# THE ASYMPTOTIC DISTRIBUTION OF TRACES OF CYCLE INTEGRALS OF THE $j$-FUNCTION

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Abstract. We establish an asymptotic formula with a power savings in the error term for traces of cycle integrals of the classical modular $j$-function

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots .$$

This implies a conjecture of Duke, Imamoḡlu, and Tóth [DIT, eq. (1.25)].

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## 1. Introduction and statement of results

Let $D$ be a nonzero integer with $D \equiv 0, 1 \mod 4$, i.e., a *discriminant*. Let $\mathcal{Q}_D$ be the set of integral binary quadratic forms $Q(X, Y) = aX^2 + bXY + cY^2$ of discriminant $b^2 - 4ac = D$ which are positive definite if $D < 0$. The group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$ in the usual way with finite quotient $\mathcal{Q}_D/\Gamma$. Let $\Gamma_Q < \Gamma$ be the group of automorphs of $Q$.

For $D < 0$, the root

$$\alpha_Q := \frac{-b + \sqrt{D}}{2a}$$

of the dehomogenized form $Q(z, 1)$ in the complex upper half-plane $\mathbb{H}$ is called a *CM point*. Let $j(z)$ be the classical modular $j$-function for $\Gamma$,

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots ,$$

where $q = e(z) = e^{2\pi i z}$. The values $j(\alpha_Q)$ for $Q \in \mathcal{Q}_D/\Gamma$ are called *singular moduli*. These are algebraic integers which play a fundamental role in number theory.

In [Z], Zagier established an important connection between traces of singular moduli and the Fourier coefficients of modular forms. Let $\mathbb{C}[j]$ be the space of weakly holomorphic modular forms of weight 0 for $\Gamma$. This space has a unique basis $\{j_m(z)\}_{m=0}^{\infty}$ where $j_m(z) = q^{-m} + O(q)$ (see [Z, section 6]). For example,

$$j_0 = 1, \quad j_1 = j - 744, \quad j_2 = j^2 - 1488j + 159768, \cdots$$

For each $m \in \mathbb{Z}_{\geq 0}$, define the trace

$$\mathrm{Tr}_D(j_m) := \sum_{Q \in \mathcal{Q}_D/\Gamma} \frac{j_m(\alpha_Q)}{\#\Gamma_Q}.$$

Zagier proved that the traces $\{\mathrm{Tr}_D(j_m)\}_{D<0}$ are Fourier coefficients of a weight $3/2$ weakly holomorphic modular form in Kohnen's plus space for $\Gamma_0(4)$. For example, when $m = 1$ the

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"generating function"

$$g(z) := q^{-1} - 2 - \sum_{\substack{D<0 \\ D\equiv 0,1 \mod 4}} \mathrm{Tr}_D(j_1) q^{|D|} \tag{1.1}$$

is a modular form in this space. In recent years there have been many striking developments related to Zagier's work. We refer the reader to the survey articles of Ono [O, O2] where many of these results are discussed.

One interesting question related to Zagier's work concerns the asymptotic distribution of the traces $\mathrm{Tr}_D(j_m)$ as $D \to -\infty$. In [BJO], Bruinier, Jenkins, and Ono established the Rademacher-type exact formula[1]

$$\mathrm{Tr}_D(j_m) = -24H^*(D)\sigma_1(m) + \sum_{c\in\mathbb{Z}^+} S_m(D;4c)\sinh\left(\pi m \frac{\sqrt{|D|}}{c}\right),$$

where $H^*(D)$ is the Hurwitz class number of $\mathbb{Q}(\sqrt{D})$, $\sigma_1(m) := \sum_{\ell|m} \ell$ is the divisor function and $S_m(D;4c)$ is the quadratic Weyl sum

$$S_m(D;4c) := \sum_{\substack{b \mod 4c \\ b^2\equiv D \mod 4c}} e\left(\frac{mb}{2c}\right).$$

Based on this exact formula, they made a conjecture concerning the asymptotic distribution of $\mathrm{Tr}_D(j_1)$ as $D \to -\infty$. This conjecture was proved by Duke [D], who established the asymptotic formula

$$\frac{1}{h(D)}\left(\mathrm{Tr}_D(j_m) - \sum_{1\le c<2\sqrt{|D|}} S_m(D;4c)\sinh\left(\pi m\frac{\sqrt{|D|}}{c}\right)\right) \to -24\sigma_1(m) \tag{1.2}$$

as $D \to -\infty$ through fundamental discriminants, where $h(D)$ is the class number of $\mathbb{Q}(\sqrt{D})$.

In light of these results, it is natural to ask whether there exist real quadratic analogs of singular moduli, and if so, whether their traces possess properties similar to those just described. These questions were recently investigated by Duke, Imamoḡlu, and Tóth in [DIT].

Let $D > 0$ be a nonsquare and let $Q \in \mathcal{Q}_D$. The group of automorphs $\Gamma_Q$ is infinite cyclic with generator $M_Q$ which for primitive $Q$ is given by

$$M_Q := \pm \begin{pmatrix} \frac{t+bu}{2} & cu \\ -au & \frac{t-bu}{2} \end{pmatrix}$$

where $(t,u)$ are the smallest positive integral solutions of Pell's equation $t^2 - Du^2 = 4$. In general, $M_Q = M_{Q/\gcd(a,b,c)}$. Let

$$S_Q := a\,|z|^2 + b\mathrm{Re}(z) + c = 0$$

be the semi-circle joining the two real roots of the dehomogenized form $Q(z,1)$. For $z \in S_Q$, let $C_Q$ be the directed arc on $S_Q \cap \mathbb{H}$ from $z$ to $M_Q z$ and let

$$dz_Q := \frac{\sqrt{D}dz}{Q(z,1)}$$

[1] Here we have made the substitution $c \mapsto 4c$.

be the hyperbolic arc length differential along $C_Q$. Define the trace of cycle integrals

$$\mathrm{Tr}_D(j_m) := \frac{1}{2\pi} \sum_{Q \in \mathcal{Q}_D/\Gamma} \int_{C_Q} j_m(z) dz_Q.$$

The cycle integral is both independent of the choice of basepoint $z \in S_Q$ and a class invariant. These traces can be viewed as the real quadratic analogues of traces of singular moduli.

In [DIT, Theorem 1], Duke, Imamoḡlu, and Tóth proved that the generating function[2]

$$\sum_{\substack{D>0 \\ D \equiv 0,1 \mod 4}} \frac{\mathrm{Tr}_D(j_1)}{\sqrt{D}} q^D$$

is a mock modular form of weight 1/2 for $\Gamma_0(4)$ with "shadow" $g(z)$ defined by (1.1). Moreover, in [DIT, eq. (1.24)] they established the exact formula[3]

$$\mathrm{Tr}_D(j_m) = -24\sigma_1(m)\mathrm{Tr}_D(1) + \sum_{c \in \mathbb{Z}^+} S_m(D; 4c) \sin\left(\pi m \frac{\sqrt{D}}{c}\right),$$

and based on this formula they conjectured that (see [DIT, eq. (1.25)])

$$\frac{\mathrm{Tr}_D(j_m)}{\mathrm{Tr}_D(1)} \to -24\sigma_1(m) \tag{1.3}$$

as $D \to \infty$.

Our main result is the following theorem which implies the conjectured asymptotic (1.3).

**Theorem 1.1.** *For all $\epsilon > 0$ we have*

$$\frac{\mathrm{Tr}_D(j_m)}{\mathrm{Tr}_D(1)} = -24\sigma_1(m) + O_{\epsilon,m}(D^{-\frac{1}{400}+\epsilon})$$

*as $D \to \infty$ through fundamental discriminants.*

**Remark 1.2.** Duke, Friedlander, and Iwaniec [DFI] proved the conjectured asymptotic (1.3) with a power savings in the error term using a different method.

**Remark 1.3.** The implied constant in $O_{\epsilon,m}$ is ineffective due to an application of Siegel's theorem.

**Remark 1.4.** Note that if $D$ is a fundamental discriminant and

$$H(D) := 2\log(\epsilon_D)h^+(D)$$

where $h^+(D)$ is the narrow class number of $\mathbb{Q}(\sqrt{D})$ and $\epsilon_D$ is the fundamental unit of norm 1, then

$$\mathrm{Tr}_D(1) = \frac{H(D)}{2\pi}.$$

[2]Here $\mathrm{Tr}_D(j_1)$ must be suitably defined when $D$ is a perfect square.

[3]The formula stated here is $\sqrt{D}$ times the formula in [DIT, eq. (2.4)] due to our normalization of the measure $dz_Q$.

From the exact formulas for the traces, one sees that the asymptotics in (1.2) and (1.3) are determined by cancellation in the weighted sum of quadratic Weyl sums

$$\sum_{c\in\mathbb{Z}^+} S_m(D;4c)\phi\left(\pi m\frac{\sqrt{|D|}}{c}\right),$$

where the "weight" function $\phi(x) := \sinh(x)$ if $D<0$ and $\phi(x) := \sin(x)$ if $D>0$. Because $\sinh(x)\sim e^x$ as $x\to\infty$, when $D<0$ cancellation occurs only if $c$ is large compared to $\sqrt{|D|}$. This reflects the fact that $j(z)$ has a pole at $\infty$, and so its values at CM points high in the cusp must contribute an exponential main term to the asymptotic for the trace. In fact, Bruinier, Jenkins, and Ono conjectured that

$$\sum_{c>\sqrt{|D|/3}} S_m(D;4c)\sinh\left(\pi m\frac{\sqrt{|D|}}{c}\right) = o(h(D))$$

as $D\to-\infty$ through fundamental discriminants, which is implied by Duke's proof of (1.2). On the other hand, because $\sin(x)=O(1)$, when $D>0$ cancellation occurs in the entire range of $c$. Indeed, our proof of Theorem 1.1 implies that

$$\sum_{c\in\mathbb{Z}^+} S_m(D;4c)\sin\left(\pi m\frac{\sqrt{D}}{c}\right) = o(H(D))$$

quantitatively as $D\to\infty$.

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## 2. Outline of the proof of Theorem 1.1

In this section we will outline the proof of Theorem 1.1, which is influenced by the approach of Duke in [D].

Let

$$\Delta = y^2(\partial_x^2+\partial_y^2)$$

be the hyperbolic Laplacian. Let $F:\mathbb{H}\to\mathbb{C}$ be a $C^\infty$, $\Gamma$-invariant function satisfying the following growth condition: for all $a\in\mathbb{Z}_{\geq 0}$,

$$\Delta^a F(z) = O(e^{-cy}) \quad \text{as} \quad y=\mathrm{Im}(z)\to\infty \tag{2.1}$$

for some $c=c(a)>0$, where

$$\Delta^a = \Delta\circ\cdots\circ\Delta \quad a\text{-times}.$$

Define the $L^2$-norm

$$||\Delta^a F||_2 := \left(\int_X |\Delta^a F(z)|^2\, d\mu\right)^{1/2}$$

where $d\mu = (3/\pi)dxdy/y^2$ is the normalized hyperbolic measure on the modular curve $X=\mathbb{H}/\Gamma$.

Using period relations and subconvexity bounds for twisted $L$–functions, we will show in Proposition 3.1 that

$$\frac{1}{H(D)}\sum_{Q\in\mathcal{Q}_D/\Gamma}\int_{C_Q}F(z)dz_Q=\int_X F(z)d\mu+O_\epsilon(||\Delta^2F||_2D^{-\frac{1}{16}+\epsilon}) \tag{2.2}$$

as $D\to\infty$ through fundamental discriminants.

One can construct for each $\varepsilon>0$ a $C^\infty$, $\Gamma$-invariant Poincaré series $\mathscr{P}_{m,\varepsilon}(z)$ with growth coinciding precisely with that of $j_m(z)$ in the cusp at $\infty$ of $X$. Because the "regularized" function

$$F_{m,\varepsilon}(z):=\frac{1}{2\pi}\left(j_m(z)-\mathscr{P}_{m,\varepsilon}(z)\right)$$

satisfies the growth condition (2.1), we will substitute $F_{m,\varepsilon}(z)$ into (2.2) to obtain the preliminary asymptotic formula

$$\frac{1}{-24\sigma_1(m)\mathrm{Tr}_D(1)}\left(\mathrm{Tr}_D\left(j_m\right)-\mathrm{Tr}_D\left(\mathscr{P}_{m,\varepsilon}\right)\right)=\int_X F_{m,\varepsilon}(z)d\mu+O_\epsilon(||\Delta^2F_{m,\varepsilon}||_2D^{-\frac{1}{16}+\epsilon})$$

as $D\to\infty$.

Using a regularization of Borcherds-type, one can show that for each $\varepsilon>0$,

$$\int_X F_{m,\varepsilon}(z)d\mu=\frac{-24\sigma_1(m)}{2\pi}.$$

Moreover, we will establish the estimate

$$||\Delta^2F_{m,\varepsilon}||_2=O_m(\varepsilon^{-12})$$

as $\varepsilon\to 0$. By letting $\varepsilon=D^{-\nu}$ for $\nu>0$, we will obtain an error term which is

$$O_{\epsilon,m}(D^{-(\frac{1}{16}-12\nu)+\epsilon}).$$

Given these results, the proof of Theorem 1.1 is reduced to showing that

$$\frac{1}{-24\sigma_1(m)\mathrm{Tr}_D(1)}\mathrm{Tr}_D\left(\mathscr{P}_{m,D^{-\nu}}\right)=O_{\epsilon,m}(D^{-\delta+\epsilon}) \tag{2.3}$$

for some absolute constant $\delta>0$.

Roughly speaking, using a careful decomposition of $\mathrm{Tr}_D\left(\mathscr{P}_{m,D^{-\nu}}\right)$ and a quantitative unsmoothing argument, we will reduce (2.3) to the uniform estimate

$$\sum_{1\le c\le T}S_m(D;4c)=O_{\epsilon,m}(D^{\frac{1}{2}-\theta+\epsilon})$$

for $T=O(\sqrt{D})$ and some absolute constant $\theta>0$. To establish such an estimate for the sum of quadratic Weyl sums, we will use a quantitative form of equidistribution of integral points on the hyperboloid

$$H_D(\mathbb{R}):y^2-4xz=D,$$

which is implied by Duke's theorem on the equidistribution of closed geodesics on the unit tangent bundle of $X$ (see e.g. [D3, Ch, ELMV, LRS]).

## 3. Quantitative equidistribution of closed geodesics

**Proposition 3.1.** *Let $F : \mathbb{H} \to \mathbb{C}$ be a $C^\infty$, $\Gamma$-invariant function satisfying the growth condition (2.1). Then for all $\epsilon > 0$,*

$$\frac{1}{H(D)} \sum_{Q \in \mathcal{Q}_D/\Gamma} \int_{C_Q} F(z) dz_Q = \int_X F(z) d\mu + O_\epsilon(||\Delta^2 F||_2 D^{-\frac{1}{16}+\epsilon})$$

*as $D \to \infty$ through fundamental discriminants.*

*Proof.* The spectral decomposition of $L^2(X)$ with respect to $\Delta$ yields the expansion

$$F(z) = \langle F, 1\rangle_2 + \sum_{n=1}^{\infty} \langle F, u_n\rangle_2 u_n(z) + \frac{1}{4\pi} \int_{\mathbb{R}} \langle F, E(\cdot, \frac{1}{2} + it)\rangle_2 E(z, \frac{1}{2} + it) dt,$$

where $\{u_n(z)\}_{n \in \mathbb{Z}^+}$ is an orthonormal basis of Maass cusp forms satisfying

$$\Delta u_n = \lambda_n u_n \quad \text{for} \quad n \in \mathbb{Z}^+$$

with eigenvalues $\lambda_n = \frac{1}{4} + t_n^2$ ordered so that $0 < \lambda_1 \leq \lambda_2 \cdots$, and $E(z, s)$ is the real-analytic Eisenstein series

$$E(z, s) := \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s, \quad z \in \mathbb{H}, \quad \mathrm{Re}(s) > 1$$

which is an eigenfunction for $\Delta$ with eigenvalue $s(1-s)$. By the growth condition (2.1), the spectral expansion converges pointwise absolutely and uniformly on compact subsets of $X$. From the spectral expansion we obtain

$$\frac{1}{H(D)} \sum_{Q \in \mathcal{Q}_D/\Gamma} \int_{C_Q} F(z) dz_Q = \tag{3.1}$$
$$\langle F, 1\rangle_2 + \sum_{n=1}^{\infty} \langle F, u_n\rangle_2 W(u_n, D) + \frac{1}{4\pi} \int_{\mathbb{R}} \langle F, E(\cdot, \frac{1}{2} + it)\rangle_2 W(t, D) dt,$$

where the “hyperbolic” Weyl sums are defined by

$$W(u_n, D) := \frac{1}{H(D)} \sum_{Q \in \mathcal{Q}_D/\Gamma} \int_{C_Q} u_n(z) dz_Q$$

and

$$W(t, D) := \frac{1}{H(D)} \sum_{Q \in \mathcal{Q}_D/\Gamma} \int_{C_Q} E(z, \frac{1}{2} + it) dz_Q.$$

By Popa [P, Theorem 6.3.1] one has the Waldspurger-type formula[4]

$$\left| \sum_{Q \in \mathcal{Q}_D/\Gamma} \int_{C_Q} u_n(z) dz_Q \right|^2 = \frac{\sqrt{D}}{2} |a_n(1)|^2 \Lambda(u_n, \frac{1}{2}) \Lambda(u_n \otimes (\frac{D}{\cdot}), \frac{1}{2}), \tag{3.2}$$

[4]Here we multiply the right hand side by $|a_n(1)|^2$ because Popa normalizes the Maass form $u_n$ to have $a_n(1) = 1$.

where $a_n(1)$ is the first Fourier coefficient of $u_n$ and $\Lambda(\Pi, s) := L_\infty(\Pi, s)L(\Pi, s)$ is the completed $L$–function. By Blomer and Harcos [BH, Theorem 2] one has the hybrid subconvexity bound

$$L(u_n \otimes (\frac{D}{\cdot}), 1/2) \ll_\epsilon \lambda_n^{\frac{7}{4}+\epsilon} D^{\frac{1}{2}-\frac{1}{8}+\epsilon}.$$

Then using the following estimate of Hoffstein and Lockhart [HL]

$$|a_n(1)|^2 \ll_\epsilon \lambda_n^\epsilon e^{\pi|t|}, \tag{3.3}$$

and the fact that the contribution from the infinite parts of the $L$–functions in (3.2) cancels the exponential factor in (3.3), we obtain the estimate

$$W(u_n, D) \ll_\epsilon \lambda_n^{\frac{7}{8}+\epsilon} \frac{D^{\frac{1}{2}-\frac{1}{16}+\epsilon}}{H(D)}.$$

Similarly, one has the following classical formula due to Siegel (see [P, p. 865])

$$\sum_{Q \in \mathcal{Q}_D/\Gamma} \int_{C_Q} E(z, \frac{1}{2}+it) dz_Q = D^{\frac{1}{4}+\frac{it}{2}} \frac{\Gamma(\frac{1}{4}+\frac{it}{2})^2}{\Gamma(\frac{1}{2}+it)} \zeta(\frac{1}{2}+it) L((\frac{D}{\cdot}), \frac{1}{2}+it).$$

By Heath-Brown [HB] one has the hybrid subconvexity bound

$$L((\frac{D}{\cdot}), \frac{1}{2}+it) \ll_\epsilon (\frac{1}{4}+t^2)^{\frac{3}{32}+\epsilon} D^{\frac{1}{4}-\frac{1}{16}+\epsilon}.$$

Thus by Stirling's formula and the convexity bound for $\zeta(s)$ we obtain the estimate

$$W(t, D) \ll_\epsilon (\frac{1}{4}+t^2)^{\frac{11}{32}+\epsilon} \frac{D^{\frac{1}{2}-\frac{1}{16}+\epsilon}}{H(D)}.$$

By Siegel's theorem, for all $\epsilon > 0$ we have the (ineffective) lower bound $H(D) \gg_\epsilon D^{\frac{1}{2}-\epsilon}$. Thus we obtain the estimates

$$W(u_n, D) \ll_\epsilon \lambda_n^{\frac{7}{8}+\epsilon} D^{-\frac{1}{16}+\epsilon} \tag{3.4}$$

and

$$W(t, D) \ll_\epsilon (\frac{1}{4}+t^2)^{\frac{11}{32}+\epsilon} D^{-\frac{1}{16}+\epsilon}. \tag{3.5}$$

By the growth condition (2.1), a repeated application of Stokes' theorem (see e.g. [I, Lemma 1.18]) yields the following identities which are valid for any $a \in \mathbb{Z}_{\geq 0}$,

$$\langle F, u_n \rangle_2 = \lambda_n^{-a} \langle \Delta^a F, u_n \rangle_2$$

and

$$\langle F, E(\cdot, \frac{1}{2}+it) \rangle_2 = (\frac{1}{4}+t^2)^{-a} \langle \Delta^a F, E(\cdot, \frac{1}{2}+it) \rangle_2.$$

By Parseval's formula (see [IK, (15.17)]),

$$\sum_{n=1}^\infty |\langle \Delta^a F, u_n \rangle_2|^2 + \frac{1}{4\pi} \int_{\mathbb{R}} \left| \langle \Delta^a F, E(\cdot, \frac{1}{2}+it) \rangle_2 \right|^2 dt = ||\Delta^a F||_2^2.$$

Then by the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}\sum_{n=1}^{\infty} |\langle F, u_n\rangle_2| \, \lambda_n^{\frac{7}{8}+\epsilon} &= \sum_{n=1}^{\infty} \left|\langle \Delta^2 F, u_n\rangle_2\right| \frac{\lambda_n^{\frac{7}{8}+\epsilon}}{\lambda_n^2} \\ &\leq \sqrt{\sum_{n=1}^{\infty} |\langle \Delta^2 F, u_n\rangle_2|^2} \cdot \sqrt{\sum_{n=1}^{\infty} \frac{\lambda_n^{\frac{14}{8}+\epsilon}}{\lambda_n^4}} \\ &\ll ||\Delta^2 F||_2, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\frac{1}{4\pi}\int_{\mathbb{R}} \left|\langle F, E(\cdot, \frac{1}{2}+it)\rangle_2\right| (\frac{1}{4}+t^2)^{\frac{11}{32}+\epsilon} dt \\ &= \frac{1}{4\pi}\int_{\mathbb{R}} \left|\langle \Delta^2 F, E(\cdot, \frac{1}{2}+it)\rangle_2\right| \frac{(\frac{1}{4}+t^2)^{\frac{11}{32}+\epsilon}}{(\frac{1}{4}+t^2)^2} dt \\ &\leq \sqrt{\frac{1}{4\pi}\int_{\mathbb{R}} \left|\langle \Delta^2 F, E(\cdot, \frac{1}{2}+it)\rangle_2\right|^2 dt} \cdot \sqrt{\int_{\mathbb{R}} \frac{(\frac{1}{4}+t^2)^{\frac{22}{32}+\epsilon}}{(\frac{1}{4}+t^2)^4} dt} \\ &\ll ||\Delta^2 F||_2. \end{aligned} \tag{3.7}$$

The proposition now follows by combining the estimates (3.4)-(3.7) with (3.1). □

## 4. Poincaré series

Let $\lambda : \mathbb{R} \to [0,1]$ be a $C^\infty$ function such that

$$\lambda(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t \geq 1. \end{cases}$$

For $\varepsilon > 0$ define the function

$$\lambda_\varepsilon(t) := \lambda\left(\frac{t-1}{\varepsilon}\right).$$

Then $\lambda_\varepsilon : \mathbb{R} \to [0,1]$ is $C^\infty$ and satisfies

$$\lambda_\varepsilon(t) = \begin{cases} 0, & t \leq 1 \\ 1, & t \geq 1+\varepsilon. \end{cases}$$

For $m \in \mathbb{Z}^+$ define the Poincaré series

$$\mathscr{P}_{m,\varepsilon}(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \lambda_\varepsilon(\mathrm{Im}(\gamma z)) e(-m\gamma z), \quad z \in \mathbb{H}.$$

Then define the "regularized" function

$$F_{m,\varepsilon}(z) := \frac{1}{2\pi}\left(j_m(z) - \mathscr{P}_{m,\varepsilon}(z)\right).$$

**Lemma 4.1.** *The function $F_{m,\varepsilon}(z)$ satisfies the growth condition* (2.1).

*Proof.* The Poincaré series $\mathscr{P}_{m,\varepsilon}(z)$ has the Fourier expansion (see [I, p. 60])

$$\mathscr{P}_{m,\varepsilon}(z) = \lambda_\varepsilon(y)e(-mz) + \sum_{n\in\mathbb{Z}} e(nx) \sum_{c=1}^{\infty} S(m,n;c) \int_{\mathbb{R}} \lambda_\varepsilon \left(\frac{c^{-2}y}{t^2+y^2}\right) e\left(\frac{mc^{-2}}{t+iy} - nt\right) dt, \tag{4.1}$$

where $S(m,n;c)$ is the usual Kloosterman sum

$$S(m,n;c) := \sum_{a \mod c} e\left(\frac{ma+n\bar{a}}{c}\right).$$

For $y \geq 1+\varepsilon$ we have

$$\frac{c^{-2}y}{t^2+y^2} \leq 1$$

for all $c \in \mathbb{Z}^+$ and $t \in \mathbb{R}$. Then by definition of $\lambda_\varepsilon(t)$ it follows that

$$\mathscr{P}_{m,\varepsilon}(z) = e(-mz)$$

for $y \geq 1+\varepsilon$. The lemma now follows from the Fourier expansion

$$j_m(z) = e(-mz) + \sum_{n=1}^{\infty} c_m(n)e(nz). \tag{4.2}$$

□

## 5. A preliminary asymptotic formula

By Lemma 4.1 the function

$$F_{m,\varepsilon}(z) := \frac{1}{2\pi}\left(j_m(z) - \mathscr{P}_{m,\varepsilon}(z)\right)$$

satisfies the growth condition (2.1). Then by Proposition 3.1 with $F := F_{m,\varepsilon}$ we find that

$$\frac{1}{H(D)}\left(\mathrm{Tr}_D\left(j_m\right) - \mathrm{Tr}_D\left(\mathscr{P}_{m,\varepsilon}\right)\right) = \int_X F_{m,\varepsilon}(z)d\mu + O_\epsilon(||\Delta^2 F_{m,\varepsilon}||_2 D^{-\frac{1}{16}+\epsilon})$$

as $D \to \infty$ through fundamental discriminants.

By a straightforward generalization of the argument in [D2, pp. 17-18], one finds that for all $\varepsilon > 0$,

$$\int_X F_{m,\varepsilon}(z)d\mu = \frac{-24\sigma_1(m)}{2\pi}.$$

Now, by Lemma 5.1 we have

$$||\Delta^2 F_{m,\varepsilon}||_2 = O_m(\varepsilon^{-12}).$$

Then if we let $\varepsilon = D^{-\nu}$ for $\nu > 0$, we find that

$$\frac{1}{-24\sigma_1(m)\mathrm{Tr}_D(1)}\left(\mathrm{Tr}_D\left(j_m\right) - \mathrm{Tr}_D\left(\mathscr{P}_{m,D^{-\nu}}\right)\right) = 1 + O_{\epsilon,m}(D^{-(\frac{1}{16}-12\nu)+\epsilon}).$$

Define

$$\phi_{m,D^{-\nu}}(t) := \lambda_{D^{-\nu}}(t)e(-imt) = \lambda_{D^{-\nu}}(t)\exp\left(2\pi mt\right).$$

Then we can write the Poincaré series $\mathscr{P}_{m,D^{-\nu}}(z)$ as

$$\mathscr{P}_{m,D^{-\nu}}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \phi_{m,D^{-\nu}}(\mathrm{Im}(\gamma z)) e(-m \mathrm{Re}(\gamma z)).$$

By [DIT, Lemma 7], we obtain

$$\mathrm{Tr}_D(\mathscr{P}_{m,D^{-\nu}}) = \frac{1}{2\pi} \sum_{c \in \mathbb{Z}^+} S_m(D; 4c) \Phi_{m,D^{-\nu}} \left( \frac{\sqrt{D}}{2c} \right),$$

where

$$\Phi_{m,D^{-\nu}} \left( \frac{\sqrt{D}}{2c} \right) := \int_0^\pi \phi_{m,D^{-\nu}} \left( 2 \frac{\sqrt{D}}{c} \sin(\theta) \right) \cos \left( \pi m \frac{\sqrt{D}}{c} \cos(\theta) \right) \frac{d\theta}{\sin(\theta)}.$$

Since $\phi_{m,D^{-\nu}}(t) = 0$ for $t \leq 1$, and

$$\frac{\sqrt{D}}{2c} \sin(\theta) \leq 1$$

for $c \geq \sqrt{D}/2$, we have

$$\mathrm{Tr}_D(\mathscr{P}_{m,D^{-\nu}}) = \frac{1}{2\pi} \sum_{1 \leq c < \sqrt{D}/2} S_m(D; 4c) \Phi_{m,D^{-\nu}} \left( \frac{\sqrt{D}}{2c} \right).$$

This yields the asymptotic formula

$$\frac{1}{-24\sigma_1(m)\mathrm{Tr}_D(1)} \left( \mathrm{Tr}_D(j_m) - \frac{1}{2\pi} \sum_{1 \leq c < \sqrt{D}/2} S_m(D; 4c) \Phi_{m,D^{-\nu}} \left( \frac{\sqrt{D}}{2c} \right) \right) = 1 + O_{\epsilon,m}(D^{-(\frac{1}{16} - 12\nu) + \epsilon}) \tag{5.1}$$

as $D \to \infty$.

**Lemma 5.1.** *For each $a \in \mathbb{Z}^+$ and $\varepsilon < 1/4$ we have*

$$||\Delta^a F_{m,\varepsilon}||_2 = O_m(\varepsilon^{-(4a+4)}).$$

*Proof.* Write the Fourier expansion (4.1) as

$$\mathscr{P}_{m,\varepsilon}(z, s) = \lambda_\varepsilon(y) e(-mz) + f_{m,\varepsilon}(x, y),$$

where

$$f_{m,\varepsilon}(x, y) := \sum_{n \in \mathbb{Z}} e(nx) \sum_{c=1}^\infty S(m, n; c) \int_{\mathbb{R}} \lambda_\varepsilon \left( \frac{c^{-2} y}{t^2 + y^2} \right) e \left( \frac{mc^{-2}}{t + iy} - nt \right) dt.$$

Using the standard fundamental domain for $X$, we have

$$||\Delta^a F_{m,\varepsilon}||_2^2 \leq \int_{\sqrt{3}/2}^\infty \int_0^1 |\Delta^a F_{m,\varepsilon}(z)|^2 \, dx \frac{dy}{y^2}.$$

For $y \geq 1$ we have

$$\frac{c^{-2} y}{t^2 + y^2} \leq 1$$

for all $c \in \mathbb{Z}^+$ and $t \in \mathbb{R}$. Then by definition of $\lambda_\varepsilon(t)$ it follows that

$$F_{m,\varepsilon}(z) = \begin{cases} j_m(z) - f_{m,\varepsilon}(x,y), & \sqrt{3}/2 \le y < 1 \\ j_m(z) - \lambda_\varepsilon(y)e(-mz), & 1 \le y < 1+\varepsilon \\ j_m(z) - e(-mz), & y \ge 1+\varepsilon. \end{cases}$$

We thus obtain

$$\begin{aligned} \int_{\sqrt{3}/2}^{\infty} \int_0^1 |\Delta^a F_{m,\varepsilon}(z)|^2 \, dx \frac{dy}{y^2} &= \int_{\sqrt{3}/2}^{1} \int_0^1 |\Delta^a (j_m(z) - f_{m,\varepsilon}(x,y))|^2 \, dx \frac{dy}{y^2} \\ &+ \int_1^{1+\varepsilon} \int_0^1 |\Delta^a (j_m(z) - \lambda_\varepsilon(y)e(-mz))|^2 \, dx \frac{dy}{y^2} \\ &+ \int_{1+\varepsilon}^{\infty} \int_0^1 |\Delta^a (j_m(z) - e(-mz))|^2 \, dx \frac{dy}{y^2} \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

By the Fourier expansion (4.2) for $j_m(z)$ we see that

$$I_3 = O(1).$$

We now estimate $I_2$ and $I_1$ in turn.

**Estimation of** $I_2$. By linearity of $\Delta^a$ and the triangle inequality we have

$$\begin{aligned} I_2 \le& \int_1^{1+\varepsilon} \int_0^1 |\Delta^a j_m(z)|^2 \, dx \frac{dy}{y^2} \\ &+ 2 \int_1^{1+\varepsilon} \int_0^1 |\Delta^a j_m(z)| \cdot |\Delta^a (\lambda_\varepsilon(y)e(-mz))| \, dx \frac{dy}{y^2} \\ &+ \int_1^{1+\varepsilon} \int_0^1 |\Delta^a (\lambda_\varepsilon(y)e(-mz))|^2 \, dx \frac{dy}{y^2}. \end{aligned}$$

Recall that

$$\lambda_\varepsilon(t) := \lambda\left(\frac{t-1}{\varepsilon}\right)$$

where $\lambda : \mathbb{R} \to [0,1]$ is a $C^\infty$ function such that

$$\lambda(t) = \begin{cases} 0, & t \le 0 \\ 1, & t \ge 1. \end{cases}$$

Then it is clear that

$$\max_{y \in [1,1+\varepsilon]} \left| \frac{d^{2a}}{dy^{2a}} \lambda_\varepsilon(y) \right| = O(\varepsilon^{-2a}).$$

Therefore (using that $\varepsilon < 1/4$),

$$I_2 = O(1) + O(\varepsilon^{-2a}) + O(\varepsilon^{-4a}).$$

**Estimation of $I_1$.** Similarly, we have

$$I_1 \leq \int_{\sqrt{3}/2}^{1} \int_0^1 |\Delta^a j_m(z)|^2 \, dx \frac{dy}{y^2} + 2 \int_{\sqrt{3}/2}^{1} \int_0^1 |\Delta^a j_m(z)| \cdot |\Delta^a f_{m,\varepsilon}(x,y)| \, dx \frac{dy}{y^2} + \int_{\sqrt{3}/2}^{1} \int_0^1 |\Delta^a f_{m,\varepsilon}(x,y)|^2 \, dx \frac{dy}{y^2}.$$

Observe that for $y \geq \sqrt{3}/2$ and $c \geq 2$ we have

$$\frac{y}{c^2(t^2+y^2)} \leq \frac{2}{c^2\sqrt{3}} \leq 1.$$

Since $\lambda_\varepsilon(u) = 0$ for $u \leq 1$, it follows that for $y \geq \sqrt{3}/2$ and $c \geq 2$,

$$\lambda_\varepsilon\left(\frac{c^{-2}y}{t^2+y^2}\right) = 0.$$

Therefore, for $y \geq \sqrt{3}/2$ the function $f_{m,\varepsilon}(x,y)$ simplifies to

$$f_{m,\varepsilon}(x,y) = \sum_{n\in\mathbb{Z}} e(nx) \int_{\mathbb{R}} \lambda_\varepsilon\left(\frac{y}{t^2+y^2}\right) e\left(\frac{m}{t+iy} - nt\right) dt.$$

Define the function

$$g_{m,y,\varepsilon}(t) := \lambda_\varepsilon\left(\frac{y}{t^2+y^2}\right) e\left(\frac{m}{t+iy}\right).$$

Then by definition of $\lambda_\varepsilon(u)$ we have

$$\lim_{|t|\to\infty} \frac{d^k}{dt^k} g_{m,y,\varepsilon}(t) = 0$$

for all integers $k \in \mathbb{Z}_{\geq 0}$. Integrating by parts $(2a+2)$-times yields

$$\int_{\mathbb{R}} g_{m,y,\varepsilon}(t) e(-nt) \, dt = \frac{1}{(2\pi i n)^{2a+2}} \int_{\mathbb{R}} g_{m,y,\varepsilon}^{(2a+2)}(t) e(-nt) \, dt.$$

In fact, since

$$\frac{y}{t^2+y^2} \leq 1 \quad \Leftrightarrow \quad |t| \geq \sqrt{y(1-y)},$$

we have

$$g_{m,y,\varepsilon}^{(2a+2)}(t) = 0$$

for $|t| \geq \sqrt{y(1-y)}$. Thus in the range $\sqrt{3}/2 \leq y \leq 1$ we obtain

$$f_{m,\varepsilon}(x,y) = \sum_{n\in\mathbb{Z}} \frac{e(nx)}{(2\pi i n)^{2a+2}} \int_{-1}^{1} g_{m,y,\varepsilon}^{(2a+2)}(t) e(-nt) \, dt.$$

We now analyze

$$\Delta^a f_{m,\varepsilon}(x,y),$$

and show that

$$I_1 = O(1) + O(\varepsilon^{-(2a+2)}) + O(\varepsilon^{-2(2a+2)}).$$

For clarity, we first give the argument for $a = 1$. We find that

$$y^2(\partial_x^2 + \partial_y^2) f_{m,\varepsilon}(x,y) = y^2 \sum_{n\in\mathbb{Z}} \frac{e(nx)}{(2\pi i n)^2} \int_{-1}^{1} g_{m,y,\varepsilon}^{(4)}(t) e\left(-nt\right) dt$$
$$+ y^2 \sum_{n\in\mathbb{Z}} \frac{e(nx)}{(2\pi i n)^4} \int_{-1}^{1} \frac{\partial^2}{\partial y^2} g_{m,y,\varepsilon}^{(4)}(t) e\left(-nt\right) dt.$$

Therefore

$$\Delta f_{m,\varepsilon}(x,y) \ll y^2 \max_{t\in[-1,1]} \left| g_{m,y,\varepsilon}^{(4)}(t) \right| + y^2 \max_{t\in[-1,1]} \left| \frac{\partial^2}{\partial y^2} g_{m,y,\varepsilon}^{(4)}(t) \right|.$$

It is clear from the definition of $g_{m,y,\varepsilon}$ that

$$\max_{t\in[-1,1]} \left| g_{m,y,\varepsilon}^{(4)}(t) \right| = O_y(\varepsilon^{-4})$$

and

$$\max_{t\in[-1,1]} \left| \frac{\partial^2}{\partial y^2} g_{m,y,\varepsilon}^{(4)}(t) \right| = O_y(\varepsilon^{-4}).$$

Hence

$$\Delta f_{m,\varepsilon}(x,y) = O_y(\varepsilon^{-4}).$$

Finally, it follows that

$$I_1 = O(1) + O(\varepsilon^{-4}) + O(\varepsilon^{-8}).$$

The preceding argument generalizes in a straightforward way to higher derivatives, and thus we conclude that

$$I_1 = O(1) + O(\varepsilon^{-(2a+2)}) + O(\varepsilon^{-2(2a+2)}).$$

□

## 6. Decomposition of $\mathrm{Tr}_D(\mathscr{P}_{m,D^{-\nu}})$

In this section we decompose

$$\mathrm{Tr}_D(\mathscr{P}_{m,D^{-\nu}}) = \frac{1}{2\pi} \sum_{1\le c<\sqrt{D}/2} S_m(D;4c)\Phi_{m,D^{-\nu}}\left(\frac{\sqrt{D}}{2c}\right)$$

into two terms which we will estimate separately in the next two sections.

For real numbers $u \ge 1$ and $0 \le \alpha < \beta$, define the sets

$$A_{\alpha,\beta}^{D,u} := \left\{\theta \in [0,\pi] : \ \frac{2u}{\sqrt{D}}\alpha \le \sin(\theta) \le \frac{2u}{\sqrt{D}}\beta\right\}$$

and

$$B_{\beta}^{D,u} := \left\{\theta \in [0,\pi] : \ \sin(\theta) \ge \frac{2u}{\sqrt{D}}\beta\right\}.$$

Fix an integer $c$ in the range $1 \leq c < \sqrt{D}/2$. Because $\lambda_{D^{-\nu}}(t) = 0$ for $t \leq 1$ and $\lambda_{D^{-\nu}}(t) = 1$ for $t \geq 1 + D^{-\nu}$, we have

$$\begin{aligned}
&\Phi_{m,D^{-\nu}}\left(\frac{\sqrt{D}}{2c}\right) = \\
&\int_{A^{D,c}_{1,1+D^{-\nu}}} \lambda_{D^{-\nu}}\left(\frac{\sqrt{D}}{2c}\sin(\theta)\right)\cos\left(\pi m\frac{\sqrt{D}}{c}\cos(\theta)\right)\exp\left(\pi m\frac{\sqrt{D}}{c}\sin(\theta)\right)\frac{d\theta}{\sin(\theta)} \\
&+ \int_{B^{D,c}_{1+D^{-\nu}}} \cos\left(\pi m\frac{\sqrt{D}}{c}\cos(\theta)\right)\exp\left(\pi m\frac{\sqrt{D}}{c}\sin(\theta)\right)\frac{d\theta}{\sin(\theta)} \\
&=: I_1(m,\nu,D,c) + I_2(m,\nu,D,c).
\end{aligned}$$

This yields the decomposition

$$\mathrm{Tr}_D(\mathscr{P}_{m,D^{-\nu}}) = C_1(m,\nu,D) + C_2(m,\nu,D), \tag{6.1}$$

where

$$C_1(m,\nu,D) := \frac{1}{2\pi}\sum_{1\leq c<\sqrt{D}/2} S_m(D;4c) I_1(m,\nu,D,c)$$

and

$$C_2(m,\nu,D) := \frac{1}{2\pi}\sum_{1\leq c<\sqrt{D}/2} S_m(D;4c) I_2(m,\nu,D,c).$$

## 7. Estimation of $C_1(m,\nu,D)$

In this section we estimate

$$C_1(m,\nu,D) := \frac{1}{2\pi}\sum_{1\leq c<\sqrt{D}/2} S_m(D;4c) I_1(m,\nu,D,c).$$

**Lemma 7.1.** *We have*

$$I_1(m,\nu,D,c) = O_m(D^{-\frac{\nu}{2}}).$$

*Proof.* Let

$$Z := \min\left\{1, \frac{2c(1+D^{-\nu})}{\sqrt{D}}\right\}$$

and

$$Y := \frac{2c}{\sqrt{D}} =: \frac{Z}{\alpha},$$

where $\alpha = 1 + O(D^{-\nu}) \geq 1$. Then

$$\begin{aligned}
I_1(m,\nu,D,c) &\ll_m \int_{A^{D,c}_{1,1+D^{-\nu}}} \frac{d\theta}{\sin(\theta)} = \int_Y^Z \frac{dx}{x\sqrt{1-x^2}} \\
&\leq \frac{1}{Y}\int_Y^Z \frac{dx}{\sqrt{1-x}} \\
&= \frac{2\alpha}{Z}\left(\sqrt{1-(Z/\alpha)} - \sqrt{1-Z}\right) \\
&= \frac{2(\alpha-1)}{\sqrt{1-(Z/\alpha)} + \sqrt{1-Z}} \\
&\leq 2\sqrt{\alpha(\alpha-1)} \\
&\ll D^{-\frac{\nu}{2}}.
\end{aligned}$$

□

For all $\epsilon > 0$, one has the trivial estimate

$$S_m(D;4c) \ll_\epsilon c^\epsilon. \tag{7.1}$$

It follows from (7.1) and Lemma 7.1 that

$$C_1(m,\nu,D) = O_{\epsilon,m}(D^{\frac{1}{2}-\frac{\nu}{2}+\epsilon}). \tag{7.2}$$

## 8. Estimation of $C_2(m,\nu,D)$

In this section we estimate

$$C_2(m,\nu,D) := \frac{1}{2\pi}\sum_{1\leq c<\sqrt{D}/2} S_m(D;4c) I_2(m,\nu,D,c).$$

**Lemma 8.1.** *For $1 \leq t < \sqrt{D}/2$ we have*

$$I_2(m,\nu,D,t) \ll_m \log(D).$$

*Proof.* Write

$$I_2(m,\nu,D,t) = \int_{B^{D,t}_{1+D^{-\nu}}} g_{m,D,t}(\theta)d\theta,$$

where

$$g_{m,D,t}(\theta) := \cos\left(\pi m\frac{\sqrt{D}}{t}\cos(\theta)\right)\exp\left(\pi m\frac{\sqrt{D}}{t}\sin(\theta)\right)\frac{1}{\sin(\theta)}.$$

Define

$$H(t) := \arcsin\left(\min\left\{1, \frac{2t}{\sqrt{D}}(1+D^{-\nu})\right\}\right).$$

Because $\sin(\theta)$ is symmetric about $\pi/2$ on $[0,\pi]$, we have

$$I_2(m,\nu,D,t) = \int_{H(t)}^{\pi-H(t)} g_{m,D,t}(\theta)d\theta. \tag{8.1}$$

By a straightforward calculation,

$$g_{m,D,t}(\theta) = \frac{\exp\left(\pi m \frac{\sqrt{D}}{t} i e^{-i\theta}\right) - \exp\left(-\pi m \frac{\sqrt{D}}{t} i e^{i\theta}\right)}{e^{i\theta} - e^{-i\theta}}. \tag{8.2}$$

Then the change of variables $z = e^{i\theta}$ yields

$$I_2(m,\nu,D,t) = \frac{1}{i}\int_{C(t)} \frac{\exp\left(\pi m \frac{\sqrt{D}}{t} i\bar{z}\right) - \exp\left(-\pi m \frac{\sqrt{D}}{t} iz\right)}{z^2-1} dz,$$

where $C(t)$ is the counter-clockwise path on the unit circle from $e^{iH(t)}$ to $e^{i(\pi - H(t))}$.

For $a \in \mathbb{R}^+$ we have

$$\exp(ai\bar{z}) - \exp(-aiz) = 2i \exp(a\mathrm{Im}(z)) \sin(a\mathrm{Re}(z)),$$

thus

$$I_2(m,\nu,D,t) = 2\int_{C(t)} \frac{\exp\left(\pi m \frac{\sqrt{D}}{t} \mathrm{Im}(z)\right) \sin\left(\pi m \frac{\sqrt{D}}{t} \mathrm{Re}(z)\right)}{z^2-1} dz.$$

Deform $C(t)$ to the horizontal line $L(t)$ connecting $e^{iH(t)}$ to $e^{i(\pi - H(t))}$. Since we pass no poles of the integrand, the integral in unchanged. Moreover, since $\mathrm{Im}(z)$ is constant on $L(t)$,

$$I_2(m,\nu,D,t) = 2\exp\left(\pi m \frac{\sqrt{D}}{t} \mathrm{Im}\left(e^{iH(t)}\right)\right) \int_{L(t)} \frac{\sin\left(\pi m \frac{\sqrt{D}}{t} \mathrm{Re}(z)\right)}{z^2-1} dz.$$

Then because

$$\mathrm{Im}(e^{iH(t)}) = \sin(H(t)) = \min\left\{1, \frac{2t}{\sqrt{D}}(1 + D^{-\nu})\right\} < \frac{4t}{\sqrt{D}},$$

we obtain

$$|I_2(m,\nu,D,t)| \leq 2\exp(4\pi m) \int_{L(t)} \frac{1}{|z+1|\,|z-1|} |dz| .$$

Write $L(t) = L_1(t) \cup L_2(t)$ where

$$L_1(t) := \{z \in L(t) :\ 0 \leq \mathrm{Re}(z) \leq 1\} \quad \text{and} \quad L_2(t) := \{z \in L(t) :\ -1 \geq \mathrm{Re}(z) < 0\}.$$

Then

$$\int_{L_1(t)} \frac{1}{|z+1|\,|z-1|} |dz| \leq \int_{L_1(t)} \frac{1}{|z-1|} |dz| .$$

Evaluating the latter integral and estimating trivially yields

$$\begin{aligned}\int_{L_1(t)} \frac{1}{|z-1|} |dz| &= \int_0^1 \frac{1}{\sqrt{(x-1)^2 + \mathrm{Im}(e^{iH(t)})^2}} dx \\ &= \log\left(\frac{\min\left\{1, \frac{2t}{\sqrt{D}}(1+D^{-\nu})\right\}}{\sqrt{1 + \min\left\{1, \frac{2t}{\sqrt{D}}(1+D^{-\nu})\right\}^2} - 1}\right) \ll \log(D).\end{aligned}$$

A similar argument shows that

$$\int_{L_2(t)} \frac{1}{|z+1|\,|z-1|}\,|dz| \ll \log(D).$$

□

Fix $0<\eta<1/2$ and write

$$C_2(m,\nu,D) = \frac{1}{2\pi}\left(\sum_{1\le c<\frac{D^{\frac12-\eta}}{2}} + \sum_{\frac{D^{\frac12-\eta}}{2}\le c<\sqrt{D}/2}\right) S_m(D;4c) I_2(m,\nu,D,c).$$

Using the estimate (7.1) and Lemma 8.1, we have

$$\sum_{1\le c<\frac{D^{\frac12-\eta}}{2}} S_m(D;4c)I_2(m,\nu,D,c) \ll_{\epsilon,m} \log(D)D^{\frac12-\eta+\epsilon}. \tag{8.3}$$

Next, by summation by parts

$$\sum_{\frac{D^{\frac12-\eta}}{2}\le c<\sqrt{D}/2} S_m(D;4c)I_2(m,\nu,D,c) =$$

$$I_2(m,\nu,D,\frac{\sqrt{D}}{2})\sum_{1\le c<\sqrt{D}/2} S_m(D;4c) - I_2(m,\nu,D,\frac{D^{\frac12-\eta}}{2})\sum_{1\le c<\frac{D^{\frac12-\eta}}{2}} S_m(D;4c)$$

$$-\int_{\frac{D^{\frac12-\eta}}{2}}^{\sqrt{D}/2}\sum_{1\le c\le t} S_m(D;4c)\frac{d}{dt}I_2(m,\nu,D,t)dt.$$

By Lemma 8.1 and Proposition 9.5, we have

$$I_2(m,\nu,D,\frac{\sqrt{D}}{2})\sum_{1\le c<\sqrt{D}/2} S_m(D;4c) - I_2(m,\nu,D,\frac{D^{\frac12-\eta}}{2})\sum_{1\le c<\frac{D^{\frac12-\eta}}{2}} S_m(D;4c)$$

$$\ll_{\epsilon,m} \log(D)D^{\frac12-\frac{1}{32}+\epsilon}. \tag{8.4}$$

It remains to estimate

$$\int_{\frac{D^{\frac12-\eta}}{2}}^{\sqrt{D}/2}\sum_{1\le c\le t} S_m(D;4c)\frac{d}{dt}I_2(m,\nu,D,t)dt.$$

Because $g_{m,D,t}(\theta)$ is symmetric about $\pi/2$ on $[0,\pi]$, we have

$$g_{m,D,t}(\pi-H(t)) = g_{m,D,t}(H(t)).$$

Then using (8.1), the fundamental theorem of calculus, and the chain rule,

$$\begin{aligned}\frac{d}{dt}I_2(m,\nu,D,t) &= -2g_{m,D,t}(H(t))H'(t) + \int_{H(t)}^{\pi-H(t)}\frac{\partial}{\partial t}g_{m,D,t}(\theta)d\theta\\ &=: A_1(t)+A_2(t).\end{aligned}$$

If

$$\min\left\{1, \frac{2t}{\sqrt{D}}(1+D^{-\nu})\right\} = 1,$$

then $H(t) = \pi/2$ and

$$\frac{d}{dt} I_2(m, \nu, D, t) = 0,$$

so we may assume that

$$t < \frac{\sqrt{D}}{2(1+D^{-\nu})}.$$

First we estimate

$$\int_{\frac{D^{\frac{1}{2}-\eta}}{2}}^{\frac{\sqrt{D}}{2(1+D^{-\nu})}} \sum_{1\le c\le t} S_m(D; 4c) A_1(t) dt.$$

We have

$$g_{m,D,t}(H(t)) \ll_m \frac{\sqrt{D}}{t}$$

and

$$H'(t) = \frac{2}{\sqrt{D}}(1+D^{-\nu}) \frac{1}{\sqrt{1-\left(\frac{2t}{\sqrt{D}}(1+D^{-\nu})\right)^2}}.$$

Then by Proposition 9.5,

$$\int_{\frac{D^{\frac{1}{2}-\eta}}{2}}^{\frac{\sqrt{D}}{2(1+D^{-\nu})}} \sum_{1\le c\le t} S_m(D; 4c) A_1(t) dt \ll_{\epsilon,m} D^{\frac{1}{2}-\frac{1}{32}+\epsilon} \int_{\frac{D^{\frac{1}{2}-\eta}}{2}}^{\frac{\sqrt{D}}{2(1+D^{-\nu})}} \frac{1}{t} \frac{1}{\sqrt{1-\left(\frac{2t}{\sqrt{D}}(1+D^{-\nu})\right)^2}} dt.$$

By an argument similar to that in the proof of Lemma 7.1, we find that

$$\int_{\frac{D^{\frac{1}{2}-\eta}}{2}}^{\frac{\sqrt{D}}{2(1+D^{-\nu})}} \frac{1}{t} \frac{1}{\sqrt{1-\left(\frac{2t}{\sqrt{D}}(1+D^{-\nu})\right)^2}} dt \ll D^{\eta}.$$

Therefore

$$\int_{\frac{D^{\frac{1}{2}-\eta}}{2}}^{\frac{\sqrt{D}}{2(1+D^{-\nu})}} \sum_{1\le c\le t} S_m(D; 4c) A_1(t) dt \ll_{\epsilon,m} D^{\frac{1}{2}-(\frac{1}{32}-\eta)+\epsilon}. \tag{8.5}$$

Next we estimate

$$\int_{\frac{D^{\frac{1}{2}-\eta}}{2}}^{\frac{\sqrt{D}}{2(1+D^{-\nu})}} \sum_{1\le c\le t} S_m(D; 4c) A_2(t) dt.$$

**Lemma 8.2.** *For $1 \le t < \sqrt{D}/2$ we have*

$$A_2(t) := \int_{H(t)}^{\pi-H(t)} \frac{\partial}{\partial t} g_{m,D,t}(\theta) d\theta \ll_m \log(D) \frac{\sqrt{D}}{t^2}.$$

*Proof.* Using (8.2) we calculate

$$\frac{\partial}{\partial t} g_{m,D,t}(\theta) = -i\frac{\sqrt{D}}{t^2}\left\{\frac{e^{-i\theta}\exp\left(\pi m\frac{\sqrt{D}}{t}ie^{-i\theta}\right)+e^{i\theta}\exp\left(-\pi m\frac{\sqrt{D}}{t}ie^{i\theta}\right)}{e^{i\theta}-e^{-i\theta}}\right\}.$$

Then the change of variables $z = e^{i\theta}$ yields

$$\int_{H(t)}^{\pi-H(t)} \frac{\partial}{\partial t} g_{m,D,t}(\theta)d\theta = -\frac{\sqrt{D}}{t^2}\int_{C(t)} \frac{\bar{z}\exp\left(\pi m\frac{\sqrt{D}}{t}i\bar{z}\right)+z\exp\left(-\pi m\frac{\sqrt{D}}{t}iz\right)}{z^2-1}dz,$$

where $C(t)$ is defined as in the proof of Lemma 8.1.

For $a \in \mathbb{R}^+$ we have

$$\begin{aligned}
&\bar{z}\exp(ai\bar{z}) + z\exp(-aiz)\\
&= \mathrm{Re}(z)\{\exp(ai\bar{z}) + \exp(-aiz)\} - i\mathrm{Im}(z)\{\exp(ai\bar{z}) - \exp(-aiz)\}\\
&= 2\mathrm{Re}(z)\exp(a\mathrm{Im}(z))\cos(a\mathrm{Re}(z)) + 2\mathrm{Im}(z)\exp(a\mathrm{Im}(z))\sin(a\mathrm{Re}(z)).
\end{aligned}$$

Then arguing as in the proof of Lemma 8.1, we find that

$$\int_{C(t)} \frac{\mathrm{Re}(z)\exp\left(\pi m\frac{\sqrt{D}}{t}\mathrm{Im}(z)\right)\cos\left(\pi m\frac{\sqrt{D}}{t}\mathrm{Re}(z)\right)}{z^2-1}dz \ll_m \log(D)$$

and

$$\int_{C(t)} \frac{\mathrm{Im}(z)\exp\left(\pi m\frac{\sqrt{D}}{t}\mathrm{Im}(z)\right)\sin\left(\pi m\frac{\sqrt{D}}{t}\mathrm{Re}(z)\right)}{z^2-1}dz \ll_m \log(D).$$

It follows that

$$\int_{H(t)}^{\pi-H(t)} \frac{\partial}{\partial t} g_{m,D,t}(\theta)d\theta \ll \log(D)\frac{\sqrt{D}}{t^2}.$$

□

By Lemma 8.2 and Proposition 9.5,

$$\begin{aligned}
\int_{\frac{D^{\frac{1}{2}-\eta}}{2}}^{\frac{\sqrt{D}}{2(1+D^{-\nu})}} \sum_{1\le c\le t} S_m(D;4c)A_2(t)dt &\ll_{\epsilon,m} \log(D)D^{\frac{1}{2}-\frac{1}{32}+\epsilon}\sqrt{D}\int_{\frac{D^{\frac{1}{2}-\eta}}{2}}^{\frac{\sqrt{D}}{2(1+D^{-\nu})}} \frac{1}{t^2}dt\\
&\ll_{\epsilon,m} \log(D)D^{\frac{1}{2}-(\frac{1}{32}-\eta)+\epsilon}. \qquad (8.6)
\end{aligned}$$

Finally, by combining estimates (8.3)–(8.6), we conclude that

$$C_2(m,\nu,D) = O_{\epsilon,m}(D^{\frac{1}{2}-\eta+\epsilon}) + O_{\epsilon,m}(D^{\frac{1}{2}-(\frac{1}{32}-\eta)+\epsilon}). \qquad (8.7)$$

## 9. Estimation of the sum of quadratic Weyl sums

In this section we use a quantitative equidistribution theorem for integral points on hyperboloids to obtain a uniform estimate for the sum of quadratic Weyl sums

$$\sum_{1\le c\le T} S_m(D;4c)$$

for $T = O(\sqrt{D})$.

We first describe the equidistribution results we will need, following closely the discussion in [ELMV, sections 1 and 2]. For a discriminant $D > 0$, define the hyperboloid

$$H_D(\mathbb{R}) : y^2 - 4xz = D.$$

Let $\mathrm{r}_D$ be the radial projection

$$\mathrm{r}_D : H_D(\mathbb{R}) \to H_1(\mathbb{R})$$

given by

$$\mathrm{r}_D(x, y, z) = (\frac{x}{\sqrt{D}}, \frac{y}{\sqrt{D}}, \frac{z}{\sqrt{D}}) =: (x_1, y_1, z_1).$$

For a convex domain $\Omega \subset H_1(\mathbb{R})$, let $\omega$ be the Lebesque measure in $\mathbb{R}^3$ of the solid cone whose vertex is the origin and base is $\Omega$.

Skubenko [S] proved that the set of integral points

$$H_D(\mathbb{Z}) := H_D(\mathbb{R}) \cap \mathbb{Z}^3$$

becomes equidistributed as $D \to \infty$ in the following sense (see e.g. [ELMV, Theorem 1.2]).

**Theorem 9.1** (Skubenko)**.** *Let $p > 2$ be a fixed prime. For any two convex domains $\Omega_1, \Omega_2 \subset H_1(\mathbb{R})$ with $\omega(\Omega_2) \neq 0$, one has*

$$\frac{\#\left(\mathrm{r}_D(H_D(\mathbb{Z})) \cap \Omega_1\right)}{\#\left(\mathrm{r}_D(H_D(\mathbb{Z})) \cap \Omega_2\right)} = \frac{\omega(\Omega_1)}{\omega(\Omega_2)} + o(1)$$

*as $D \to \infty$ through discriminants such that $(\frac{D}{p}) = 1$.*

The splitting condition $(\frac{D}{p}) = 1$ in Theorem 9.1 can be removed by reformulating the equidistribution problem in a "dual form" and using harmonic analysis on $X = \mathbb{H}/\Gamma$. Let $D > 0$ be a nonsquare and let $Q \in \mathcal{Q}_D$. The geodesic semi-circle in $\mathbb{H}$ joining the two real roots of $Q(z, 1)$ can be lifted to the unit tangent bundle $T^1(\mathbb{H})$ and projected to a geodesic orbit $\gamma_Q$ on the unit tangent bundle $T^1(X)$. The orbit $\gamma_Q$ is compact and depends only on the $\mathrm{SL}_2(\mathbb{Z})$-orbit of $Q$. This yields $h^+(D)$ compact geodesic orbits

$$\mathscr{G}_D := \bigcup_{Q \in \mathcal{Q}_D/\Gamma} \gamma_Q \subset T^1(X).$$

The set $\mathscr{G}_D$ has a natural probability measure $\mu_D$ which is invariant under the geodesic flow (see [ELMV, section 2.4.2]). As a consequence of an estimate of Duke [D3] for Fourier coefficients of half-integral weight forms, the set $\mathscr{G}_D$ becomes equidistributed on $T^1(X)$ with respect to the Liouville probability measure $\mu_{\mathrm{L}} = \frac{3}{\pi}\frac{dxdy}{y^2}\frac{\theta}{2\pi}$ as $D \to \infty$ through fundamental discriminants (see e.g. [ELMV, Theorem 1.3]).

**Theorem 9.2** (Duke)**.** *For any $\phi \in C_c^\infty(T^1(X))$,*

$$\int_{\mathscr{G}_D} \phi(t) d\mu_D(t) = \int_{T^1(X)} \phi(u) d\mu_{\mathrm{L}}(u) + o(1)$$

*as $D \to \infty$ through fundamental discriminants.*

**Remark 9.3.** In [ELMV, section 2], Einsiedler, Lindenstrauss, Michel, and Venkatesh give a proof of the equivalence of the equidistribution statements in Theorems 9.1 and 9.2.

The harmonic-analytic proof of Theorem 9.2 actually yields a power savings in the error term. To prove that $\mu_D$ weak-$*$ converges to $\mu_{\mathrm{L}}$ as $D \to \infty$, one must verify Weyl's criterion by showing that for $\phi$ ranging over a fixed orthonormal basis of $L_0^2(T^1(X))$, the Weyl sum

$$W(\phi, D) := \int_{\mathscr{G}_D} \phi(t) d\mu_D(t) \to 0$$

as $D \to \infty$. Arguing as in section 3, we obtain the estimate

$$W(\phi, D) \ll_{\phi,\epsilon} D^{-\frac{1}{16}+\epsilon},$$

where we use the period formula of Popa [P, Theorem 6.3.1] and the subconvexity bound of Blomer and Harcos [BH, Theorem 2] to estimate the contribution from the holomorphic cusp forms $\phi$ of weight $2k$ which appear in the spectral decomposition of $L_0^2(T^1(X))$ (see e.g. [LRS, section 1.3, eqs. (1.9)–(1.11)]). The precise relationship between the Weyl sum $W(\phi, D)$ and Popa's formula can be deduced from [LRS, sections 2.4, 6.1, and 6.2].

We summarize the preceding discussion in the following theorem.

**Theorem 9.4.** *For any two convex domains $\Omega_1, \Omega_2 \subset H_1(\mathbb{R})$ with $\omega(\Omega_2) \neq 0$,*

$$\frac{\#\left(\mathrm{r}_D(H_D(\mathbb{Z})) \cap \Omega_1\right)}{\#\left(\mathrm{r}_D(H_D(\mathbb{Z})) \cap \Omega_2\right)} = \frac{\omega(\Omega_1)}{\omega(\Omega_2)} + O_\epsilon(D^{-\frac{1}{16}+\epsilon})$$

*as $D \to \infty$ through fundamental discriminants.*

By adapting a method of Golubeva [G] we will establish the following estimate.

**Proposition 9.5.** *For $T = O(\sqrt{D})$, we have*

$$\sum_{1 \le c \le T} S_m(D; 4c) = O_{m,\epsilon}(D^{\frac{1}{2}-\frac{1}{32}+\epsilon}).$$

*Proof.* Let $h = 1/K$ and partition the interval $[0, 4c]$ into $K$ intervals

$$\left[\frac{k}{K}4c, \left(\frac{k}{K}+h\right)4c\right), \quad k = 0, 1, \ldots, K-1,$$

of length $4c/K$. This yields the decomposition

$$\sum_{1 \le c \le T} S_m(D; 4c) = \sum_{k=0}^{K-1} e\left(\frac{2mk}{K}\right) \sum_{1 \le c \le T} \sum_{\substack{\frac{k}{K}4c \le b < \left(\frac{k}{K}+h\right)4c \\ b^2 \equiv D \mod 4c}} e\left(\frac{2m\left(b - \frac{k}{K}4c\right)}{4c}\right).$$

Let $\Omega_k$ be the subset of $H_1(\mathbb{R})$ defined by the inequalities

$$0 < z_1 \le \frac{T}{\sqrt{D}}, \qquad \frac{k}{K}4z_1 \le y_1 < \left(\frac{k}{K}+h\right)4z_1.$$

Then a straightforward calculation gives (see e.g. [G, eq. (3) and Lemma 2])

$$\omega(\Omega_k) = \frac{2}{3}\iint_{\Omega_k} \frac{dy_1 dz_1}{|z_1|} = \frac{2}{3}\int_0^{T/\sqrt{D}} \int_{\frac{k}{K}4z_1}^{\left(\frac{k}{K}+h\right)4z_1} dy_1 \frac{dz_1}{z_1} = \frac{8}{3}h\frac{T}{\sqrt{D}}.$$

Let

$$\Omega_{\mathrm{red}} := \mathrm{r}_D(F_{\mathrm{red}}) \subset H_1(\mathbb{R})$$

where $F_{\text{red}}$ is the classical reduction domain for $\mathcal{Q}_D$ on $H_D(\mathbb{R})$. Then

$$\#(H_D(\mathbb{Z}) \cap F_{\text{red}}) \ll \sqrt{D} L_D(1) \ll_\epsilon D^{\frac{1}{2}+\epsilon},$$

where $L_D(s)$ is the Dirichlet $L$–function of $(\frac{D}{\cdot})$. It follows from Theorem 9.4 with the choice of sets $\Omega_k$ and $\Omega_{\text{red}}$ that

$$\# \left(H_D(\mathbb{Z}) \cap \mathrm{r}_D^{-1}(\Omega_k)\right) = \frac{8}{3} h \frac{T}{\sqrt{D}} \frac{\#(H_D(\mathbb{Z}) \cap F_{\text{red}})}{\omega(\Omega_{\text{red}})} + O_\epsilon(D^{\frac{1}{2}-\frac{1}{16}+\epsilon}). \tag{9.1}$$

Using the Taylor expansion

$$e\left(\frac{2m\left(b - \frac{k}{K}4c\right)}{4c}\right) = 1 + O_m\left(\frac{\left(b - \frac{k}{K}4c\right)}{4c}\right)$$

and the asymptotic formula (9.1), we obtain

$$\begin{aligned}
\sum_{1\le c\le T} S_m(D;4c) &= \sum_{k=0}^{K-1} e\left(\frac{2mk}{K}\right) \sum_{1\le c\le T} \sum_{\substack{\frac{k}{K}4c \le b < \left(\frac{k}{K}+h\right)4c \\ b^2\equiv D \mod 4c}} e\left(\frac{2m\left(b-\frac{k}{K}4c\right)}{4c}\right) \\
&= \sum_{k=0}^{K-1} e\left(\frac{2mk}{K}\right) \sum_{1\le c\le T} \sum_{\substack{\frac{k}{K}4c \le b < \left(\frac{k}{K}+h\right)4c \\ b^2\equiv D \mod 4c}} (1 + O_m(h)) \\
&= \sum_{k=0}^{K-1} e\left(\frac{2mk}{K}\right) \# \left(H_D(\mathbb{Z}) \cap \mathrm{r}_D^{-1}(\Omega_k)\right) + O_m\left(L_D(1) hT\right) \\
&= \frac{8}{3} h \frac{T}{\sqrt{D}} \frac{\#(H_D(\mathbb{Z}) \cap F_{\text{red}})}{\omega(\Omega_{\text{red}})} \sum_{k=0}^{K-1} e\left(\frac{2mk}{K}\right) + O_\epsilon(KD^{\frac{1}{2}-\frac{1}{16}+\epsilon}) + O_m\left(L_D(1)hT\right) \\
&= O_\epsilon(KD^{\frac{1}{2}-\frac{1}{16}+\epsilon}) + O_{m,\epsilon}\left(\frac{D^{\frac{1}{2}+\epsilon}}{K}\right),
\end{aligned}$$

where we used

$$\sum_{k=0}^{K-1} e\left(\frac{2mk}{K}\right) = 0$$

and the bounds $L_D(1) \ll_\epsilon D^\epsilon$ and $T = O(\sqrt{D})$. Let $K \asymp D^{1/32}$ and substitute this bound to complete the proof. □

## 10. Proof of Theorem 1.1

By combining the asymptotic formula (5.1), the decomposition (6.1), and the estimates (7.2) and (8.7), we have

$$\frac{\mathrm{Tr}_D\,(j_m)}{-24\sigma_1(m)\mathrm{Tr}_D(1)} = 1 + \frac{1}{-24\sigma_1(m)\mathrm{Tr}_D(1)} E(m, D, \nu, \eta) + O_{\epsilon,m}(D^{-(\frac{1}{16}-12\nu)+\epsilon}),$$

where

$$E(m,D,\nu,\eta) = O_{\epsilon,m}(D^{\frac{1}{2}-\frac{\nu}{2}+\epsilon}) + O_{\epsilon,m}(D^{\frac{1}{2}-\eta+\epsilon}) + O_{\epsilon,m}(D^{\frac{1}{2}-(\frac{1}{32}-\eta)+\epsilon}).$$

It follows from Siegel's theorem $\mathrm{Tr}_D(1) \gg_\epsilon D^{\frac{1}{2}-\epsilon}$ that

$$\frac{\mathrm{Tr}_D\,(j_m)}{-24\sigma_1(m)\mathrm{Tr}_D(1)} = 1 + O_{m,\epsilon}(D^{-\delta+\epsilon}),$$

where

$$\delta := \max_{\substack{0<\nu<1/192 \\ 0<\eta<1/32}} \min\{\frac{1}{16} - 12\nu, \frac{\nu}{2}, \eta, \frac{1}{32} - \eta\} = \frac{1}{400}.$$

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