2}-\epsilon}. \tag{4.2}$$

By Weil's converse theorem, $\psi_{d,k,u}$ is associated to a normalized, primitive, CM newform $f_{\psi_{d,k,u}}$ of weight $2k$ and level d^2D^2 with *trivial* nebentypus $\chi_{f_{\psi_{d,k,u}}} \equiv \mathbf{1}$ (see [I, pg. 213]). Then by a deep theorem of Duke, Friedlander, and Iwaniec [DFI, Theorem 3], one has the subconvexity bound

$$L^{(j)}(f_{\psi_{d,k,u}}, 1/2) \ll_{d,\epsilon} D^{\frac{1}{2}-\frac{1}{60}+\epsilon}.$$

A straightforward estimate now yields

$$\sum_{\psi_{d,k,u}\in\Psi_{d,k,u}} |L^{(j)}(\psi_{d,k,u},1/2)| \\ \ll_{d,k} |\{\psi_{d,k,u}\in\Psi_{d,k,u}:\ L^{(j)}(\psi_{d,k,u},1/2)\neq 0\}| D^{\frac{1}{2}-\frac{1}{60}+\epsilon}. \tag{4.3}$$

Combine estimates (4.2) and (4.3) to obtain

$$|\{\psi_{d,k,u}\in\Psi_{d,k,u}:\ L^{(j)}(\psi_{d,k,u},1/2)\neq 0\}| \gg_{d,k,\delta} |a_D^{(j)}(d,k)| D^{\delta} \tag{4.4}$$

for all $\delta < 1/60$.

Proof of Theorem 1.4. Assume that $w(\psi_{d,k}) = 1$. Using the estimates for I_1 and I_2 given in [LX, Sections 2 and 3], we find that

$$|a_D^{(0)}(d,k)| \geq 2(I_1 - |I_2|) \geq 2f_1(D) \tag{4.5}$$

where

$$f_1(D) = c_1(\epsilon,k)\,|d|^{-\epsilon}\,D^{-\epsilon} - c_2(\epsilon)k4^k\,|d|^{\frac{3}{4}+\epsilon}\,D^{-\frac{1}{16}+\epsilon}.$$

Note that for $\epsilon < 1/16$ we have

$$f_1(D) \to 0 \quad \text{as} \quad D\to\infty.$$

However, for $\epsilon < \delta$ ($< 1/60$),

$$g(D) := f_1(D)D^{\delta} \sim c_1(\epsilon,k)D^{\delta-\epsilon} \quad \text{as} \quad D\to\infty.$$

It follows from estimates (4.4) and (4.5) that for all $\epsilon < \delta < 1/60$,

$$|\{\psi_{d,k,u}\in\Psi_{d,k,u}:\ L(\psi_{d,k,u},1/2)\neq 0\}| \gg_{d,k,\delta} g(D)$$

where

$$g(D) \sim c_1(\epsilon,k)D^{\delta-\epsilon} \quad \text{as} \quad D\to\infty.$$

Proof of Theorem 1.5. Assume that $w(\psi_{d,k}) = -1$. Using the estimates for R_k and C given in [LX, Sections 5 and 6], we find that

$$\begin{aligned} |a_D^{(1)}(d,k)| &\geq 2(|R_k| - |C|) \\ &\geq 2\left[.0351 - c_3(\epsilon)(|d|\,D)^{-\frac{1}{16}+\epsilon} - c_4(k,\epsilon)\,|d|^{\frac{3}{4}+\epsilon}\,D^{-\frac{1}{16}+\epsilon}\right] \\ &= 2\left[.0351 - D^{-\frac{1}{16}+\epsilon}\{c_3(\epsilon)\,|d|^{-\frac{1}{16}+\epsilon} + c_4(k,\epsilon)\,|d|^{\frac{3}{4}+\epsilon}\}\right] \\ &\geq 2f_2(D) \end{aligned} \tag{4.6}$$

where

$$f_2(D) = .0351 - c_5(k,\epsilon)\,|d|^{\frac{3}{4}+\epsilon}\,D^{-\frac{1}{16}+\epsilon}.$$

For $\epsilon < 1/16$ we have

$$f_2(D) \to .0351 \quad \text{as} \quad D\to\infty.$$

Therefore, from estimate (4.6) we find that for $\epsilon < 1/16$,

$$\inf_{D^{\frac{1}{12}-\epsilon}\gg|d|} |a_D^{(1)}(d,k)| \geq 2 \inf_{D^{\frac{1}{12}-\epsilon}\gg|d|} f_2(D) > 0. \tag{4.7}$$

Finally, it follows from estimates (4.4) and (4.7) that for all $\delta < 1/60$,

$$|\{\psi_{d,k,u} \in \Psi_{d,k,u} :\ L'(\psi_{d,k,u}, 1/2) \neq 0\}| \gg_{d,k,\delta,\epsilon} D^{\delta}$$

as $D \to \infty$.

Appendix

In this Appendix we calculate the root number $w(\psi_{d,k})$. As remarked in the introduction, the calculation is a mild generalization of [G, Theorem 9.1.1], the differences being that in [G] the quantity D is a prime > 3 rather than a product of distinct odd primes, and the quantity k is 1 rather than an arbitrary positive integer.

Let $\psi_{k,u}$ be a unitary Hecke character of K of conductor $\sqrt{-D}\mathcal{O}_D$ satisfying

$$\psi_{k,u}(\alpha\mathcal{O}_D) = \varepsilon(\alpha)\left(\frac{\alpha}{|\alpha|}\right)^{2k-1}, \quad \text{for} \quad (\alpha\mathcal{O}_D, \sqrt{-D}\mathcal{O}_D) = 1, \quad k \in \mathbb{Z}_{\geq 1}.$$

Assume in addition that $\psi_{k,u}$ satisfies (1.1). Let

$$\psi_{d,k,u} = \left(\frac{d}{N(\cdot)}\right)\psi_{k,u}$$

be the Hecke character of K of conductor $d\sqrt{-D}\mathcal{O}_K$ given by the quadratic twist of $\psi_{k,u}$ by $(d/N(\cdot))$. Clearly, $\psi_{d,k,u}$ also satisfies (1.1).

Let $\Psi_{d,k,u}$ be the set of all Hecke characters of the form $\psi_{d,k,u}$. This set consists of $h(-D)$ characters, and if $\psi_{d,k,u}$ is any one of them,

$$\Psi_{d,k,u} = \{\psi_{d,k,u}\xi :\ \xi \in \mathrm{CL}(K)^\wedge\}.$$

Now, a straightforward calculation yields

$$L(\psi_{d,k}, s + k - \frac{1}{2}) = L(\psi_{d,k,u}, s), \tag{4.8}$$

so that

$$L(\psi_{d,k}, k) = L(\psi_{d,k,u}, 1/2).$$

Let

$$\Lambda(\psi_{d,k,u}, s) := (2\pi)^{-s}(|d|\, D)^s\Gamma(s + k - \frac{1}{2})L(\psi_{d,k,u}, s).$$

Then $\Lambda(\psi_{d,k,u}, s)$ is entire, bounded in vertical strips, and satisfies the functional equation

$$\Lambda(\psi_{d,k,u}, s) = w(\psi_{d,k,u})\Lambda(\overline{\psi_{d,k,u}}, 1 - s)$$

where $w(\psi_{d,k,u})$ is the root number of $\psi_{d,k,u}$ (which is the same as the root number of $\psi_{d,k}$). Because $\psi_{d,k,u}$ satisfies (1.1),

$$L(\psi_{d,k,u}, s) = L(\overline{\psi_{d,k,u}}, s).$$

Therefore,

$$\Lambda(\psi_{d,k,u}, s) = w(\psi_{d,k,u})\Lambda(\psi_{d,k,u}, 1 - s) \tag{4.9}$$

with $w(\psi_{d,k,u}) = \pm 1$.

Because the Hecke character $\psi_{d,k,u}$ is primitive of weight $2k-1$ and conductor $d\sqrt{-D}\mathcal{O}_D$, it follows from [I, pg. 211, equation (12.26)] that

$$\begin{aligned} w(\psi_{d,k,u}) &= i^{-(2k-1)} W(\psi_{d,k,u}) N(d\sqrt{-D}\mathcal{O}_D)^{-1/2} \\ &= i^{-(2k-1)} W(\psi_{d,k,u}) (|d|^2\, D)^{-1/2}, \end{aligned} \tag{4.10}$$

where $W(\psi_{d,k,u})$ is the Gauss sum of $\psi_{d,k,u}$.

Recall that if ψ is a primitive Hecke character of K of weight r and conductor $\mathfrak{f}$, the Gauss sum $W(\psi)$ is defined as follows (see [I, pg. 210, equation (12.21)]). Let $\mathfrak{c}$ be an integral ideal in the class of $\mathfrak{f}^{-1}$ such that $(\mathfrak{c}, \mathfrak{f}) = \mathcal{O}_D$. Thus $\mathfrak{c}\mathfrak{f}$ is principal. Let $\mathfrak{c}\mathfrak{f}\mathfrak{d} = \gamma\mathcal{O}_D$ for some $\gamma \in \mathcal{O}_D$ (here $\mathfrak{d}$ is the different of K, which is a principal ideal). Then

$$W(\psi) = \sum_{\alpha \in \mathfrak{c}/\mathfrak{c}\mathfrak{f}} \psi\left(\frac{\alpha\mathcal{O}_D}{\mathfrak{c}}\right) \left(\frac{\alpha/\gamma}{|\alpha/\gamma|}\right)^{-r} e\left(\mathrm{tr}\frac{\alpha}{\gamma}\right), \tag{4.11}$$

where $e(z) = e^{2\pi i z}$. The sum is independent of the choice of $\mathfrak{c}$ and γ.

We aim to compute $W(\psi_{d,k,u})$ by reducing this (in essence) to the computation of the Gauss sum $W(\psi_{k,u})$.

First, suppose that ξ_1 and ξ_2 are primitive Hecke characters of K of conductors $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively, and that $(\mathfrak{m}_1, \mathfrak{m}_2) = 1$. It follows from [I, pg. 210, equation (12.23)] that

$$W(\xi_1\xi_2) = \xi_1(\mathfrak{m}_2)\xi_2(\mathfrak{m}_1) W(\xi_1) W(\xi_2). \tag{4.12}$$

By definition of $\psi_{d,k,u}$ we have

$$\psi_{d,k,u} = \xi_1\xi_2,$$

where

$$\xi_1 = \left(\frac{d}{N(\cdot)}\right)$$

is a primitive Hecke character of K of conductor $\mathfrak{m}_1 = d\mathcal{O}_D$, and

$$\xi_2 = \psi_{k,u}$$

is a primitive Hecke character of K of conductor $\mathfrak{m}_2 = \sqrt{-D}\mathcal{O}_D$. Furthermore, we have assumed that $(d, D) = 1$, thus $(\mathfrak{m}_1, \mathfrak{m}_2) = 1$. Then using (4.12), the definition of $\psi_{k,u}$, and quadratic reciprocity, we obtain

$$\begin{aligned} W(\psi_{d,k,u}) &= \left(\frac{d}{N(\sqrt{-D}\mathcal{O}_D)}\right) \varepsilon((d)) \left(\frac{d}{|d|}\right)^{2k-1} W(\left(\frac{d}{N(\cdot)}\right)) W(\psi_{k,u}) \\ &= \left(\frac{d}{D}\right)\left(\frac{-D}{d}\right)\left(\frac{d}{|d|}\right)^{2k-1} W(\left(\frac{d}{N(\cdot)}\right)) W(\psi_{k,u}) \\ &= \left(\frac{d}{|d|}\right)^{2k-1} W(\left(\frac{d}{N(\cdot)}\right)) W(\psi_{k,u}). \end{aligned} \tag{4.13}$$

Now, because the Dirichlet character $(d/\cdot)$ is primitive of conductor $|d|$, it follows from [I, Lemma 12.4] that

$$\begin{aligned} W(\left(\frac{d}{N(\cdot)}\right)) &= \left(\frac{-D}{|d|}\right)\left(\frac{d}{D}\right)\tau(\left(\frac{d}{\cdot}\right))^2 \\ &= \left(\frac{-D}{|d|}\right)\left(\frac{d}{D}\right) d, \end{aligned} \tag{4.14}$$

where we have used that the Gauss sum $\tau((d/\cdot))$ is given by

$$\tau(\left(\frac{d}{\cdot}\right)) = \begin{cases} d^{\frac{1}{2}}, & d > 0 \\ i\,|d|^{\frac{1}{2}}, & d < 0. \end{cases}$$

We need to compute the Gauss sum $W(\psi_{k,u})$ (see also [FI, pg. 673]). Because the Hecke character $\psi_{k,u}$ is of weight $r = 2k-1$ and has conductor which is principal and given by $\mathfrak{d} = \sqrt{-D}\mathcal{O}_D$, equation (4.11) simplifies to

$$W(\psi_{k,u}) = \sum_{\alpha \bmod \mathfrak{d}} \psi_{k,u}((\alpha))\left(\frac{\alpha}{|\alpha|}\right)^{-(2k-1)} e\left(\frac{\alpha+\bar{\alpha}}{\gamma}\right).$$

Furthermore, because all classes $\alpha \bmod \mathfrak{d}$ are represented by rational integers modulo D,

$$W(\psi_{k,u}) = \sum_{n \bmod D} \psi_{k,u}((n))\left(\frac{n}{|n|}\right)^{-(2k-1)} e\left(\frac{2n}{D}\right). \tag{4.15}$$

Substitute

$$\psi_{k,u}((n)) = \varepsilon(n)\left(\frac{n}{|n|}\right)^{2k-1}$$

in (4.15) to obtain

$$W(\psi_{k,u}) = \sum_{n \bmod D} \varepsilon(n) e\left(\frac{2n}{D}\right).$$

It is well-known that

$$\sum_{n \bmod D} \varepsilon(n) e\left(\frac{2n}{D}\right) = \varepsilon(2)\tau(\varepsilon),$$

where $\tau(\varepsilon)$ is the Gauss sum of the primitive Dirichlet character $\varepsilon(\cdot) = \left(\frac{-D}{\cdot}\right)$ of conductor D. Because $\left(\frac{-D}{\cdot}\right)$ is odd,

$$\left(\frac{-D}{2}\right) = (-1)^{\frac{D+1}{4}} \quad \text{and} \quad \tau(\varepsilon) = iD^{\frac{1}{2}}.$$

Therefore

$$W(\psi_{k,u}) = (-1)^{\frac{D+1}{4}} iD^{\frac{1}{2}}. \tag{4.16}$$

Substitute (4.14) and (4.16) in (4.13) to obtain

$$W(\psi_{d,k,u}) = \left(\frac{d}{|d|}\right)^{2k-1}\left(\frac{-D}{|d|}\right)\left(\frac{d}{D}\right) d(-1)^{\frac{D+1}{4}} iD^{\frac{1}{2}}. \tag{4.17}$$

Then substitute (4.17) in (4.10) to obtain

$$\begin{aligned} w(\psi_{d,k,u}) &= i^{-(2k-1)} \left(\frac{d}{|d|}\right)^{2k-1} \left(\frac{-D}{|d|}\right) \left(\frac{d}{D}\right) d(-1)^{\frac{D+1}{4}} i D^{\frac{1}{2}} (|d|^2 D)^{-1/2} \\ &= (i^2)^{-(k-1)} \left(\frac{d}{|d|}\right)^{2k} \left(\frac{-D}{|d|}\right) \left(\frac{d}{D}\right) (-1)^{\frac{D+1}{4}} \\ &= (-1)^{k-1} \left(\frac{-D}{|d|}\right) \left(\frac{d}{D}\right) (-1)^{\frac{D+1}{4}}. \end{aligned} \tag{4.18}$$

Suppose that $d > 0$ in (4.18). Then using $d = |d|$ and quadratic reciprocity we find that

$$w(\psi_{d,k,u}) = (-1)^{k-1} \mathrm{sign}(d) (-1)^{\frac{D+1}{4}}.$$

Finally, suppose that $d < 0$ in (4.18). Then using $d = -|d|$, quadratic reciprocity, and the fact that

$$\left(\frac{-1}{D}\right) = (-1)^{\frac{D-1}{2}} = -1$$

for $D \equiv 3 \bmod 4$, we find that

$$\begin{aligned} w(\psi_{d,k,u}) &= (-1)^{k-1} \left(\frac{-D}{|d|}\right) \left(\frac{-|d|}{D}\right) (-1)^{\frac{D+1}{4}} \\ &= (-1)^{k-1} \left(\frac{-D}{|d|}\right) \left(\frac{|d|}{D}\right) \left(\frac{-1}{D}\right) (-1)^{\frac{D+1}{4}} \\ &= (-1)^{k-1} \mathrm{sign}(d) (-1)^{\frac{D+1}{4}}. \end{aligned}$$

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