THE AVERAGE OF THE DIVISOR FUNCTION OVER VALUES OF A QUADRATIC POLYNOMIAL

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Abstract. We establish a uniform asymptotic formula with a power saving error term for the average of the divisor function $\tau(n) := \sum_{k|n} 1$ over values of the quadratic polynomial $x^2 + |D|$ where $D < 0$ is a fundamental discriminant.

1. Introduction and statement of results

It is an important problem in number theory to study the asymptotic distribution of averages like

$$\sum_{1 \leq n \leq M} f(P(n))$$

as $M \to \infty$ where f is some arithmetical function and $P \in \mathbb{Z}[x]$ is a quadratic polynomial. A prototype for this problem is the classical divisor function

$$\tau(n) := \sum_{k|n} 1.$$

For D a *fixed* integer such that $-D$ is not a perfect square, Hooley [H] established the asymptotic formula

$$\sum_{1 \leq n \leq M} \tau(n^2 + D) = a_{1,D} \log(M) M + a_{2,D} M + O_D(\log(M)^3 M^{\frac{8}{9}}),$$

where $a_{1,D}$ and $a_{2,D}$ are explicit non-zero constants depending on D, and the implied constant in the error term depends on D (here one sets $\tau(m) = 0$ if $m \leq 0$). The error term in Hooley's result was improved by Deshouillers-Iwaniec [DI] and Bykovskii [By] to $O_{\varepsilon,D}(M^{\frac{2}{3}+\varepsilon})$ for any $\varepsilon > 0$ using methods from the spectral theory of automorphic forms.

Assuming that $D < 0$ is a fundamental discriminant, we will establish a uniform asymptotic formula for the average of $\tau(n)$ over the quadratic polynomial $x^2 + |D|$ in which we save a power of both M and $|D|$ in the error term. This problem gives rise to considerable new challenges, particularly in the critical range $M \asymp \sqrt{|D|}$. Our main result is the following

Theorem 1.1. *Let $M \geq 1$ and $D < 0$ be a fundamental discriminant. Then*

$$\sum_{M \leq |n| \leq 2M} \tau(n^2 + |D|) = c_{1,D} \log(\sqrt{|D|}) \sqrt{|D|} + c_{2,D} \sqrt{|D|}$$

$$+ O_\varepsilon \left((M|D|)^\varepsilon |D|^{\frac{167}{336}} \max \left\{ 1, \left(\frac{M}{\sqrt{|D|}} \right)^{\frac{41}{42}} \right\} \right),$$

where

$$c_{1,D} := \frac{8L(\chi_D,1)}{\zeta(2)} \int_X^Y (1-u^{-2})^{-1/2} du$$

and

$$c_{2,D} := \frac{8L(\chi_D,1)}{\zeta(2)} \int_X^Y \left(\log(u) + \gamma - 2\frac{\zeta'(2)}{\zeta(2)} + \frac{L'(\chi_D,1)}{L(\chi_D,1)} - \log(2)\frac{2-\chi_D(2)}{3-\chi_D(2)} \right) (1-u^{-2})^{-1/2} du$$

with $X = \sqrt{1+\frac{M^2}{|D|}}$ *and* $Y = \sqrt{1+\frac{4M^2}{|D|}}$*. Here* γ *is Euler's constant and* χ_D *is the Kronecker symbol.*

Remark 1.2. The asymptotic formula in Theorem 1.1 is meaningful for

$$M > |D|^{\frac{1}{2}-\frac{1}{672}+\varepsilon}$$

with a power saving error term.

There has recently been a great amount of interest in asymptotic formulas or upper bounds for averages like

$$\sum_{1\le n\le M} f(n^2+D),$$

with a particular emphasis on obtaining a wide range of uniformity in both M and D (see e.g. [DFI2], [FI, section 14.8], [FI2], [M], [Te2], [TT]). Such uniformity is often crucial for number-theoretic applications. For example, Duke, Friedlander, and Iwaniec [DFI2] established a uniform bound for averages of Weyl sums for quadratic roots and used it to study the distribution of prime points on the sphere and cycle integrals of the j–function, among other things. Friedlander and Iwaniec [FI2] later used this bound to study representations of integers by indefinite ternary quadratic forms. Namely, if $r(n)$ denotes the number of representations of n as the sum of two squares and $D > 0$, they established a uniform asymptotic formula for the average

$$\sum_{M\le n\le 2M} r(n^2+D)F(n)$$

where $F(x)$ is suitable smooth function supported on $[M, 2M]$. Their asymptotic formula is meaningful for

$$M > D^{\frac{1}{2}-\frac{1}{1332}}$$

with a power saving error term.

To conclude the introduction, we briefly outline how Theorem 8.1 (which is the smooth version of Theorem 1.1) can be used to give an asymptotic formula with a power saving error term for the second moment

$$\sum_{\chi} |L(\chi, \tfrac{1}{2})|^2,$$

where χ varies over the ideal class group characters of an imaginary quadratic field $\mathbb{Q}(\sqrt{D})$ and $L(\chi, s)$ is the L–function of χ. A beautiful asymptotic formula for the second moment was first given by Duke, Friedlander, and Iwaniec [DFI] using a different method (see also [Bl], [Te]).

Let $E'(z,1/2)$ be the central derivative of the weight zero Eisenstein series for $PSL_2(\mathbb{Z})$, and let Θ_χ be the weight one theta series of level $|D|$ associated to χ. One has the Fourier expansion

$$E'(z,\tfrac{1}{2}) = 2\sqrt{y}\log\left(\frac{e^\gamma y}{4\pi}\right) + 4\sqrt{y}\sum_{n\geq 1}\tau(n)\cos(2\pi nx)K_0(2\pi ny).$$

The Rankin-Selberg L–function of $E'(z,1/2)$ and Θ_χ is given by

$$L(E'\times\chi,s) = \sum_{n\geq 1}\frac{a_n(\chi)}{n^s}, \qquad \mathrm{Re}(s)>1$$

where

$$a_n(\chi) := \sum_{m^2k=n}\chi_D(m)\sum_{N(\mathfrak{a})=k}\chi(\mathfrak{a})\tau(k),$$

the inner sum being over integral ideals $\mathfrak{a}$ of norm $N(\mathfrak{a})=k$ in $\mathbb{Q}(\sqrt{D})$. Using the identity

$$L(E'\times\chi,s) = L(\chi,s)^2,$$

we see that the asymptotic evaluation of the second moment is equivalent to the asymptotic evaluation of

$$\sum_\chi L(E'\times\chi,\tfrac{1}{2}).$$

After an application of the approximate functional equation [IK, Theorem 5.3] and the orthogonality relations for the characters χ, we split the sum into diagonal and off-diagonal terms. The analysis of the diagonal term reduces to a standard contour shift and application of Burgess's [Bu] subconvexity bound for $L(\chi_D,s)$, while the analysis of the off-diagonal term reduces to an asymptotic formula for a smooth average with the Fourier coefficients $\tau(n)$ of the type considered in Theorem 8.1.

Acknowledgments. We would like to thank Matt Young for very helpful discussions, Professors Friedlander and Iwaniec for their interest in this work, and the referee for suggestions which improved the exposition. The second author was supported by the NSF grant DMS-1162535 during the preparation of this work.

2. Outline of the proof

In this section we outline the proof of Theorem 8.1, from which we will deduce Theorem 1.1 by un-smoothing. Let $s_0\in\mathbb{C}$ and $\phi:(1,\infty)\to\mathbb{C}$ be a C^∞ function with compact support. Define the absolutely convergent series

$$\Phi_{s_0}(\phi;z) := \sum_{\sigma\in\Gamma}\mathrm{Im}(\sigma z)^{s_0}\phi\left(\frac{|\sigma z|}{\mathrm{Im}(\sigma z)}\right),$$

where $\Gamma = PSL_2(\mathbb{Z})$. Bykovskii [By, Lemma 6] established the identity

$$\sum_{n\in\mathbb{Z}}\sigma_{-s_0}(n^2+|D|)\phi\left(\sqrt{\frac{n^2+|D|}{|D|}}\right) = |D|^{-s_0/2}\mathrm{Tr}_D^*(\Phi_{s_0}(\phi;\cdot)), \tag{2.1}$$

where $D<0$ is an integer,

$$\sigma_{-s_0}(n) := \sum_{d|n} d^{-s_0},$$

and

$$\mathrm{Tr}_D^*(\Phi_{s_0}(\phi;\cdot)) := \sum_{z_Q\in\Lambda_D} \frac{\Phi_{s_0}(\phi;z_Q)}{\#\Gamma_Q}$$

is the trace of $\Phi_{s_0}(\phi,z)$ over the CM points $z_Q\in\Lambda_D$ of determinant D in the standard fundamental domain $\mathcal{F}$ for Γ (see section 3).

In Theorem 4.1 we establish an alternative version of Bykovskii [By, Theorem 2 (b)], which gives the spectral expansion of $\Phi_{s_0}(\phi;z)$ in $L^2(\mathcal{F})$. We then substitute this expansion into (2.1), take the trace, and let $s_0\to 0$ to obtain an identity of the form (see Theorem 4.3)

$$\sum_{n\in\mathbb{Z}} \tau(n^2+|D|)\phi\left(\sqrt{\frac{n^2+|D|}{|D|}}\right) = M_D(\phi)+E_D(\phi).$$

To evaluate the main term $M_D(\phi)$, we establish a Chowla-Selberg type formula for any negative discriminant (see Lemma 6.1). To estimate the contribution of the discrete spectrum to error term $E_D(\phi)$, we use a period formula of Katok and Sarnak [KS] for Fourier coefficients of weight zero Maass forms, an explicit version of Waldspurger's formula due to Baruch and Mao [BM] for Fourier coefficients of half-integral weight Maass forms, and subconvexity bounds of Conrey-Iwaniec [CI] and Blomer-Harcos [BH] for twisted L–functions on GL_2. To estimate the contribution of the continuous spectrum to error term $E_D(\phi)$, we use a period formula for Eisenstein series on $PSL_2(\mathbb{Z})$ and a subconvexity bound of Burgess [Bu] for Dirichlet L–functions. Considerable care must be taken in each of these arguments because of the presence of non-fundamental discriminants.

3. CM points

Let $D<0$ be an integer and let $\mathcal{Q}_D$ be the set of positive definite, integral, binary quadratic forms $Q(X,Y)=aX^2+2bXY+cY^2$ with even middle coefficient and determinant $b^2-ac=D$ (recall that the *determinant* of Q is $\frac{1}{4}\Delta$ where $\Delta\equiv 0 \mod 4$ is the discriminant of Q). The two roots of the dehomogenized form $Q(X,1)$ are complex conjugates. Let

$$z_Q := \frac{-b+\sqrt{D}}{a}$$

be the root in the complex upper half-plane $\mathbb{H}$. The group $\Gamma = PSL_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$ with finite quotient $\mathcal{Q}_D/\Gamma$ of class number $h(D) := \#(\mathcal{Q}_D/\Gamma)$. Let $\mathcal{F}$ be the standard fundamental domain for Γ in $\mathbb{H}$. Each class in $\mathcal{Q}_D/\Gamma$ contains exactly one form Q with $z_Q\in\mathcal{F}$. We denote the set of all such *CM points* by

$$\Lambda_D := \{z_Q\in\mathcal{F}:\ Q\in\mathcal{Q}_D/\Gamma\}.$$

Let $\Gamma_Q<\Gamma$ be the stabilizer of Q. We define the trace of a Γ-invariant function $h:\mathbb{H}\to\mathbb{C}$ by

$$\mathrm{Tr}_D^*(h) := \sum_{z_Q\in\Lambda_D} \frac{h(z_Q)}{\#\Gamma_Q}.$$

For the analysis in this paper, we will also need traces defined using CM points associated to quadratic forms of any negative discriminant $\Delta < 0$, and in particular, the relationship between these traces and $\mathrm{Tr}_D^*(h)$ when $\Delta \equiv 0 \mod 4$ (see e.g. the proof of Theorem 8.1). So, let $\Delta < 0$ be an integer with $\Delta \equiv 0, 1 \mod 4$ (i.e., Δ is negative discriminant) and let $\mathcal{Q}'_\Delta$ be the set of positive definite, integral, binary quadratic forms $Q'(X,Y) = a'X^2 + b'XY + c'Y^2$ of discriminant $(b')^2 - 4a'c' = \Delta$. The two roots of the dehomogenized form $Q'(X,1)$ are complex conjugates, and we let

$$z_{Q'} := \frac{-b' + \sqrt{\Delta}}{2a'}$$

be the root in $\mathbb{H}$. The group Γ acts on $\mathcal{Q}'_\Delta$ with finite quotient $\mathcal{Q}'_\Delta/\Gamma$ of class number $h(\Delta) := \#(\mathcal{Q}'_\Delta/\Gamma)$, and each class in $\mathcal{Q}'_\Delta/\Gamma$ contains exactly one form Q' with $z_{Q'} \in \mathcal{F}$. We let

$$\Lambda'_\Delta := \{z_{Q'} \in \mathcal{F} : \; Q' \in \mathcal{Q}'_\Delta/\Gamma\}$$

and define the trace

$$\mathrm{Tr}_\Delta(h) := \sum_{z_{Q'} \in \Lambda'_\Delta} \frac{h(z_{Q'})}{\#\Gamma_{Q'}}.$$

Finally, suppose that $\Delta \equiv 0 \mod 4$ and write $\Delta = 4D$ for some integer $D < 0$. Then $\mathcal{Q}'_\Delta = \mathcal{Q}_D$, $\Lambda'_\Delta = \Lambda_D$ and

$$\mathrm{Tr}_\Delta(h) = \mathrm{Tr}_D^*(h). \tag{3.1}$$

4. A formula of Bykovskii

Define the hyperbolic Laplacian

$$\Delta := -y^2(\partial_x^2 + \partial_y^2), \qquad z = x + iy \in \mathbb{H}.$$

Let $\{u_j(z)\}_{j=1}^\infty$ be an orthonormal basis of weight zero Hecke-Maass cusp forms for Γ with Δ-eigenvalues λ_j and let

$$E(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s, \qquad \mathrm{Re}(s) > 1$$

be the weight zero Eisenstein series for Γ. The Hecke-Maass cusp form $u_j(z)$ has the Fourier expansion

$$u_j(x+iy) = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} a_j(n) W_{\frac{1}{2}+it_j}(n(x+iy)),$$

where $t_j := \sqrt{\lambda_j - \frac{1}{4}}$ is the spectral parameter,

$$W_{\frac{1}{2}+it_j}(x+iy) = 2\sqrt{|y|} K_{it_j}(2\pi\,|y|) e^{2\pi i x}$$

is the Whittaker function and $K_s(u)$ is the modified Bessel function of the second kind.

Let $\lambda_j(n)$ be the n-th Hecke eigenvalue of $u_j(z)$. Then $\lambda_j(1) = 1$ and

$$a_j(\pm n) = a_j(\pm 1)\lambda_j(n)|n|^{-1/2}.$$

Iwaniec [I] and Hoffstein-Lockhart [HL] proved that

$$|t_j|^{-\varepsilon} \ll_\varepsilon \frac{|a_j(1)|^2}{\cosh(\pi t_j)} \ll_\varepsilon |t_j|^{\varepsilon}. \tag{4.1}$$

Define the Dirichlet series

$$L_j(s) := 2\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\overline{a_j(n)}}{|n|^{s-\frac{1}{2}}} = 2\overline{a_j(1)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\overline{\lambda_j(n)}}{|n|^{s}}, \qquad \mathrm{Re}(s) > 1.$$

Let $s_0 \in \mathbb{C}$ and $\phi : (1,\infty) \to \mathbb{C}$ be a C^∞ function with compact support. Define the absolutely convergent series

$$\Phi_{s_0}(\phi; z) := \sum_{\sigma\in\Gamma} \mathrm{Im}(\sigma z)^{s_0}\phi\left(\frac{|\sigma z|}{\mathrm{Im}(\sigma z)}\right).$$

Bykovskii calculated the spectral expansion of $\Phi_{s_0}(\phi; z)$ in [By, Theorem 2 (b)]. The integral transform appearing in Bykovskii's formula is a somewhat complicated expression involving hypergeometric functions. We will establish the following alternative version of Bykovskii's formula in which the integral transform is expressed in a way which makes it easier to estimate. The proof is a modification and elaboration of Bykovskii's argument.

Theorem 4.1. *For $0 < \mathrm{Re}(s_0) < 1/2$ we have*

$$\begin{aligned}\Phi_{s_0}(\phi; z) &= 4E(z, 1+s_0)\int_1^\infty \phi(u)(1-u^{-2})^{-1/2}du \\ &+ 4E(z, 1-s_0)\int_1^\infty \phi(u)u^{-2s_0}(1-u^{-2})^{-1/2}du \\ &+ \sum_{j=1}^{\infty} G_{s_0}(\phi; t_j)L_j(\tfrac{1}{2}+s_0)u_j(z) \\ &+ \frac{1}{2\sqrt{\pi}}\int_{\mathbb{R}} G_{s_0}(\phi; t)\pi^{-it}\frac{\zeta(\frac{1}{2}+it+s_0)\zeta(\frac{1}{2}-it+s_0)}{\Gamma(\frac{1}{2}-it)\zeta(1-2it)}E(z, \tfrac{1}{2}+it)dt,\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}G_{s_0}(\phi; t) := \frac{\pi^{-s_0}}{4\pi i}&\Gamma(\tfrac{1}{4}+\tfrac{it}{2}+\tfrac{s_0}{2})\Gamma(\tfrac{1}{4}-\tfrac{it}{2}+\tfrac{s_0}{2}) \quad \times \\ &\int_{(\varepsilon)} \frac{\Gamma(\frac{s}{2}+it+\frac{s_0}{2})\Gamma(\frac{s}{2}+\frac{s_0}{2})}{\Gamma(\frac{s}{2}+\frac{1}{4}+\frac{it}{2})\Gamma(\frac{s}{2}+\frac{1}{4}+\frac{it}{2}+s_0)}\tilde{\phi}\left(s+\tfrac{1}{2}+it\right)ds, \quad \varepsilon > 0, \quad t\in\mathbb{R}\end{aligned}$$

and $\tilde{\phi}$ is the Mellin transform

$$\tilde{\phi}(s) := \int_0^\infty \phi(u)u^{s-1}du.$$

Proof. Define the function

$$\phi_1(P; u) := \begin{cases} P-u, & 0 < u \le P \\ 0, & u > P.\end{cases}$$

Then a short calculation yields

$$\Phi_{s_0}(\phi_1(P;\cdot);z) = \sum_{\substack{\sigma\in\Gamma \\ 0<\frac{|\sigma z|}{\mathrm{Im}(\sigma z)}\leq P}} (\mathrm{Im}(\sigma z))^{s_0}\frac{|\sigma z|}{\mathrm{Im}(\sigma z)}\left(\frac{P}{\frac{|\sigma z|}{\mathrm{Im}(\sigma z)}}-1\right).$$

For $c>1$, we have the formula

$$\frac{1}{2\pi i}\int_{(c)}\frac{1}{s(s+1)}X^{s+1}ds = \begin{cases} X-1, & X\geq 1 \\ 0, & 0\leq X<1. \end{cases}$$

Apply this formula with

$$X = \frac{P}{\frac{|\sigma z|}{\mathrm{Im}(\sigma z)}}$$

to obtain

$$\Phi_{s_0}(\phi_1(P;\cdot);z) = \frac{1}{2\pi i}\int_{(c)}\frac{1}{s(s+1)}P^{s+1}\Pi(z;s,s_0)ds, \tag{4.3}$$

where the series

$$\Pi(z;s,s_0) := \sum_{\sigma\in\Gamma}(\mathrm{Im}(\sigma z))^{s_0}\left(\frac{|\sigma z|}{\mathrm{Im}(\sigma z)}\right)^{-s}$$

is absolutely convergent and holomorphic for $\mathrm{Re}(s_0)>0$ and $\mathrm{Re}(s)>1$ (see [By, p. 12]).

Let $\phi:(1,\infty)\to\mathbb{C}$ be a C^∞ function with compact support. Then a straightforward calculation yields the identity

$$\Phi_{s_0}(\phi;z) = \int_1^\infty \phi^{(2)}(P)\Phi_{s_0}(\phi_1(P;\cdot);z)dP. \tag{4.4}$$

Substitute (4.3) into (4.4), reverse the order of integration, and integrate by parts twice with respect to P to get

$$\Phi_{s_0}(\phi;z) = \frac{1}{2\pi i}\int_{(c)}\Pi(z;s,s_0)\tilde{\phi}(s)ds, \tag{4.5}$$

where $\tilde{\phi}$ is the Mellin transform

$$\tilde{\phi}(s) := \int_0^\infty \phi(P)P^{s-1}dP.$$

From [By, Lemma 2], if $0 < \mathrm{Re}(s_0) < 1/2$ and $\mathrm{Re}(s) > 1$ then

$$\begin{aligned}\Pi(z; s, s_0) &= 2\sqrt{\pi}\frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}E(z, 1+s_0) + 2\sqrt{\pi}\frac{\Gamma(\frac{s-1}{2}+s_0)}{\Gamma(\frac{s}{2}+s_0)}E(z, 1-s_0) \\ &+ \frac{\pi^{-s_0}}{2\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)}\sum_{j=1}^{\infty}\Gamma(s, s_0; t_j)L_j(\tfrac{1}{2}+s_0)u_j(z) \\ &+ \frac{1}{2\sqrt{\pi}}\frac{\pi^{-s_0}}{2\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)} \\ &\quad\times \int_{\mathbb{R}}\Gamma(s, s_0; t)\pi^{-it}\frac{\zeta(\frac{1}{2}+it+s_0)\zeta(\frac{1}{2}-it+s_0)}{\Gamma(\frac{1}{2}-it)\zeta(1-2it)}E(z, \tfrac{1}{2}+it)dt,\end{aligned} \tag{4.6}$$

where

$$\Gamma(s, s_0, t) := \Gamma(\tfrac{1}{4}+\tfrac{it}{2}+\tfrac{s_0}{2})\Gamma(\tfrac{1}{4}-\tfrac{it}{2}+\tfrac{s_0}{2})\Gamma(\tfrac{s}{2}-\tfrac{1}{4}+\tfrac{it}{2}+\tfrac{s_0}{2})\Gamma(\tfrac{s}{2}-\tfrac{1}{4}-\tfrac{it}{2}+\tfrac{s_0}{2}).$$

Next substitute (4.6) into (4.5) to get

$$\begin{aligned}\Phi_{s_0}(\phi; z) &= 2\sqrt{\pi}E(z, 1+s_0)I_0(\phi) + 2\sqrt{\pi}E(z, 1-s_0)I_{s_0}(\phi) \\ &+ \frac{\pi^{-s_0}}{4\pi i}\sum_{j=1}^{\infty}L_j(\tfrac{1}{2}+s_0)u_j(z)\Gamma(\tfrac{1}{4}+\tfrac{it_j}{2}+\tfrac{s_0}{2})\Gamma(\tfrac{1}{4}-\tfrac{it_j}{2}+\tfrac{s_0}{2})I(\phi; t_j) \\ &+ \frac{1}{2\sqrt{\pi}}\frac{\pi^{-s_0}}{4\pi i}\int_{\mathbb{R}}\pi^{-it}\Gamma(\tfrac{1}{4}+\tfrac{it}{2}+\tfrac{s_0}{2})\Gamma(\tfrac{1}{4}-\tfrac{it}{2}+\tfrac{s_0}{2})I(\phi; t) \\ &\qquad\times\frac{\zeta(\frac{1}{2}+it+s_0)\zeta(\frac{1}{2}-it+s_0)}{\Gamma(\frac{1}{2}-it)\zeta(1-2it)}E(z, \tfrac{1}{2}+it)dt,\end{aligned}$$

where

$$I_\alpha(\phi) := \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\frac{s-1}{2}+\alpha)}{\Gamma(\frac{s}{2}+\alpha)}\tilde{\phi}(s)ds, \quad \alpha \in \mathbb{C}$$

and

$$I(\phi; t) := \int_{(c)}\frac{\Gamma(\frac{s}{2}-\frac{1}{4}+\frac{it}{2}+\frac{s_0}{2})\Gamma(\frac{s}{2}-\frac{1}{4}-\frac{it}{2}+\frac{s_0}{2})}{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)}\tilde{\phi}(s)ds.$$

To complete the proof, we use Lemma 4.2 and the identity

$$I(\phi; t) = \int_{(\varepsilon)}\frac{\Gamma(\frac{s}{2}+it+\frac{s_0}{2})\Gamma(\frac{s}{2}+\frac{s_0}{2})}{\Gamma(\frac{s}{2}+\frac{1}{4}+\frac{it}{2})\Gamma(\frac{s}{2}+\frac{1}{4}+\frac{it}{2}+s_0)}\tilde{\phi}\left(s+\tfrac{1}{2}+it\right)ds, \quad \varepsilon > 0$$

which follows by changing variables $s \mapsto s + \frac{1}{2} + it$ and shifting the contour (c) to (ε).

□

Lemma 4.2. *We have*

$$I_\alpha(\phi) = \frac{2}{\sqrt{\pi}}\int_1^\infty \phi(u)u^{-2\alpha}(1-u^{-2})^{-1/2}du.$$

Proof. Recall that $\Gamma(\frac{s-1}{2}+\alpha)$ has simple poles at $s=-(2n+1)-2\alpha$ for $n=-1,0,\ldots$ with residue $2(-1)^{n+1}/(n+1)!$. Thus

$$I_\alpha(\phi):=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\frac{s-1}{2}+\alpha)}{\Gamma(\frac{s}{2}+\alpha)}\tilde{\phi}(s)ds=2\frac{\tilde{\phi}(1-2\alpha)}{\Gamma(\frac{1}{2})}+2\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)!}\frac{\tilde{\phi}(-(2n+1)-2\alpha)}{\Gamma\left(\frac{-(2n+1)}{2}\right)}.$$

Using the identities $\Gamma(1/2)=\sqrt{\pi}$ and

$$\Gamma\left(\tfrac{-(2n+1)}{2}\right)=\sqrt{\pi}(-1)^{n+1}2^{2n+1}\frac{n!}{(2n+1)!},$$

along with the Taylor expansion

$$(1+x)^{-1/2}=1+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)!}{2^{2n+1}(n+1)!n!}x^{n+1},\quad |x|<1$$

we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\frac{s-1}{2}+\alpha)}{\Gamma(\frac{s}{2}+\alpha)}\tilde{\phi}(s)ds&=\frac{2}{\sqrt{\pi}}\int_1^\infty\phi(u)u^{-2\alpha}\left(1+\sum_{n=0}^{\infty}\frac{(2n+1)!}{2^{2n+1}(n+1)!n!}u^{-2n-2}\right)du\\&=\frac{2}{\sqrt{\pi}}\int_1^\infty\phi(u)u^{-2\alpha}(1-u^{-2})^{-1/2}du.\end{aligned}$$

□

Now, define the Eisenstein series

$$\widetilde{E}(z,s):=\frac{1}{2}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}}\frac{y^s}{|mz+n|^{2s}},\qquad z=x+iy\in\mathbb{H},\quad \mathrm{Re}(s)>1.$$

By Kronecker's first limit formula (see e.g. [L, chapter 20]),

$$\widetilde{E}(z,s)=\frac{\pi}{2}\frac{1}{s-1}+\pi\left(\gamma-\log(2)-F(z)\right)+O(s-1)\tag{4.7}$$

where

$$F(z):=\log(\sqrt{y}|\eta(z)|^2)$$

and

$$\eta(z):=q^{1/24}\prod_{n=1}^{\infty}(1-q^n),\quad q:=e^{2\pi iz}$$

is the Dedekind eta function. Using the identity

$$\widetilde{E}(z,s)=\zeta(2s)E(z,s),$$

we obtain the expansion

$$E(z,s)=\frac{3}{\pi}\frac{1}{s-1}+\frac{6}{\pi}\left(\gamma-\log(2)-\frac{\zeta'(2)}{\zeta(2)}-F(z)\right)+O(s-1).\tag{4.8}$$

Bykovskii [By, Lemma 6] established the identity

$$\sum_{n\in\mathbb{Z}}\sigma_{-s_0}(n^2+|D|)\phi\left(\sqrt{\frac{n^2+|D|}{|D|}}\right)=|D|^{-s_0/2}\mathrm{Tr}_D^*(\Phi_{s_0}(\phi;\cdot)),\tag{4.9}$$

where $D<0$ is an integer and

$$\sigma_{-s_0}(n) := \sum_{d|n} d^{-s_0}.$$

Apply (4.8) to the first two terms on the right hand side of the spectral expansion (4.2), then substitute the resulting expression for $\Phi_{s_0}(\phi;z)$ into the trace in (4.9) and let $s_0 \to 0$ to obtain the following analog of [By, Theorem 5 (c)].

Theorem 4.3. *Let $D<0$ be an integer and $\phi:(1,\infty)\to\mathbb{C}$ be a C^∞ function with compact support. Then*

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \tau(n^2+|D|)\phi\left(\sqrt{\frac{n^2+|D|}{|D|}}\right) = \qquad\qquad (4.10)\\ \mathrm{Tr}_D^*(1)\int_1^\infty \left(\frac{24}{\pi}\log(u)+\frac{48}{\pi}\left(\gamma-\log(2)-\frac{\zeta'(2)}{\zeta(2)}\right)\right)\phi(u)\left(1-u^{-2}\right)^{-1/2}du\\ -\frac{48}{\pi}\mathrm{Tr}_D^*(F)\int_1^\infty \phi(u)\left(1-u^{-2}\right)^{-1/2}du\\ +\sum_{j=1}^\infty G_0(\phi;t_j)L_j(\tfrac{1}{2})\mathrm{Tr}_D^*(u_j)\\ +\frac{1}{2\sqrt{\pi}}\int_{\mathbb{R}} G_0(\phi;t)\pi^{-it}\frac{\left|\zeta(\frac{1}{2}+it)\right|^2}{\Gamma(\frac{1}{2}-it)\zeta(1-2it)}\mathrm{Tr}_D^*(E(\cdot,\tfrac{1}{2}+it))dt.\end{aligned}$$

Finally, we will repeatedly use the following estimate for $G_0(\phi;t)$.

Lemma 4.4. *For $1\le X<Y\le 2X$ and $P\ge 1$, let $\phi:(1,\infty)\to\mathbb{C}$ be a C^∞ function supported on $[X,Y]$ such that*

$$\phi^{(A)} \ll \left(\frac{X}{P}\right)^{-A}, \quad A=0,1,\ldots.$$

Then

$$G_0(\phi;t) \ll X^{\frac{1}{2}+\varepsilon}P^A(1+|t|)^{-A}e^{-\frac{\pi}{2}|t|}.$$

Proof. Integrating by parts A-times and using the bound for $\phi^{(A)}$ yields

$$\tilde{\phi}(s) \ll X^{\mathrm{Re}(s)}P^A(1+|s|)^{-A}. \tag{4.11}$$

The result now follows from (4.11) and Stirling's formula. □

5. Periods of Eisenstein series

Let $\Delta<0$ be an integer with $\Delta\equiv 0,1 \mod 4$ and write $\Delta=f^2d$ for $f\in\mathbb{Z}^+$ and $d<0$ a fundamental discriminant. Following Zagier [Z, p. 109, eq. (6)], we define the zeta function

$$\zeta_\Delta(s) := \sum_{Q'\in\mathcal{Q}'_\Delta/\Gamma}\ \sum_{\substack{(m,n)\in\mathbb{Z}^2/\Gamma_{Q'}\\ Q'(m,n)>0}} \frac{1}{Q'(m,n)^s}, \qquad \mathrm{Re}(s)>1.$$

Then a standard calculation yields the identity (see e.g. [Z2, pp. 280-281])

$$\mathrm{Tr}_\Delta(E(\cdot,s)) = \frac{c_\Delta}{\zeta(2s)}\left(\frac{\sqrt{|\Delta|}}{2}\right)^s \zeta_\Delta(s) \tag{5.1}$$

where $c_\Delta > 0$ is a constant depending on $\#\Gamma_{Q'}$ which takes finitely many different values. On the other hand, Zagier [Z, Proposition 3 (iii)] showed that

$$\zeta_\Delta(s) = L_{f,d}(s)\zeta(s)L(\chi_d, s)$$

where $\chi_d(\cdot) := (\frac{d}{\cdot})$ is the Kronecker symbol,

$$L(\chi_d, s) = \sum_{n=1}^{\infty} \chi_d(n) n^{-s}, \qquad \mathrm{Re}(s) > 1$$

is the Dirichlet L–function associated to χ_d and

$$L_{f,d}(s) := \sum_{\alpha | f} \mu(\alpha)\chi_d(\alpha)\sigma_{1-2s}(f/\alpha)\alpha^{-s}$$

is a finite Dirichlet series where μ is the Möbius function and

$$\sigma_\nu(m) := \sum_{\substack{\beta | m \\ \beta > 0}} \beta^\nu, \quad m \in \mathbb{Z}^+.$$

The preceding facts imply

$$\mathrm{Tr}_\Delta(E(\cdot,s)) = c_\Delta \frac{\zeta(s)}{\zeta(2s)}\left(\frac{f\sqrt{|d|}}{2}\right)^s L_{f,d}(s) L(\chi_d, s). \tag{5.2}$$

Lemma 5.1. *Let $\Delta < 0$ be an integer with $\Delta \equiv 0,1 \mod 4$ and write $\Delta = f^2 d$ for $f \in \mathbb{Z}^+$ and $d < 0$ a fundamental discriminant. Then*

$$\mathrm{Tr}_\Delta(E(\cdot, \tfrac{1}{2}+it)) \ll_\varepsilon (1+|t|)^{\frac{7}{6}+\varepsilon} f^{1+\varepsilon} |d|^{\frac{7}{16}+\varepsilon}.$$

Proof. This follows from (5.2), the bound (see [T, Theorem 5.12])

$$\zeta(\tfrac{1}{2}+it) \ll_\varepsilon |t|^{\frac{1}{6}+\varepsilon}, \tag{5.3}$$

a standard lower bound for $\zeta(2s)$ on $\mathrm{Re}(s) = 1/2$ (see [T, eq. (3.6.5)]) and Burgess's [Bu] subconvexity bound

$$L(\chi_d, \tfrac{1}{2}+it) \ll_\varepsilon (1+|t|)|d|^{\frac{3}{16}+\varepsilon}.$$

□

6. A formula for $\mathrm{Tr}_\Delta(F)$

We will need the following Chowla-Selberg type formula.

Lemma 6.1. *Let $\Delta < 0$ be an integer with $\Delta \equiv 0,1 \mod 4$ and write $\Delta = f^2 d$ for $f \in \mathbb{Z}^+$ and $d < 0$ a fundamental discriminant. Then*

$$\mathrm{Tr}_\Delta(F) = \frac{\mathrm{Tr}_\Delta(1)}{2}\left(\gamma - \log(2) - \frac{1}{2}\log|\Delta| - \frac{L'(\chi_d,1)}{L(\chi_d,1)} - \frac{L'_{f,d}(1)}{L_{f,d}(1)}\right). \tag{6.1}$$

Proof. Recall that

$$\widetilde{E}(z,s)=\zeta(2s)E(z,s).$$

Then (5.1) yields the identity

$$\zeta_\Delta(s)=c_\Delta^{-1}\left(\frac{2}{\sqrt{|\Delta|}}\right)^s \mathrm{Tr}_\Delta(\widetilde{E}(\cdot,s)).$$

Using the expansion

$$\left(\frac{2}{\sqrt{|\Delta|}}\right)^{s-1}=1+\log\left(\frac{2}{\sqrt{|\Delta|}}\right)(s-1)+O(s-1)^2$$

and the Kronecker limit formula (4.7), we obtain the expansion

$$\zeta_\Delta(s)=\frac{\pi c_\Delta^{-1}\mathrm{Tr}_\Delta(1)}{\sqrt{|\Delta|}}\left(\frac{1}{s-1}+2\gamma-\log(2)-\frac{1}{2}\log|\Delta|-\frac{2}{\mathrm{Tr}_\Delta(1)}\mathrm{Tr}_\Delta(F)+O(s-1)\right).$$

Recall also that

$$\zeta_\Delta(s)=L_{f,d}(s)\zeta(s)L(\chi_d,s).$$

Then letting

$$Z_{f,d}(s):=L_{f,d}(s)L(\chi_d,s),$$

we obtain the expansion

$$\zeta_\Delta(s)=\frac{Z_{f,d}(1)}{s-1}+Z_{f,d}(1)\gamma+Z'_{f,d}(1)+O(s-1).$$

By comparing coefficients in the two expansions of $\zeta_\Delta(s)$, we get

$$Z_{f,d}(1)=\frac{\pi c_\Delta^{-1}\mathrm{Tr}_\Delta(1)}{\sqrt{|\Delta|}}$$

and

$$Z'_{f,d}(1)=\frac{\pi c_\Delta^{-1}\mathrm{Tr}_\Delta(1)}{\sqrt{|\Delta|}}\left(\gamma-\log(2)-\frac{1}{2}\log|\Delta|-\frac{2}{\mathrm{Tr}_\Delta(1)}\mathrm{Tr}_\Delta(F)\right).$$

Together, these yield the identity

$$\frac{Z'_{f,d}(1)}{Z_{f,d}(1)}=\gamma-\log(2)-\frac{1}{2}\log|\Delta|-\frac{2}{\mathrm{Tr}_\Delta(1)}\mathrm{Tr}_\Delta(F),$$

or equivalently

$$\mathrm{Tr}_\Delta(F)=\frac{\mathrm{Tr}_\Delta(1)}{2}\left(\gamma-\log(2)-\frac{1}{2}\log|\Delta|-\frac{L'(\chi_d,1)}{L(\chi_d,1)}-\frac{L'_{f,d}(1)}{L_{f,d}(1)}\right).$$

∎

7. Periods of Maass forms and Waldspurger's formula

Let

$$\Delta_k := \Delta - iky\partial_x, \quad k \in \frac{1}{2}\mathbb{Z}$$

be the weight k hyperbolic Laplacian. Let $u_j(z)$ be an even Hecke-Maass cusp form of weight zero for Γ with Δ-eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$ and $\langle u_j, u_j \rangle = 1$. The theta lift $g_j(\tau) := \theta(u_j, \tau)$ defined in [KS, p. 206] is a Maass form of weight $\frac{1}{2}$ for $\Gamma_0(4)$ with $\Delta_{1/2}$-eigenvalue $\frac{1}{4} + \frac{t_j^2}{4}$. The Fourier expansion of $g_j(\tau)$ is given by (see [KS, p. 196, eq. (0.10)])

$$g_j(u+iv) = \sum_{n \neq 0} \rho_j(n) W_{\frac{\mathrm{sign}(n)}{4}, \frac{it_j}{2}}(4\pi|n|v)e(nu), \qquad \tau = u + iv \in \mathbb{H}.$$

In particular, one has $\rho_j(n) = 0$ if $n \equiv 2, 3 \mod 4$. If $n < 0$ with $n \equiv 0, 1 \mod 4$, one has the following period formula of Katok and Sarnak [KS],

$$\rho_j(n) = \frac{1}{4\sqrt{2}\pi^{3/4}|n|^{3/4}} \mathrm{Tr}_n(u_j). \tag{7.1}$$

Write

$$Z_{f,d}(s) = L_{f,d}(s)L(\chi_d, s) =: \sum_{n=1}^{\infty} \frac{\varepsilon_\Delta(n)}{n^s},$$

and define the L–series

$$L(\Delta, u_j, s) := \sum_{n=1}^{\infty} \frac{\varepsilon_\Delta(n)\lambda_j(n)}{n^s} = \sum_{\alpha | f} \alpha^{1-2s} \sum_{(n, \frac{f}{\alpha})=1} \chi_d(n)\lambda_j(nf^2)n^{-s}, \qquad \mathrm{Re}(s) > 1.$$

One has the following explicit Waldspurger type formula.

Theorem 7.1. *Let $\Delta < 0$ be an integer with $\Delta \equiv 0, 1 \mod 4$. Then*

$$\frac{|\rho_j(\Delta)|^2}{\langle g_j, g_j \rangle} = (\pi|\Delta|)^{-1}\Gamma(\tfrac{3}{4} - \tfrac{it_j}{2})\Gamma(\tfrac{3}{4} + \tfrac{it_j}{2})|a_j(1)|^2 L(\Delta, u_j, \tfrac{1}{2}).$$

Proof. If $\Delta = d$ is a fundamental discriminant this follows from Baruch and Mao [BM, Theorem 1.4]. If $\Delta = f^2 d$ for $f > 1$ this follows by modifying [BM, Theorem 1.4] along the lines of the proof of Kohnen and Zagier [KZ, Corollary 4]. $\square$

Lemma 7.2. *Let $\Delta < 0$ be an integer with $\Delta \equiv 0, 1 \mod 4$ and write $\Delta = f^2 d$ for $f \in \mathbb{Z}^+$ and $d < 0$ a fundamental discriminant. Then*

$$|\rho_j(\Delta)| \ll_\varepsilon f^{-\frac{57}{64}}(1 + |t_j|)^{\frac{15}{2}}(1 + |t_j||d|)^\varepsilon |d|^{-\frac{5}{16}}.$$

Proof. By Theorem 7.1, the bound (4.1) and Stirling's formula, we have

$$|\rho_j(\Delta)|^2 \ll_\varepsilon \langle g_j, g_j \rangle \, |\Delta|^{-1} e^{-\frac{\pi}{2}|t_j|} L(\Delta, u_j, \tfrac{1}{2}).$$

Define the L–series

$$L(\chi_d \times u_j, s) := \sum_{n=1}^{\infty} \frac{\chi_d(n)\lambda_j(n)}{n^s}, \qquad \mathrm{Re}(s) > 1.$$

Then one has the factorization (see the proof of [KZ, Corollary 4])

$$L(\Delta, u_j, \tfrac{1}{2}) = L(\chi_d \times u_j, \tfrac{1}{2}) \left(\sum_{\alpha | f} \mu(\alpha)\chi_d(n)\alpha^{-1}\lambda_j(f/\alpha) \right)^2 .$$

Using the Kim-Sarnak estimate ([K, appendix 2])

$$\lambda_j(n) \ll |n|^{\frac{7}{64}+\varepsilon}$$

and the hybrid subconvexity bound of Blomer and Harcos [BH, eq. (1.3)]

$$L(\chi_d \times u_j, \tfrac{1}{2}) \ll_\varepsilon (1+|t_j|)^3(1+|t_j||d|)^\varepsilon |d|^{\frac{3}{8}},$$

we obtain

$$L(\Delta, u_j, \tfrac{1}{2}) \ll_\varepsilon f(1+|t_j|)^4(1+|t_j||d|)^\varepsilon |d|^{\frac{3}{8}}.$$

By [DFI, Proposition 3],

$$\langle g_j, g_j \rangle \ll_\varepsilon \cosh\left(\frac{\pi}{2}t_j\right)(1+|t_j|)^{12+\varepsilon}.$$

Combining the preceding estimates yields

$$|\rho_j(\Delta)| \ll_\varepsilon f^{-\frac{57}{64}}(1+|t_j|)^{\frac{15}{2}}(1+|t_j||d|)^\varepsilon |d|^{-\frac{5}{16}}.$$

□

Lemma 7.3. *Let $\Delta < 0$ be an integer with $\Delta \equiv 0, 1 \mod 4$ and write $\Delta = f^2 d$ for $f \in \mathbb{Z}^+$ and $d < 0$ a fundamental discriminant. Then*

$$\mathrm{Tr}_\Delta(u_j) \ll_\varepsilon f^{\frac{39}{64}}(1+|t_j|)^{\frac{15}{2}}(1+|t_j||d|)^\varepsilon |d|^{\frac{7}{16}+\varepsilon}.$$

Proof. By (7.1),

$$\mathrm{Tr}_\Delta(u_j) = 4\sqrt{2}\pi^{3/4}|\Delta|^{3/4}\rho_j(\Delta).$$

The result now follows from Lemma 7.2.

□

8. The smooth average of $\tau(n)$ over $x^2 + |D|$

In this section we establish an asymptotic formula for the smooth average of $\tau(n)$ over $x^2 + |D|$.

Theorem 8.1. *Let $D < 0$ be a fundamental discriminant. For $1 \le X < Y \le 2X$ and $P \ge 1$, let $\phi : (1, \infty) \to \mathbb{C}$ be a C^∞ function supported on $[X, Y]$ such that*

$$\phi^{(A)} \ll \left(\frac{X}{P}\right)^{-A}, \quad A = 0, \ldots, 12.$$

Then

$$\sum_{n\in\mathbb{Z}} \tau(n^2+|D|)\phi\left(\sqrt{\frac{n^2+|D|}{|D|}}\right) = c_{1,D}(\phi)\log(\sqrt{|D|})\sqrt{|D|} + c_{2,D}(\phi)\sqrt{|D|} + O_\varepsilon(X^{\frac{1}{2}+\varepsilon}P^{10}|D|^{\frac{7}{16}+\varepsilon})$$

where

$$c_{1,D}(\phi) := \frac{8L(\chi_D, 1)}{\zeta(2)} \int_1^\infty \phi(u)(1-u^{-2})^{-1/2}du$$

and

$$c_{2,D}(\phi) := \frac{8L(\chi_D,1)}{\zeta(2)} \int_1^\infty \left(\log(u) + \gamma - 2\frac{\zeta'(2)}{\zeta(2)} + \frac{L'(\chi_D,1)}{L(\chi_D,1)} - \log(2)\frac{2-\chi_D(2)}{3-\chi_D(2)} \right) \phi(u)(1-u^{-2})^{-1/2} du.$$

Proof. Suppose that $D < 0$ is a fundamental discriminant, and let $\Delta = 4D$. Then $\Delta \equiv 0 \bmod 4$ is a negative discriminant, and if we write $\Delta = f^2 d$ for $f \in \mathbb{Z}^+$ and $d < 0$ a fundamental discriminant, we have $f = 2$ and $d = D$. Recall also that from (3.1), we have

$$\mathrm{Tr}_D^*(h) = \mathrm{Tr}_\Delta(h)$$

for any Γ-invariant function $h : \mathbb{H} \to \mathbb{C}$.

For notational convenience, we write Theorem 4.3 as

$$\sum_{n\in\mathbb{Z}} \tau(n^2+|D|)\phi\left(\sqrt{\frac{n^2+|D|}{|D|}}\right) = M_D(\phi) + E_D(\phi)$$

where $M_D(\phi)$ consists of the first and second terms on the right hand side of (4.10) and $E_D(\phi)$ consists of the third and fourth terms on the right hand side of (4.10).

To evaluate the main term $M_D(\phi)$, we use the identity (see [Z, eq. (20)])

$$\mathrm{Tr}_D^*(1) = \frac{2}{\pi} L(\chi_D,1)\sqrt{|D|}$$

and the following identity which is implied by (6.1),

$$\mathrm{Tr}_D^*(F) = \frac{1}{\pi} L(\chi_D,1)\sqrt{|D|}\left(\gamma - \log(2) - \frac{1}{2}\log(4|D|) - \frac{L'(\chi_D,1)}{L(\chi_D,1)} - \frac{L'_{2,D}(1)}{L_{2,D}(1)}\right).$$

Then a straightforward calculation yields

$$M_D(\phi) = c_{1,D}(\phi)\log(\sqrt{|D|})\sqrt{|D|} + c_{2,D}(\phi)\sqrt{|D|}$$

where

$$c_{1,D}(\phi) := \frac{8L(\chi_D,1)}{\zeta(2)} \int_1^\infty \phi(u)(1-u^{-2})^{-1/2} du$$

and

$$c_{2,D}(\phi) := \frac{8L(\chi_D,1)}{\zeta(2)} \int_1^\infty \left(\log(u) + \gamma - 2\frac{\zeta'(2)}{\zeta(2)} + \frac{L'(\chi_D,1)}{L(\chi_D,1)} + \frac{L'_{2,D}(1)}{L_{2,D}(1)} \right) \phi(u)(1-u^{-2})^{-1/2} du.$$

To evaluate

$$\frac{L'_{2,D}(1)}{L_{2,D}(1)}$$

observe that

$$L_{2,D}(s) = \sum_{\alpha|2} \mu(\alpha)\chi_D(\alpha) \sum_{k|\frac{2}{\alpha}} k^{1-2s}\alpha^{-s},$$

hence another straightforward calculation yields

$$\frac{L'_{2,D}(1)}{L_{2,D}(1)} = -\log(2)\frac{2-\chi_D(2)}{3-\chi_D(2)}.$$

We now estimate the error term $E_D(\phi)$. For $u_j(z)$ even, we have

$$L_j(s) = 4\overline{a_j(1)}L(u_j,s)$$

where

$$L(u_j,s) := \sum_{n=1}^{\infty} \frac{\overline{\lambda_j(n)}}{n^s}, \qquad \mathrm{Re}(s) > 1.$$

Then Lemmas 4.4 and 7.3 yield

$$\begin{aligned}\sum_{j=1}^{\infty} G(\phi,t_j)L_j(\tfrac{1}{2})\mathrm{Tr}_D^*(u_j) &\ll_\varepsilon (X|D|P)^\varepsilon X^{\frac{1}{2}}|D|^{\frac{7}{16}}P^8 \sum_{t_j \ll P(X|D|P)^\varepsilon} L(u_j,\tfrac{1}{2}) \\ &\ll_\varepsilon (X|D|P)^\varepsilon X^{\frac{1}{2}}P^{10}\,|D|^{\frac{7}{16}}. \qquad (8.1)\end{aligned}$$

Similarly, using Lemmas 4.4 and 5.1, the bound (5.3), the lower bound for $\zeta(2s)$ on $\mathrm{Re}(s) = 1/2$ and Stirling's formula, we obtain

$$\int_{\mathbb{R}} G(\phi,t)\frac{\pi^{-it}\left|\zeta(\frac{1}{2}+it)\right|^2}{\Gamma(\frac{1}{2}-it)\zeta(1-2it)}\mathrm{Tr}_D^*(E(\cdot,\tfrac{1}{2}+it))dt \ll_\varepsilon X^{\frac{1}{2}+\varepsilon}P^4|D|^{\frac{7}{16}+\varepsilon}. \qquad (8.2)$$

Finally, by combining these estimates we get

$$E_D(\phi) = O_\varepsilon((X|D|P)^\varepsilon X^{\frac{1}{2}}P^{10}\,|D|^{\frac{7}{16}}).$$

$\square$

9. Proof of Theorem 1.1

For $1 \le X < Y \le 2X$ and $P \ge 1$, let $\phi : (1,\infty) \to \mathbb{C}$ be a C^∞ function such that

(1) $0 \le \phi \le 1$,
(2) ϕ is supported on $[X,Y]$,
(3) $\phi \equiv 1$ on the interval $[X+\frac{X}{P}, Y-\frac{X}{P}]$,

and ϕ satisfies the bound

$$\phi^{(A)} \ll \left(\frac{X}{P}\right)^{-A}, \qquad A = 0,\ldots,12.$$

Using properties (1)–(3) and the trivial estimate

$$\tau(c) \ll_\varepsilon c^\varepsilon,$$

we find that

$$\sum_{M\le|n|\le 2M} \tau(n^2+|D|) = \sum_{n\in\mathbb{Z}} \tau(n^2+|D|)\phi\left(\sqrt{\frac{n^2+|D|}{|D|}}\right) + O_\varepsilon(X^{1+\varepsilon}P^{-\frac{1}{2}}\,|D|^{\frac{1}{2}+\varepsilon}),$$

with $X = \sqrt{1+\frac{M^2}{|D|}}$ and $Y = \sqrt{1+\frac{4M^2}{|D|}}$.

By Theorem 8.1 we have

$$\sum_{n\in\mathbb{Z}} \tau(n^2+|D|)\phi\left(\sqrt{\frac{n^2+|D|}{|D|}}\right) = c_{1,D}(\phi)\log(\sqrt{|D|})\sqrt{|D|} + c_{2,D}(\phi)\sqrt{|D|} + O_\varepsilon(X^{\frac{1}{2}+\varepsilon}P^{10}\,|D|^{\frac{7}{16}+\varepsilon}).$$

Next we un-smooth the coefficients $c_{1,D}(\phi)$ and $c_{2,D}(\phi)$. Using properties (1)–(3), a straightforward estimate yields

$$\int_1^\infty \phi(u)\log(u)(1-u^{-2})^{-1/2}du = \int_X^Y \log(u)(1-u^{-2})^{-1/2}du + O(X^{1+\varepsilon}P^{-1}).$$

The same estimate holds with $\log(u)$ replaced by 1. Then using the upper bound

$$L^{(j)}(\chi_D,1) \ll_\varepsilon D^\varepsilon, \qquad j=0,1$$

we obtain

$$c_{1,D}(\phi) = c_{1,D} + O_\varepsilon(X^{1+\varepsilon}P^{-1}\,|D|^\varepsilon),$$
$$c_{2,D}(\phi) = c_{2,D} + O_\varepsilon(X^{1+\varepsilon}P^{-1}\,|D|^\varepsilon),$$

where $c_{1,D}$ and $c_{2,D}$ are defined as in the statement of Theorem 1.1.

Combining the preceding estimates yields

$$\sum_{M\le|n|\le 2M} \tau(n^2+|D|) = c_{1,D}\sqrt{|D|} + c_{2,D}\log(\sqrt{|D|})\sqrt{|D|}$$
$$+ O_\varepsilon(X^{1+\varepsilon}P^{-\frac12}\,|D|^{\frac12+\varepsilon}) + O_\varepsilon(X^{\frac12+\varepsilon}P^{10}\,|D|^{\frac{7}{16}+\varepsilon}) + O_\varepsilon(X^{1+\varepsilon}P^{-1}\,|D|^{\frac12+\varepsilon}).$$

Finally, to optimize the error term we choose

$$P = |D|^{\frac{1}{168}} \max\left\{1, \left(\frac{M}{|D|^{\frac12}}\right)^{\frac{1}{21}}\right\}.$$

□

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