THE DISTRIBUTION OF G-WEYL CM FIELDS AND THE COLMEZ CONJECTURE

ADRIAN BARQUERO-SANCHEZ, RIAD MASRI, AND FRANK THORNE

Abstract. Let G be a transitive subgroup of S_d, and let E be a CM field of degree $2d$ with a maximal totally real G-field. If the Galois group of the closure E^c is isomorphic to the wreath product $C_2 \wr G$, then we say that E is a *G-Weyl CM field*. This refines the notion of a Weyl CM field introduced by Chai and Oort in their study of special points on Shimura varieties.

Assuming a weak form of the upper bound in Malle's conjecture which is known unconditionally in many cases, we prove an asymptotic formula with a power-saving error term for the number of G-Weyl CM fields E of degree $2d$ and discriminant $|d_E| \leq X$. This asymptotic formula implies that the G-Weyl CM fields comprise an asymptotic density of 100% of all CM fields of degree $2d$.

We apply these distribution results to study the Colmez conjecture. More precisely, using the recently proved averaged Colmez conjecture, we prove that the Colmez conjecture is true for G-Weyl CM fields. Combined with our distribution results, we conclude (conditional on the upper bound mentioned above) that the Colmez conjecture is true for 100% of CM fields of degree $2d$; in other words, the Colmez conjecture is true for a *random* CM field.

1. Introduction and statement of results

In the important papers [Mal02, Mal04], Malle made some very precise conjectures for the asymptotic growth of functions which count number fields of fixed degree and bounded discriminant with prescribed Galois group. In this paper we will study this distribution problem for CM fields. As is well known, CM fields are of fundamental importance in number theory, and their Galois groups encode deep arithmetic information.

Recall that a *CM field* E of degree $2d$ is a totally imaginary quadratic extension of a totally real number field F of degree d over $\mathbb{Q}$. Let E^c be the Galois closure of E and let S_d be the symmetric group. Then the Galois group $\mathrm{Gal}(E^c/\mathbb{Q})$ embeds as a subgroup of the wreath product $C_2 \wr S_d$. In their study of special points on Shimura varieties, Chai and Oort [CO12] introduced the notion of a *Weyl CM field*, which is a CM field E with Galois group $\mathrm{Gal}(E^c/\mathbb{Q}) \cong C_2 \wr S_d$ (that is, E has maximal possible Galois group). The Weyl CM fields are associated to special CM points on the moduli space $\mathcal{A}_d$ of principally polarized abelian varieties of dimension d called *Weyl CM points* (see [CO12, Definition 2.11]).

As remarked by Oort [Oor12, p. 5], results of Gallagher [Gal73], Chavdarov [Cha97], Kowalski [Kow06], and Ellenberg, Elsholtz, Hall, and Kowalski [EEHK09] suggest that "most" CM fields should be Weyl CM fields (see also Footnote 2 on p. 591 of [CO12]). Here we will introduce a more refined notion of a Weyl CM field, and prove that this expectation does not hold in general if one orders CM fields by discriminant.

More precisely, if G is the transitive subgroup of S_d with $\mathrm{Gal}(F^c/\mathbb{Q}) \cong G$, then $\mathrm{Gal}(E^c/\mathbb{Q})$ embeds as a subgroup of the wreath product $C_2 \wr G$ (see Proposition 3.1). This is a more refined version of the above mentioned embedding into $C_2 \wr S_d$. We say that E is a *G-Weyl CM field* if $\mathrm{Gal}(E^c/\mathbb{Q}) \cong C_2 \wr G$ (that is, E has maximal possible Galois group, subject to the restriction that its maximal totally real subfield have Galois group G). With this terminology, a Weyl CM field in the sense of Chai and Oort is an *S_d-Weyl CM field*.

Assuming a weak form of the upper bound in the Malle conjecture which is known unconditionally in many cases (see Hypothesis 1.1 and Corollary 1.3), we will prove an asymptotic formula with a power-saving error term for the number of G-Weyl CM fields E of degree $2d$ with discriminant $|d_E| \leq X$. Our strategy is to count pairs (E, F), where F is the maximal totally real subfield; by transitivity of the discriminant we have

$$|d_E| = d_F^2 \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{D}_{E/F}).$$

We find that for each F, 100% of CM extensions E/F are G-Weyl. We also find, subject to our weak Malle conjecture, that each F contributes a positive proportion to our asymptotics.

Given this conjecture, we can thus conclude that: (1) for each pair (d, G) such that there exists at least one totally real G-field, the G-Weyl CM fields comprise a positive proportion of all CM fields of degree $2d$; and (2) when taken together, the G-Weyl CM fields comprise an asymptotic density of 100% of all CM fields of degree $2d$ (see Theorem 1.9).

For example, when $d = 3, 4$ or 5, the S_d-Weyl CM fields comprise approximately 31%, 20%, and 70% of CM fields of degree $2d$, respectively (see Table 1). These statistics are largely determined by the distribution of Galois groups among those degree d fields of smallest discriminant.

Our approach to counting G-Weyl CM fields is inspired by work of Klüners [Klu12], which established asymptotics for the counting function of number fields with Galois group $C_2 \wr G$, without signature conditions. We will adapt Klüners' work to handle the signature conditions needed to count CM fields. We will also give asymptotic formulas with power-saving error terms which incorporate recent progress on non-trivial bounds for 2-torsion in class groups of number fields [BSTTTZ17] and subconvexity bounds for ray class L–functions of totally real fields [ELMV11], and determine the weakest form of the upper bound in Malle's conjecture needed for our results.

We next discuss the connection between the distribution of G-Weyl CM fields and the Colmez conjecture [Col93], which relates the Faltings height of CM abelian variety to logarithmic derivatives of Artin L–functions at $s = 0$ (see Section 1.2). In fact, this paper was motivated by our effort to answer the following:

Question. *Is the Colmez conjecture true for a random CM field?*

In order to address this question, the first two authors [BSM16] developed a plan to study the Colmez conjecture from an arithmetic statistical point of view. First, we used an averaged version of the Colmez conjecture proved independently by Andreatta, Goren, Howard, and Madapusi Pera [AGHM15] and Yuan and Zhang [YZ15] to show that the Colmez conjecture is true for S_d-Weyl CM fields. We then applied work of Cohen, Diaz y Diaz, and Olivier [CDO02] to conclude that 100% of quartic CM fields are S_2-Weyl, and consequently, satisfy the Colmez conjecture. Due to the well known difficulties which arise when counting number fields with Galois group S_d, this line of attack seemed limited initially to CM fields of small degree.

To overcome these difficulties, we will use the Galois theory of CM fields to refine the argument in [BSM16] and prove that the Colmez conjecture is true for *any* G-Weyl CM field; in particular, if X is an abelian variety of dimension d with complex multiplication by a G-Weyl CM field E of degree $2d$ with maximal totally real subfield F, then the Faltings height of X is given by

$$h_{\mathrm{Fal}}(X) = -\frac{1}{2}\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} - \frac{1}{4}\log\left(\frac{d_E}{d_F}\right) - \frac{d}{2}\log(2\pi), \tag{1.1}$$

where $L(\chi_{E/F}, s)$ is the L–function of the Hecke character $\chi_{E/F}$ associated to the quadratic extension E/F (see Theorem 1.13). Combined with our results on the distribution of G-Weyl CM fields, this allows us to conclude (conditional on Hypothesis 1.1) that the Colmez conjecture is true for 100% of CM fields of any degree $2d$; in other words, the Colmez conjecture is true for a random CM field (see Theorem 1.17). Moreover, given the pairs (d, G) for which Hypothesis 1.1 is known unconditionally, we are able to produce infinitely many density-one families of non-abelian CM fields which satisfy the Colmez conjecture (see the results of Section 1.3).

Finally, we note that the identity (1.11) has many applications. For example, if E is a non-abelian quartic CM field (or equivalently, an S_2-Weyl CM field), the first two authors [BSM18] used (1.11) to give an explicit non-abelian analog of the classical Chowla-Selberg formula for CM abelian surfaces; this is an identity which evaluates the Faltings height of an abelian surface with complex multiplication by E in terms of values of the Barnes double Gamma function at algebraic arguments. The first two authors, jointly with Wei-Lun Tsai, are in the process of generalizing this analog of the Chowla-Selberg formula to any abelian variety with complex multiplication by a G-Weyl CM field.

1.1. The distribution of G-Weyl CM fields.

In order to state our results, we first define some of the counting functions that will be used throughout the paper (all number fields are counted up to $\mathbb{Q}$-isomorphism).

- Let $N_d(X, G)$ count all number fields K of degree d and discriminant $|d_K| \le X$ with $\mathrm{Gal}(K^c/\mathbb{Q}) \cong G$.
- Let $N_{2d}^{\mathrm{cm}}(X, G)$ count all CM fields E of degree $2d$ and discriminant $|d_E| \le X$ which have a maximal totally real subfield F with $\mathrm{Gal}(F^c/\mathbb{Q}) \cong G$.
- Let $N_{2d}^{\mathrm{Weyl}}(X, G)$ count all CM fields E of degree $2d$ and discriminant $|d_E| \le X$ which are G-Weyl.
- Let
$$N_{2d}^{\mathrm{Weyl}}(X) := \sum_{G \le S_d} N_{2d}^{\mathrm{Weyl}}(X, G).$$
- Let $N_{2d}^{\mathrm{cm}}(X)$ count all CM fields E of degree $2d$ and discriminant $|d_E| \le X$.

In [Mal02], Malle gave conjectural bounds for $N_d(X, G)$. If $g \in S_d$, the *index* of g is defined by

$$\mathrm{ind}(g) := d - \#(\{1, \dots, d\}/\langle g \rangle).$$

Let

$$\mathrm{ind}(G) := \min\{\mathrm{ind}(g) : \ 1 \ne g \in G\}$$

and define the constant $0 < a(G) := \mathrm{ind}(G)^{-1} \le 1$. Malle conjectured that for any $\epsilon > 0$, there exist positive constants $c_1(G), c_2(G, \epsilon)$ such that

$$c_1(G)X^{a(G)} \le N_d(X, G) \le c_2(G, \epsilon)X^{a(G)+\epsilon}. \tag{1.2}$$

Malle [Mal04] later refined this and gave a precise conjectural asymptotic formula for $N_d(X,G)$ as $X \to \infty$.

For our purposes, we only need an upper bound for $N_d(X,G)$ with an exponent which is *much weaker* than what is predicted by (1.2). This exponent will depend on bounds for 2-torsion in class groups.

For a number field K, let $\mathrm{Cl}(K)[2]$ be the 2-torsion subgroup of the ideal class group. Let $\delta_d \geq 0$ be a variable such that

$$|\mathrm{Cl}(K)[2]| \ll_{\epsilon,d} |d_K|^{\delta_d+\epsilon} \tag{1.3}$$

for all number fields K of degree d. By the Brauer-Siegel theorem, the bound (1.3) holds with $\delta_d = 1/2$. Any bound (1.3) with $0 < \delta_d < 1/2$ is called a non-trivial bound, and $\delta_d = 0$ is the conjectured optimal bound.

If $d = 2$, then it is a classical result that (1.3) holds with $\delta_2 = 0$. The first non-trivial bounds in (1.3) for $d \geq 3$ were recently proved by Bhargava, Shankar, Taniguchi, Thorne, Tsimerman, and Zhang [BSTTTZ17]. In particular, they proved that if $d = 3, 4$, then (1.3) holds with $\delta_d = 0.2784$, and if $d \geq 5$, then (1.3) holds with $\delta_d = 1/2 - 1/2d$.

With the variable δ_d as in (1.3), we state the following weak form of the upper bound in Malle's conjecture (1.2).

Hypothesis 1.1. *For a fixed pair (d,G) and $0 \leq \delta_d \leq 1/2$ satisfying (1.3), we have*

$$N_d(X,G) \ll X^{M(G)} \tag{1.4}$$

for some $M(G) > 0$ such that

$$\delta_d + M(G) < 2.$$

Our first result gives an asymptotic formula with a power-saving error term for the density of those CM fields counted by $N_{2d}^{\mathrm{cm}}(X,G)$ which are G-Weyl.

Theorem 1.2. *Assume that Hypothesis 1.1 is true for (d,G). Then*

$$\frac{N_{2d}^{\mathrm{Weyl}}(X,G)}{N_{2d}^{\mathrm{cm}}(X,G)} = 1 + O_{d,G,\epsilon}(X^{-C_1(\delta_d,M(G))+\epsilon}), \tag{1.5}$$

where

$$C_1(\delta_d, M(G)) := \begin{cases} 1/2, & \text{if } \delta_d + M(G) \leq 1 \\ 1 - \dfrac{\delta_d + M(G)}{2}, & \text{if } 1 < \delta_d + M(G) < 2. \end{cases} \tag{1.6}$$

The Hypothesis 1.1 is known in many cases due to work of the following authors on the Malle conjectures: [DH71, Mak85, CDO02, KM04, Bha05, KY05, EV06, BW08, Bha10, CT17, Wan17]. For convenience, we have summarized these results in Table 2 of Section 4.

Given these known cases of Hypothesis 1.1, we get the following unconditional results.

Corollary 1.3. *The asymptotic formula (1.5) holds unconditionally for the following pairs (d,G):*

- *Any (d,G) with G abelian.*
- *Any (d,G) with $d = \ell$ prime and $G = D_\ell$ dihedral.*
- *Any (d,G) with G a p-group.*
- *Any (d,G) with $d \geq 5$ and $|G| = d$.*
- *Any (d,G) with $d \leq 5$.*

- *Any* (d, G) *with* $d = 3|A|$ *and* $G = S_3 \times A$ *with* A *abelian.*
- *Any* (d, G) *with* $d = 4|A|$ *and* $G = S_4 \times A$ *with* A *abelian.*

Remark 1.4. In the 4th bullet of Corollary 1.3, the condition $|G| = d$ is equivalent to all number fields counted by $N_d(X, G)$ being Galois over $\mathbb{Q}$. This case follows from Ellenberg and Venkatesh [EV06, Proposition 1.3].

Remark 1.5. The Malle conjecture can be formulated more generally for degree d extensions L/K of any global field K with $\mathcal{N}_{K/\mathbb{Q}}(\mathfrak{D}_{L/K}) < X$ and Galois group $\mathrm{Gal}(L/K) \cong G$ for a transitive subgroup $G \leq S_d$ (here $\mathfrak{D}_{L/K}$ is the relative discriminant). In this setting, Ellenberg, Tran, and Westerland [ETW17] recently proved the upper bound in Malle's Conjecture when $K = \mathbb{F}_q(t)$ is the rational function field.

Example 1.6. If $(d, G) = (5, S_5)$, then Hypothesis 1.1 is true for the pair $(\delta_5, M(S_5)) = (2/5, 1)$. Since $C_1(2/5, 1) = 3/10$, we have

$$\frac{N_{10}^{\mathrm{Weyl}}(X, S_5)}{N_{10}^{\mathrm{cm}}(X, S_5)} = 1 + O_\epsilon(X^{-\frac{3}{10}+\epsilon}).$$

Remark 1.7. Assuming a sufficiently strong value for the exponent δ_d appearing in the 2-torsion bound (1.3), the asymptotic formula (1.5) is known for some additional pairs (d, G); see Table 3.

Our next goal is to give an asymptotic formula with a power-saving error term for the density of those CM fields counted by $N_{2d}^{\mathrm{cm}}(X)$ which are G-Weyl. This will involve the subconvexity problem for a certain family of Hecke L–functions for totally real fields.

Let F be a totally real field of degree d. Let $\mathfrak{c}$ be an integral ideal of F dividing 2, and let $\mathfrak{c}_\infty \subset \mathfrak{m}_\infty$ be a subset of the set $\mathfrak{m}_\infty$ of real places of F. Suppose that χ is a primitive character of the ray class group $\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F)$ modulo $\mathfrak{c}^2\mathfrak{c}_\infty$. The L–function of χ is defined by

$$L_F(\chi, s) := \prod_{\mathfrak{p}} \left(1 - \chi(\mathfrak{p})\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{p})^{-s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

The completed L–function is defined by (see e.g. [IK04, p. 129])

$$\Lambda_F(\chi, s) := q(\chi)^{s/2}\gamma(\chi, s)L_F(\chi, s),$$

where $q(\chi) := d_F \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{c}^2)$ and

$$\gamma(\chi, s) := \pi^{-ds/2}\Gamma\left(\frac{s}{2}\right)^{d-|\mathfrak{c}_\infty|}\Gamma\left(\frac{s+1}{2}\right)^{|\mathfrak{c}_\infty|}.$$

The completed L–function satisfies the functional equation

$$\Lambda_F(\chi, s) = \varepsilon(\chi)\Lambda_F(\overline{\chi}, 1-s),$$

where the root number $\varepsilon(\chi)$ is a complex number of modulus 1 which can be written explicitly as a normalized Gauss sum for χ. Given this data, we calculate the analytic conductor of $L_F(\chi, s)$ as (a slightly weaker version of [IK04, eq. (5.7)])

$$\mathfrak{q}(F, \chi, s) = d_F \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{c}^2)(|s| + 4)^d.$$

Let $\delta' \geq 0$ be a variable such that

$$\left(\frac{s-1}{s+1}\right)^{a(\chi)} L_F(\chi, s) \ll_{\epsilon,d} \mathfrak{q}(F,\chi,s)^{\delta'(1-\sigma)+\epsilon}, \quad 1/2 \leq \sigma := \mathrm{Re}(s) \leq 1+\epsilon, \tag{1.7}$$

where $a(\chi) = 1$ if χ is the trivial character and $a(\chi) = 0$ if χ is non-trivial. The bound (1.7) holds when $\delta' = 1/2$ (the convexity bound). Any bound (1.7) with $0 < \delta' < 1/2$ is called a subconvexity bound, and $\delta' = 0$ is the Lindelöf hypothesis. For more details concerning these facts, see [IK04, Chapter 5].

Remark 1.8. A subconvexity bound of the form (1.7) is known, for example, if F is either abelian or cubic (see e.g. the summary of results in [ELMV11, Appendix A]).

Now, let $D_G^{\mathrm{cm}}(s)$ be the Dirichlet series which enumerates all fields counted by $N_{2d}^{\mathrm{cm}}(X, G)$ (see (2.3)). In Theorem 2.2, we will prove that if Hypothesis 1.1 is true for (d, G) and $0 \leq \delta' \leq 1/2$ satisfies (1.7), then $D_G^{\mathrm{cm}}(s)$ has a meromorphic continuation to a half-plane $\mathrm{Re}(s) > \alpha$ for some $\alpha < 1$ (depending on $\delta_d, M(G)$ and δ') with only a single (simple) pole at $s = 1$. Moreover, the residue of $D_G^{\mathrm{cm}}(s)$ at $s = 1$ is given by the convergent series

$$r_d(G) := \sum_{F \in \mathcal{F}_G^+} \frac{\mathrm{Res}_{s=1}\zeta_F(s)}{2^d d_F^2 \zeta_F(2)} > 0, \tag{1.8}$$

where

$$\mathcal{F}_G^+ := \{F/\mathbb{Q} : \ F \text{ totally real of degree } d, \ \mathrm{Gal}(F^c/\mathbb{Q}) \cong G\}.$$

Using properties of the Dirichlet series $D_G^{\mathrm{cm}}(s)$ and an upper bound for the number of CM fields counted by $N_{2d}^{\mathrm{cm}}(X)$ which are *not* G-Weyl for any transitive subgroup $G \leq S_d$, we will prove the following asymptotic formulas with power-saving error terms.

Theorem 1.9. *Assume that Hypothesis 1.1 is true for every pair (d, G) where G ranges over all transitive subgroups $G \leq S_d$. Moreover, assume that $0 \leq \delta' \leq 1/2$ satisfies (1.7). Then for any such $G_0 \leq S_d$, we have*

$$\frac{N_{2d}^{\mathrm{Weyl}}(X, G_0)}{N_{2d}^{\mathrm{cm}}(X)} = \frac{r_d(G_0)}{\sum_{G \leq S_d} r_d(G)} + O_{d,G_0,\epsilon}(X^{-C_2(\delta_d, M(G_0), \delta')+\epsilon}), \tag{1.9}$$

and

$$\frac{N_{2d}^{\mathrm{Weyl}}(X)}{N_{2d}^{\mathrm{cm}}(X)} = 1 + O_{d,\epsilon}(X^{-C_3(\delta_d, \delta')+\epsilon}), \tag{1.10}$$

where $C_2(\delta_d, M(G), \delta') > 0$ and $C_3(\delta_d, \delta') > 0$ are explicit constants defined in (2.20) *and* (2.21)*, respectively.*

In Table 1 we give numerical computations for the residue $r_d(G)$, and hence for the relative density of G-Weyl CM fields, for each transitive $G \leq S_d$ with $d \leq 5$. We computed these by summing the series of (1.8) over the first n fields $F \in \mathcal{F}_G^+$, for n listed in the table. The basic field data was downloaded from the website `lmfdb.org` [LMFDB], and the remaining computations, including the L-function computations in (1.8), were handled with PARI/GP [PARI]. The (short) PARI/GP source code with which we put these computations together may be downloaded at the third author's website[1].

[1] `http://people.math.sc.edu/thornef`

From (1.8) we see that the residues are (very) approximately given by $2^{-d}\sum_F d_F^{-2}$. Assuming Malle's conjecture (1.2), the series converge relatively rapidly; and indeed it is known that $N_d(X,G) \ll X$ for all (d,G) listed in the table. With some effort, it should be possible to explicitly bound the error in our residue computations below; numerics suggest that these values are likely to be accurate within approximately ± 1 in the least significant digit listed.

We observe: each totally real F contributes a positive proportion to its respective residue, with those of smallest discriminant making the largest contribution; also, the residues are decreasing with d – a pattern which should persist, in light of lower bounds on d_F which are exponential in d [Odl90].

TABLE 1. Values of $r_d(G)$ for $d \le 5$

d	G	Number of fields	Minimal discriminant	Residue	Proportion in (1.9)
2	C_2	100,000	5	0.009856	-
3		25,000	49	3.30×10^{-5}	-
	C_3	107	49	2.29×10^{-5}	0.69
	S_3	24,893	148	1.01×10^{-5}	0.31
4		25,000	725	1.24×10^{-7}	-
	C_4	75	1125	2.41×10^{-8}	0.19
	V_4	289	1600	1.56×10^{-8}	0.13
	D_4	8147	725	5.9×10^{-8}	0.48
	A_4	45	26569	9.3×10^{-11}	0.0008
	S_4	16,444	1957	2.5×10^{-8}	0.20
5		25,000	14641	1.05×10^{-10}	-
	C_5	5	14641	3.08×10^{-11}	0.29
	D_5	28	160801	4.24×10^{-13}	0.003
	F_5	15	2382032	9×10^{-15}	0.00009
	A_5	21	3104644	5×10^{-15}	0.00005
	S_5	24,931	24217	7.4×10^{-11}	0.70

Example 1.10. Let (d,G) be any pair with $d=5$. Then Hypothesis 1.1 is true for the pair $(\delta_5, M(G)) = (2/5, 1)$. If we take $\delta' = 1/2$, then we have

$$C_1(\delta_5, M(G)) \ge \frac{1}{4},\ \alpha(\delta_5, M(G), \delta') \le \frac{19}{25},\ \beta(\delta_5,\delta') = \frac{17}{20},\ C_2(\delta_d, M(G), \delta') = \frac{3}{20},\ C_3(\delta_d, \delta') = \frac{3}{20},$$

these constants being defined in (1.6), (2.4), (2.11), (2.20), and (2.21) respectively. We conclude that

$$\frac{N_{10}^{\mathrm{Weyl}}(X)}{N_{10}^{\mathrm{cm}}(X)} = 1 + O_\epsilon(X^{-\frac{3}{20}+\epsilon}).$$

Moreover, if we assume the conjectured optimal bound in (1.3) and the Lindelöf hypothesis in (1.7) (so that Hypothesis 1.1 is true for the pair $(\delta_5, M(G)) = (0,1)$ and $\delta' = 0$), then

$$\frac{N_{10}^{\mathrm{Weyl}}(X)}{N_{10}^{\mathrm{cm}}(X)} = 1 + O_\epsilon(X^{-\frac{1}{2}+\epsilon}).$$

Since Hypothesis 1.1 is known for every pair (d, G) with $d \leq 5$ (see Corollary 1.3), we get the following unconditional result.

Corollary 1.11. *If $d \leq 5$, then* (1.9) *and* (1.10) *hold unconditionally. In particular, if $d \leq 5$, then the set*

$$\bigcup_{G \leq S_d} \{G\text{-}Weyl\ CM\ fields\ of\ degree\ 2d\}$$

of all Weyl CM fields of degree $2d$ comprises 100% of all CM fields of degree $2d$.

1.2. **The Colmez conjecture.** The Colmez conjecture [Col93] relates the Faltings height of a CM abelian variety to logarithmic derivatives of Artin L–functions at $s = 0$. In preparation for our applications, we briefly recall the statement of this conjecture.

Let F be a totally real number field of degree d. Let E/F be a CM extension of F and $\Phi \in \Phi(E)$ be a CM type for E. Let X_Φ be an abelian variety defined over $\overline{\mathbb{Q}}$ with complex multiplication by $\mathcal{O}_E$ and CM type Φ. We call X_Φ a CM abelian variety of type $(\mathcal{O}_E, \Phi)$.

Let $K \subseteq \overline{\mathbb{Q}}$ be a number field over which X_Φ has everywhere good reduction, and choose a Néron differential $\omega \in H^0(X_\Phi, \Omega^d_{X_\Phi})$. Then the *Faltings height* of X_Φ is defined by

$$h_{\mathrm{Fal}}(X_\Phi) := -\frac{1}{2[K:\mathbb{Q}]} \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log \left| \int_{X_\Phi^\sigma(\mathbb{C})} \omega^\sigma \wedge \overline{\omega^\sigma} \right|.$$

The Faltings height does not depend on the choice of K, ω, or X_Φ. In particular, the Faltings height depends only on the choice of CM type Φ, and hence is often denoted by $h_{\mathrm{Fal}}(\Phi)$.

Let $\mathbb{Q}^{\mathcal{CM}}$ be the compositum of all CM fields. Then $\mathbb{Q}^{\mathcal{CM}}/\mathbb{Q}$ is a Galois extension of infinite degree, and the Galois group $G^{\mathcal{CM}} := \mathrm{Gal}(\mathbb{Q}^{\mathcal{CM}}/\mathbb{Q})$ is a profinite group with the Krull topology. Let $H(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$ be the Hecke algebra of Schwartz functions on $G^{\mathcal{CM}}$ which take values in $\overline{\mathbb{Q}}$. This is the $\overline{\mathbb{Q}}$-algebra of locally constant, compactly supported functions $f : G^{\mathcal{CM}} \longrightarrow \overline{\mathbb{Q}}$ with multiplication of functions $f_1, f_2 \in H(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$ given by the convolution

$$(f_1 * f_2)(g) := \int_{G^{\mathcal{CM}}} f_1(h) f_2(h^{-1}g)\, d\mu(h).$$

Here μ is the left-invariant Haar measure on $G^{\mathcal{CM}}$, normalized so that

$$\mathrm{Vol}(G^{\mathcal{CM}}) = \int_{G^{\mathcal{CM}}} d\mu(g) = 1.$$

The Hecke algebra $H(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$ is an associative algebra with no identity element. For a function $f \in H(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$, the *reflex function* $f^\vee \in H(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$ is defined by $f^\vee(g) := \overline{f(g^{-1})}$. Define a Hermitian inner product on $H(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$ by

$$\langle f_1, f_2 \rangle := \int_{G^{\mathcal{CM}}} f_1(h) \overline{f_2(h)}\, d\mu(h).$$

Let $H^0(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$ be the $\overline{\mathbb{Q}}$-subalgebra of $H(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$ consisting of class functions, i.e., the subalgebra of functions $f \in H(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$ satisfying $f(hgh^{-1}) = f(g)$ for all $h, g \in G^{\mathcal{CM}}$. It is known that an orthonormal basis for $H^0(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$ is given by the set

$$\{\chi_\pi \mid \pi \text{ an irreducible representation of } G^{\mathcal{CM}}\}$$

of Artin characters χ_π associated to the irreducible representations π of $G^{\mathcal{CM}}$. There is a projection map

$$\begin{aligned} H(G^{\mathcal{CM}}, \overline{\mathbb{Q}}) &\longrightarrow H^0(G^{\mathcal{CM}}, \overline{\mathbb{Q}}) \\ f &\longmapsto f^0 \end{aligned}$$

defined by

$$f^0(g) := \int_{G^{\mathcal{CM}}} f(hgh^{-1})\, d\mu(h).$$

As a map of $\overline{\mathbb{Q}}$-vector spaces, it corresponds to the orthogonal projection of $H(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$ onto $H^0(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$. In particular, one has

$$f^0 = \sum_{\chi_\pi} \langle f, \chi_\pi \rangle \chi_\pi.$$

Define the functions

$$Z(f^0, s) := \sum_{\chi_\pi} \langle f, \chi_\pi \rangle \frac{L'(\chi_\pi, s)}{L(\chi_\pi, s)} \quad \text{and} \quad \mu_{\mathrm{Art}}(f^0) := \sum_{\chi_\pi} \langle f, \chi_\pi \rangle \log(\mathfrak{f}_{\chi_\pi}),$$

where $L(\chi_\pi, s)$ is the (incomplete) Artin L–function of χ_π and $\mathfrak{f}_{\chi_\pi}$ is the analytic Artin conductor of χ_π.

Now, a CM type $\Phi \in \Phi(E)$ induces a function $\Phi \in H^0(G^{\mathcal{CM}}, \overline{\mathbb{Q}})$ defined by

$$\Phi(g) := \chi_\Phi(g|_E), \quad g \in G^{\mathcal{CM}}$$

where χ_Φ is the characteristic function of the set Φ and $g|_E$ is the restriction of g to E. Then define the function

$$A_{E,\Phi} := [E : \mathbb{Q}](\Phi * \Phi^\vee).$$

Colmez [Col93] made the following conjecture which provides a striking link between Faltings heights of CM abelian varieties and derivatives of L–functions.

Conjecture 1.12 (The Colmez conjecture)**.** *Let E be a CM field and X_Φ be a CM abelian variety of type $(\mathcal{O}_E, \Phi)$. Then*

$$h_{\mathrm{Fal}}(X_\Phi) = -Z(A^0_{E,\Phi}, 0) - \frac{1}{2}\mu_{\mathrm{Art}}(A^0_{E,\Phi}).$$

1.3. **Applications to the Colmez conjecture.** In [BSM16], the first two authors used the recently proved averaged Colmez conjecture [AGHM15, YZ15], along with a combination of analytic and algebraic methods, to establish many new non-abelian cases of the Colmez conjecture. Building on this work, we will prove the following:

Theorem 1.13. *If E is a G-Weyl CM field, then the Colmez conjecture is true for E. In particular, if X is an abelian variety of dimension d with complex multiplication by a G-Weyl*

CM field E of degree $2d$ with maximal totally real subfield F, then the Faltings height of X is given by

$$h_{\mathrm{Fal}}(X) = -\frac{1}{2}\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} - \frac{1}{4}\log\left(\frac{|d_E|}{d_F}\right) - \frac{d}{2}\log(2\pi), \tag{1.11}$$

where $L(\chi_{E/F},s)$ is the L–function of the Hecke character $\chi_{E/F}$ associated to the quadratic extension E/F

Remark 1.14. We note that if E is a G-Weyl CM field of degree $2d \geq 4$, then $E/\mathbb{Q}$ is *non-abelian* (see Proposition 3.2).

The following results give infinitely many density-one families of non-abelian CM fields which satisfy the Colmez conjecture.

First, we have the following result, which is an immediate consequence of Theorems 1.13 and 1.2.

Theorem 1.15. *Assume that Hypothesis 1.1 is true for the pair (d,G). Then the Colmez conjecture is true for 100% of CM fields E of degree $2d$ which have a maximal totally real subfield F with Galois group $\mathrm{Gal}(F^c/\mathbb{Q}) \cong G$.*

Next, by combining Theorem 1.15 with Corollary 1.3, we get the following unconditional result.

Corollary 1.16. *If (d,G) is any of the pairs in Corollary 1.3, then the Colmez conjecture is true for 100% of CM fields E of degree $2d$ which have a maximal totally real subfield F with Galois group $\mathrm{Gal}(F^c/\mathbb{Q}) \cong G$.*

Similarly, the following is an immediate consequence of Theorems 1.13 and 1.9.

Theorem 1.17. *Assume that Hypothesis 1.1 is true for every pair (d,G) where G ranges over all transitive subgroups $G \leq S_d$. Then the Colmez conjecture is true for 100% of CM fields of degree $2d$.*

Finally, since Hypothesis 1.1 holds for every pair (d,G) with $d \leq 5$ (as observed previously), we get the following unconditional result.

Corollary 1.18. *If $d \leq 5$, then the Colmez conjecture is true for 100% of CM fields of degree $2d$.*

Remark 1.19. The Colmez conjecture is now known to be true for quartic CM fields, sextic CM fields, and many degree 10 CM fields (see e.g. [YY16]).

We conclude by briefly summarizing some results on the Colmez conjecture.

Colmez [Col93] proved his conjecture for abelian CM fields (up to an error term which was eliminated by Obus [Obu13]). Yang [Yan10a, Yan10b, Yan13] proved the Colmez conjecture for a large class of quartic CM fields, including the first *non-abelian* cases.

The averaged Colmez conjecture, which was recently proved independently by Andreatta-Goren-Howard-Madapusi Pera [AGHM15] and Yuan-Zhang [YZ15], opened the way to prove many new non-abelian cases of the Colmez conjecture. For example, the first two authors [BSM16] proved that if F is any totally real number field of degree $d \geq 3$, then there are infinitely many effectively constructible, positive density sets of CM extensions E/F such that $E/\mathbb{Q}$ is non-abelian and the Colmez conjecture is true for E. Yang and Yin [YY16] proved that if E is a CM field of the form $E = FK$ where $K = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic field, and $\mathrm{Gal}(F^c/\mathbb{Q})$ is isomorphic to either S_d or A_d, then the Colmez conjecture is true for E. This follows from a more refined result proved by these authors in [YY16], which shows that the Colmez conjecture is true for CM types Φ of $E = FK$ of signature $(d-1, 1)$.

2. Proof of Theorems 1.2 and 1.9

In this section we prove Theorems 1.2 and 1.9, following closely Klüners's Dirichlet series approach [Klu12].

Let $N_{2d}^{\mathrm{cm}}(X, G)$ count all CM fields E of degree $2d$ and discriminant $|d_E| \leq X$ which have a maximal totally real subfield F with $\mathrm{Gal}(F^c/\mathbb{Q}) \cong G$.

Let $N_{2d}^{\neg\mathrm{Weyl}}(X, G)$ count the subset of all CM fields counted by $N_{2d}^{\mathrm{cm}}(X, G)$ which are *not* of G-Weyl type.

We first establish asymptotics for $N_{2d}^{\mathrm{cm}}(X, G)$, and then give upper bounds for $N_{2d}^{\neg\mathrm{Weyl}}(X, G)$.

2.1. **Asymptotics for $N_{2d}^{\mathrm{cm}}(X, G)$.** We begin by establishing an asymptotic formula with a power-saving error term for $N_{2d}^{\mathrm{cm}}(X, G)$. The key is a theorem of Cohen, Diaz y Diaz, and Olivier [CDO02] which expresses the Dirichlet series enumerating all quadratic extensions of a number field as a linear combination of Hecke L–functions. We will use a version of their result which incorporates signature conditions.

Fix a totally real field F of degree d and define the Dirichlet series

$$D_{F,C_2}^{-}(s) := \sum_{[E:F]=2} \frac{1}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{D}_{E/F})^s} = \sum_{n=1}^{\infty} \frac{a^{-}(n)}{n^s}, \quad \mathrm{Re}(s) > 1$$

where the sum is over all totally imaginary quadratic extensions E/F, $\mathfrak{D}_{E/F}$ is the relative discriminant, and

$$a^{-}(n) := \#\{E/F \text{ totally imaginary quadratic, } \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{D}_{E/F}) = n\}.$$

The following is a special case of [CDO02, Theorem 3.11], applied to the totally real field F (which has signature $(d, 0)$) and with $\mathfrak{m}_\infty$ equal to the set of real places of F (which are all ramified in the imaginary quadratic extension E/F).

Theorem 2.1 ([CDO02])**.** *For* $\mathrm{Re}(s) > 1$ *we have*

$$D_{F,C_2}^{-}(s) = \frac{1}{\zeta_F(2s)} \sum_{\mathfrak{c}_\infty \subset \mathfrak{m}_\infty} \sum_{\mathfrak{c} | 2} \frac{(-1)^{|\mathfrak{c}_\infty|}}{2^{|\mathfrak{c}_\infty|}} \mathcal{N}_{F/\mathbb{Q}}(2/\mathfrak{c})^{1-2s} \sum_{\chi \in Q(\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F))} L_F(\chi, s),$$

where $\mathfrak{c}$ *runs over all integral ideals of* F *dividing* 2*,* $\mathfrak{c}_\infty$ *runs over all subsets of the set of real places* $\mathfrak{m}_\infty$ *of* F*,* χ *runs over all quadratic characters* $Q(\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F))$ *of the ray class group* $\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F)$ *modulo* $\mathfrak{c}^2\mathfrak{c}_\infty$*, and* $L_F(\chi, s)$ *is the* L*–function of* χ*.*

The following result establishes some important analytic properties of the Dirichlet series $D_{F,C_2}^{-}(s)$ (this is analogous to [Klu12, Theorem 5]).

Theorem 2.2. *Assume that* $0 \leq \delta_d, \delta' \leq 1/2$ *satisfy (1.3) and (1.7), respectively. Then the Dirichlet series* $D_{F,C_2}^{-}(s)$ *has a meromorphic continuation to* $\mathrm{Re}(s) > 1/2$ *with only a single* *(simple) pole at $s = 1$ with residue*

$$R_d(F) := \frac{\mathrm{Res}_{s=1}\,\zeta_F(s)}{2^d \zeta_F(2)} > 0.$$

Moreover, the function

$$g_F(s) := D^-_{F,C_2}(s) - \frac{R_d(F)}{s-1}$$

is analytic for $\sigma := \mathrm{Re}(s) > 1/2$ and satisfies the bound

$$g_F(\sigma + it) \ll_{\epsilon,d} d_F^{\delta'(1-\sigma)+\delta_d+\epsilon} \frac{(1+|t|)^{d\delta'(1-\sigma)+\epsilon}}{(\sigma - 1/2)^d}, \quad 1/2 < \sigma \leq 1 + \epsilon. \tag{2.1}$$

Proof. Let $\chi_{0,\mathfrak{c}} \in Q(\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F))$ be the trivial character and write

$$D^-_{F,C_2}(s) = A(s) + B(s), \tag{2.2}$$

where

$$A(s) := \frac{1}{\zeta_F(2s)} \sum_{\mathfrak{c}_\infty \subset \mathfrak{m}_\infty} \frac{(-1)^{|\mathfrak{c}_\infty|}}{2^{|\mathfrak{c}_\infty|}} \sum_{\mathfrak{c}|2} \mathcal{N}_{F/\mathbb{Q}}(2/\mathfrak{c})^{1-2s} L(\chi_{0,\mathfrak{c}}, s)$$

and

$$B(s) := \frac{1}{\zeta_F(2s)} \sum_{\mathfrak{c}_\infty \subset \mathfrak{m}_\infty} \sum_{\mathfrak{c}|2} \frac{(-1)^{|\mathfrak{c}_\infty|}}{2^{|\mathfrak{c}_\infty|}} \mathcal{N}_{F/\mathbb{Q}}(2/\mathfrak{c})^{1-2s} \sum_{\substack{\chi \in Q(\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F)) \\ \chi \neq \chi_{0,\mathfrak{c}}}} L_F(\chi, s).$$

The L–function

$$L_F(\chi_{0,\mathfrak{c}}, s) = \zeta_F(s) \prod_{\mathfrak{p}|\mathfrak{c}} \left(1 - \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{p})^{-s}\right)$$

extends to a meromorphic function on $\mathbb{C}$ with only a single (simple) pole at $s = 1$, and if $\chi \neq \chi_{0,\mathfrak{c}}$ the L–function $L_F(\chi, s)$ extends to an analytic function on $\mathbb{C}$. Hence, by (2.2) the Dirichlet series $D^-_{F,C_2}(s)$ extends to a meromorphic function on $\sigma > 1/2$ with only a single (simple) pole at $s = 1$ with residue

$$R_d(F) = \mathrm{Res}_{s=1} A(s) = S \cdot \mathrm{Res}_{s=1} \zeta_F(s),$$

where

$$S := \frac{1}{\zeta_F(2)} \sum_{\mathfrak{c}_\infty \subset \mathfrak{m}_\infty} \frac{(-1)^{|\mathfrak{c}_\infty|}}{2^{|\mathfrak{c}_\infty|}} \sum_{\mathfrak{c}|2} \mathcal{N}_{F/\mathbb{Q}}(2/\mathfrak{c})^{-1} \prod_{\mathfrak{p}|\mathfrak{c}} \left(1 - \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{p})^{-1}\right).$$

As in [CDO02, Sections 3.3 and 3.4], we may compute that

$$\sum_{\mathfrak{c}|2} \mathcal{N}_{F/\mathbb{Q}}(2/\mathfrak{c})^{-1} \prod_{\mathfrak{p}|\mathfrak{c}} \left(1 - \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{p})^{-1}\right) = 1$$

and

$$\sum_{\mathfrak{c}_\infty \subset \mathfrak{m}_\infty} \frac{(-1)^{|\mathfrak{c}_\infty|}}{2^{|\mathfrak{c}_\infty|}} = \frac{1}{2^{|\mathfrak{m}_\infty|}} = \frac{1}{2^d}.$$

Therefore, we have $S = 2^{-d}$ and

$$R_d(F) = \frac{\mathrm{Res}_{s=1}\zeta_F(s)}{2^d\zeta_F(2)}.$$

By the preceding facts, the function $g_F(s) := D^-_{F,C_2}(s) - R_d(F)/(s-1)$ is analytic for $\sigma > 1/2$. Hence, it remains to establish the bound (2.1).

By (1.7), we have the bound

$$\begin{aligned}(s-1)\zeta_F(s) &\ll_{\epsilon,d} \mathfrak{q}(F,\chi_0,s)^{\delta'(1-\sigma)+\epsilon}(|s|+1)\\ &\ll_{\epsilon,d} (d_F(|s|+4)^d)^{\delta'(1-\sigma)+\epsilon}(|s|+1), \quad 1/2 < \sigma \leq 1+\epsilon,\end{aligned}$$

and we also have the bound

$$\frac{1}{\zeta_F(2s)} \ll \frac{1}{(\sigma-1/2)^d}, \quad 1/2 < \sigma \leq 1+\epsilon$$

(the implied constant is uniform in F). These bounds yield the estimate

$$(s-1)A(s) - R_d(F) \ll_{\epsilon,d} (\sigma-1/2)^{-d}(d_F(|s|+4)^d)^{\delta'(1-\sigma)+\epsilon}(|s|+1), \quad 1/2 < \sigma \leq 1+\epsilon,$$

and thus with $f(s) := A(s) - R_d(F)/(s-1)$ the estimate

$$f(\sigma+it) \ll_{\epsilon,d} (\sigma-1/2)^{-d} d_F^{\delta'(1-\sigma)+\epsilon}(1+|t|)^{d(\delta'(1-\sigma)+\epsilon)}, \quad 1/2 < \sigma \leq 1+\epsilon.$$

Next observe that if the bound (1.7) holds for some $0 \leq \delta' \leq 1/2$, then it also holds (with the same δ') if χ is imprimitive, since the L–functions of an imprimitive and primitive character differ by a finite Euler product, uniformly bounded above and below by $O(1)$ in the strip $1/2 < \sigma \leq 1+\epsilon$. Therefore, for $\chi \in Q(\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F))$ with $\chi \neq \chi_{\mathfrak{c},0}$, we have

$$\begin{aligned}L_F(\chi,s) &\ll_{\epsilon,d} \mathfrak{q}(F,\chi,s)^{\delta'(1-\sigma)+2\epsilon}\\ &\ll_{\epsilon,d} (d_F(|s|+4)^d)^{\delta'(1-\sigma)+2\epsilon}, \quad 1/2 < \sigma \leq 1+\epsilon\end{aligned}$$

with $\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{c}^2) = O(1)$ for all allowable $\mathfrak{c}$. Also, from (1.3) we have the bound

$$|Q(\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F))| = |\,\mathrm{Cl}_{\mathfrak{c}^2\mathfrak{c}_\infty}(F)[2]| \ll_d |\,\mathrm{Cl}(F)[2]| \ll_{\epsilon,d} d_F^{\delta_d+\epsilon}.$$

Then arguing as above we get

$$B(\sigma+it) \ll_{\epsilon,d} (\sigma-1/2)^{-d} d_F^{\delta'(1-\sigma)+\epsilon}(1+|t|)^{d(\delta'(1-\sigma)+\epsilon)} d_F^{\delta_d+\epsilon}, \quad 1/2 < \sigma \leq 1+\epsilon.$$

Finally, since $g_F(s) = f(s) + B(s)$, we have

$$g_F(\sigma+it) \ll_{\epsilon,d} d_F^{\delta'(1-\sigma)+\delta_d+2\epsilon}\frac{(1+|t|)^{d(\delta'(1-\sigma)+\epsilon)}}{(\sigma-1/2)^d}, \quad 1/2 < \sigma \leq 1+\epsilon.$$

□

Now, given a pair (d,G), let

$$\mathcal{F}_G^+ := \{F/\mathbb{Q} : \ F \text{ totally real of degree } d, \ \mathrm{Gal}(F^c/\mathbb{Q}) \cong G\}.$$

Define the Dirichlet series

$$D_G^{\mathrm{cm}}(s) := \sum_{F\in\mathcal{F}_G^+}\sum_{[E:F]=2}\frac{1}{|d_E|^s} = \sum_{n=1}^{\infty}\frac{a(n)}{n^s},$$

where the inner sum is over all totally imaginary quadratic extensions E/F and

$$a(n) := \#\{(E,F) :\ F \in \mathcal{F}_G^+,\ E/F \text{ totally imaginary quadratic},\ |d_E| = n\}.$$

Clearly, the Dirichlet series $D_G^{\mathrm{cm}}(s)$ enumerates all CM fields counted by $N_{2d}^{\mathrm{cm}}(X,G)$. Using the relation

$$|d_E| = d_F^2 \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{D}_{E/F}),$$

we have

$$D_G^{\mathrm{cm}}(s) = \sum_{F\in\mathcal{F}_G^+} \frac{1}{d_F^{2s}} \sum_{[E:F]=2} \frac{1}{\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{D}_{E/F})^s} = \sum_{F\in\mathcal{F}_G^+} \frac{D_{F,C_2}^-(s)}{d_F^{2s}}. \tag{2.3}$$

Theorem 2.3. *Assume that Hypothesis 1.1 is true for (d,G) and δ' satisfies (1.7). Then the Dirichlet series $D_G^{\mathrm{cm}}(s)$ has a meromorphic continuation to the half-plane $\mathrm{Re}(s) > \alpha$ where*

$$\alpha = \alpha(\delta_d, M(G), \delta') := \max\left\{\frac{\delta_d + \delta' + M(G)}{\delta' + 2}, \frac{M(G)}{2}\right\} < 1, \tag{2.4}$$

with only a single (simple) pole at $s = 1$ given by the convergent series

$$r_d(G) := \sum_{F\in\mathcal{F}_G^+} \frac{R_d(F)}{d_F^2} = \sum_{F\in\mathcal{F}_G^+} \frac{\mathrm{Res}_{s=1}\zeta_F(s)}{2^d d_F^2 \zeta_F(2)} > 0.$$

Moreover, for $\sigma := \mathrm{Re}(s) \in (\alpha, 1+\epsilon]$ and $|t| > 1$, the Dirichlet series $D_G^{\mathrm{cm}}(s)$ satisfies the bound

$$D_G^{\mathrm{cm}}(\sigma + it) \ll_{\epsilon,d,G} \frac{(1+|t|)^{d\delta'(1-\sigma)+\epsilon}}{(\sigma - 1/2)^d}. \tag{2.5}$$

Proof. Write

$$D_G^{\mathrm{cm}}(s) = g(s) + \frac{1}{s-1} h(s), \tag{2.6}$$

where

$$g(s) := \sum_{F\in\mathcal{F}_G^+} \frac{g_F(s)}{d_F^{2s}} \quad \text{and} \quad h(s) := \sum_{F\in\mathcal{F}_G^+} \frac{R_d(F)}{d_F^{2s}}.$$

Using the estimate (2.1), we have

$$g(s) \ll_{\epsilon,d} \frac{(1+|t|)^{d\delta'(1-\sigma)+\epsilon}}{(\sigma-1/2)^d} \sum_{F\in\mathcal{F}_G^+} d_F^{\delta'(1-\sigma)+\delta_d-2\sigma+\epsilon}, \quad 1/2 < \sigma \le 1+\epsilon.$$

Hence, the absolute convergence of the series $g(s)$ is guaranteed by the convergence of the series

$$\sum_{F\in\mathcal{F}_G^+} d_F^{\delta'(1-\sigma)+\delta_d-2\sigma+\epsilon}. \tag{2.7}$$

Divide the sum over F into intervals $N < d_F \le 2N$ and let N range over the integer powers of 2. Then using the estimate (1.4), we see that (2.7) converges whenever the series

$$\sum_N N^{\delta'(1-\sigma)+\delta_d-2\sigma+M(G)+\epsilon} \tag{2.8}$$

converges. The series (2.8) converges whenever the exponent is negative, i.e., whenever $\sigma > \alpha_1$ with

$$\alpha_1 := \frac{\delta_d + \delta' + M(G)}{\delta' + 2}$$

(for an appropriate choice of $\epsilon > 0$). The condition $\alpha_1 < 1$ is equivalent to the condition $\delta_d + M(G) < 2$. Therefore, we see that $g(s)$ is analytic for $\sigma > \alpha_1$ with $\alpha_1 < 1$, and that $g(s)$ satisfies the bound

$$g(\sigma + it) \ll_{\epsilon,d,G} \frac{(1+|t|)^{d\delta'(1-\sigma)+\epsilon}}{(\sigma - 1/2)^d}, \quad \alpha_1 < \sigma \leq 1 + \epsilon. \tag{2.9}$$

Next, using the estimate $R_d(F) \ll_{\epsilon,d} d_F^{\epsilon}$ we have

$$h(s) \ll_{\epsilon,d} \sum_{F \in \mathcal{F}_G^+} d_F^{-2\sigma+\epsilon}.$$

Then a similar argument using the estimate (1.4) shows that the series $h(s)$ converges for $\sigma > \alpha_2$ with $\alpha_2 := M(G)/2$. The condition $\alpha_2 < 1$ is ensured by $\delta_d + M(G) < 2$. Therefore, we see that $(s-1)^{-1}h(s)$ is meromorphic for $\sigma > \alpha_2$ with $\alpha_2 < 1$ with only a single (simple) pole at $s = 1$ with residue

$$r_d(G) := \sum_{F \in \mathcal{F}_G^+} \frac{R_d(F)}{d_F^2},$$

and that $(s-1)^{-1}h(s)$ satisfies the bound

$$\frac{1}{s-1} h(s) \ll_{\epsilon,d,G} 1, \quad \sigma > \alpha_2, \quad |t| > 1. \tag{2.10}$$

Finally, from (2.6) we conclude that $D_G^{\mathrm{cm}}(s)$ has a meromorphic continuation to $\sigma > \alpha := \max\{\alpha_1, \alpha_2\}$ with $\alpha < 1$ with only a single (simple) pole at $s = 1$ with residue $r_d(G)$. Moreover, from the estimates (2.9) and (2.10) we see that $D_G^{\mathrm{cm}}(s)$ satisfies the bound

$$D_G^{\mathrm{cm}}(\sigma + it) \ll_{\epsilon,d,G} \frac{(1+|t|)^{d\delta'(1-\sigma)+\epsilon}}{(\sigma - 1/2)^d}, \quad \alpha < \sigma \leq 1 + \epsilon, \quad |t| > 1.$$

□

Theorem 2.4. (*i*) *Under the assumptions of Theorem 2.3, we have*

$$N_{2d}^{\mathrm{cm}}(X, G) = r_d(G)X + O_{d,\epsilon}(X^{\beta(\delta_d, M(G), \delta')+\epsilon})$$

where

$$\beta(\delta_d, M(G), \delta') := 1 - \frac{1-\alpha}{1 + d\delta'(1-\alpha)}$$

with $\alpha = \alpha(\delta_d, M(G), \delta') < 1$ *defined by (2.4).*

(*ii*) *If Hypothesis 1.1 is true for every pair* (d, G) *where* G *ranges over all transitive subgroups* $G \leq S_d$*, then*

$$N_{2d}^{\mathrm{cm}}(X) = \left(\sum_{G \leq S_d} r_d(G) \right) X + O_{d,\epsilon}(X^{\beta(\delta_d, \delta')+\epsilon})$$

where

$$\beta(\delta_d, \delta') := \max_{G \leq S_d} \beta(\delta_d, M(G), \delta') < 1. \tag{2.11}$$

Proof. Fix a smooth function $\phi : [0,1] \to [0,1]$ with $\phi(0) = 1$ and $\phi(1) = 0$. Then, for each $Y > 1$, define

$$\phi_Y(t) := \begin{cases} 1, & \text{if } t \in [0,1]; \\ \phi(Y(t-1)), & \text{if } t \in [1, 1+Y^{-1}]; \\ 0 & \text{if } t \geq 1 + Y^{-1}. \end{cases}$$

Let

$$\widehat{\phi_Y}(s) := \int_0^\infty \phi_Y(t) t^{s-1} dt, \quad \mathrm{Re}(s) > 0$$

be the Mellin transform of ϕ_Y. Integrating by parts $A \geq 1$ times yields the estimate

$$\widehat{\phi_Y}(s) \ll Y^{-1} \left(\frac{Y}{1+|t|} \right)^A, \tag{2.12}$$

valid for all s in any fixed vertical strip $\sigma_0 \leq \mathrm{Re}(s) \leq \sigma_1$ with $\sigma_0 > 0$, and also valid for all real numbers $A \geq 1$ by interpolation.

By construction, and then by Mellin inversion, we have

$$N_{2d}^{\mathrm{cm}}(X, G) = \sum_{n=1}^{X} a(n) \leq \sum_{n=1}^{\infty} a(n) \phi_Y\left(\frac{n}{X}\right) = \frac{1}{2\pi i} \int_{(1+\epsilon)} D_G^{\mathrm{cm}}(s) \widehat{\phi_Y}(s) X^s ds.$$

From the estimate (2.5), we see that

$$D_G^{\mathrm{cm}}(s) \ll (1+|t|)^{d\delta'(1-\sigma)+\epsilon} \tag{2.13}$$

in any vertical strip $1/2 < \alpha + \eta < \sigma \leq 1 + \epsilon$, $|t| > 1$, where the implied constant depends on ϵ, d, F, and η. Then using the estimates (2.12) and (2.13), we may shift the contour to $\mathrm{Re}(s) = \alpha'$ with $\alpha < \alpha' < 1$ to get

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} D_G^{\mathrm{cm}}(s) \widehat{\phi_Y}(s) X^s ds = \widehat{\phi_Y}(1) r_d(G) X + \frac{1}{2\pi i} \int_{(\alpha')} D_G^{\mathrm{cm}}(s) \widehat{\phi_Y}(s) X^s ds.$$

For any $A \geq 1$ we have the estimate

$$\frac{1}{2\pi i} \int_{(\alpha')} D_G^{\mathrm{cm}}(s) \widehat{\phi_Y}(s) X^s ds \ll X^{\alpha'} Y^{-1} \int_{\mathbb{R}} (1+|t|)^{d\delta'(1-\alpha')+\epsilon} \left(\frac{Y}{1+|t|} \right)^A dt.$$

Choose $A = d\delta'(1-\alpha') + 1 + 2\epsilon$. Then

$$\frac{1}{2\pi i} \int_{(\alpha+\varepsilon)} D_G^{\mathrm{cm}}(s) \widehat{\phi_Y}(s) X^s ds \ll X^{\alpha'} Y^{d\delta'(1-\alpha')+2\epsilon}.$$

Since $\widehat{\phi_Y}(1) = 1 + O(Y^{-1})$, we have

$$\widehat{\phi_Y}(1) r_d(G) X = r_d(G) X + O(XY^{-1}).$$

Then putting things together, and replacing 2ϵ by ϵ, we get

$$N_{2d}^{\mathrm{cm}}(X, G) \leq \sum_{n=1}^{\infty} a(n) \phi_Y\left(\frac{n}{X}\right) = r_d(G) X + O(XY^{-1}) + O(X^{\alpha'} Y^{d\delta'(1-\alpha')+\epsilon}). \tag{2.14}$$

Similarly, we have

$$N_{2d}^{\mathrm{cm}}(X,G) \geq \sum_{n=1}^{\infty} a(n)\phi_Y\Big(\frac{n}{X-XY^{-1}}\Big),$$

for which the same estimate in (2.14) also holds (since we may interchange X and $X-XY^{-1}$ in (2.14), within the error terms given there), so that in fact we have

$$N_{2d}^{\mathrm{cm}}(X,G) = r_d(G)X + O(XY^{-1}) + O(X^{\alpha'}Y^{d\delta'(1-\alpha')+\epsilon}).$$

We optimize (apart from epsilon factors) by choosing $\alpha' = \alpha+\epsilon$ and $Y = X^{\frac{1-\alpha}{1+d\delta'(1-\alpha)}}$, so as to obtain for each $\epsilon > 0$ that

$$N_{2d}^{\mathrm{cm}}(X,G) = r_d(G)X + O_\epsilon(X^{\beta(\delta_d,M(G),\delta')+\epsilon}),$$

where

$$\beta(\delta_d, M(G), \delta') := 1 - \frac{1-\alpha}{1+d\delta'(1-\alpha)}.$$

This proves part (i). Part (ii) follows by summing the asymptotic formula in (i) over all transitive subgroups $G \leq S_d$. □

2.2. **Upper bounds for $N_{2d}^{\neg\mathrm{Weyl}}(X,G)$.** Let K be a number field of degree d with $\mathrm{Gal}(K^c/\mathbb{Q}) \cong G$, and let L be a quadratic extension of K. Then $\mathrm{Gal}(L^c/\mathbb{Q})$ embeds as a subgroup of the wreath product $C_2 \wr G$ (see Proposition 3.1). Clearly, we have

$$N_{2d}^{\neg\mathrm{Weyl}}(X,G) \ll Y(X,G),$$

where

$$Y(X,G) := \#\{L/K:\ \mathrm{Gal}(L^c/\mathbb{Q}) \ncong C_2 \wr G,\ \mathrm{Gal}(K^c/\mathbb{Q}) \cong G,\ [L:K]=2, |d_L| \leq X\}. \tag{2.15}$$

Therefore, it suffices to give an upper bound for $Y(X,G)$.

The extensions counted by $Y(X,G)$ are distinguished by the following fact: for each prime p unramified in $K/\mathbb{Q}$ but ramified in L/K (so that $p \mid d_L$), we must in fact have $p^2 \mid d_L$ (see [Klu12, Lemma 4]).

Let

$$\mathcal{K}_G(X^{1/2}) := \{K/\mathbb{Q}:\ \mathrm{Gal}(K^c/\mathbb{Q}) \cong G,\ |d_K| \leq X^{1/2}\}.$$

As in [Klu12, p. 9-10], we have the bound

$$Y(X,G) \leq \sum_{K\in\mathcal{K}_G(X^{1/2})} O_{\epsilon,d}\left(\frac{X^{\frac{1}{2}+\epsilon}}{|d_K|^2}|\mathrm{Cl}(K)[2]|\right). \tag{2.16}$$

We briefly recall the proof. Each L counted in (2.15) satisfies

$$d_L = d_K^2\mathcal{N}_{K/\mathbb{Q}}(\mathfrak{D}_{L/K})$$

with $\mathcal{N}_{K/\mathbb{Q}}(\mathfrak{D}_{L/K}) = ab^2$, where a is only divisible by primes dividing d_K. Since each such prime can only divide a with bounded multiplicity, the problem is reduced to proving (for each positive integer n) that the number of quadratic extensions L/K with $\mathcal{N}_{K/\mathbb{Q}}(\mathfrak{D}_{L/K}) = n$ is $O_{d,\epsilon}(|\operatorname{Cl}(K)[2]|n^\epsilon)$, and this is done by bounding the 2-torsion in the relevant ray class group.

Continuing then, applying the bound (1.3) to (2.16) gives

$$N_{2d}^{\neg\,\mathrm{Weyl}}(X,G) \ll_{\epsilon,d} X^{\frac{1}{2}+\epsilon} \sum_{K\in\mathcal{K}_G(X^{1/2})} |d_K|^{-1+\delta_d}.$$

Again, divide the sum over K into intervals with $N < |d_K| \leq 2N$ and let N range over the integer powers of 2. Then applying the estimate (1.4) gives

$$N_{2d}^{\neg\,\mathrm{Weyl}}(X,G) \ll_{\epsilon,d} X^{\frac{1}{2}+\epsilon} \sum_{N} N^{-1+\delta_d+M(G)}.$$

If $\delta_d + M(G) \leq 1$ then

$$N_{2d}^{\neg\,\mathrm{Weyl}}(X,G) \ll_{\epsilon,d} X^{\frac{1}{2}+\epsilon}, \tag{2.17}$$

while if $\delta_d + M(G) > 1$ then

$$N_{2d}^{\neg\,\mathrm{Weyl}}(X,G) \ll_{\epsilon,d} X^{\frac{1}{2}+\frac{-1+\delta_d+M(G)}{2}+\epsilon}. \tag{2.18}$$

The exponent in (2.18) is less than 1 (for an appropriate choice of $\epsilon > 0$) provided that $\delta_d + M(G) < 2$.

2.3. **Proof of Theorem 1.2.** Using Theorem 2.4 and estimates (2.17) and (2.18), we have

$$\frac{N_{2d}^{\mathrm{Weyl}}(X,G)}{N_{2d}^{\mathrm{cm}}(X,G)} = \frac{N_{2d}^{\mathrm{cm}}(X,G) - N_{2d}^{\neg\,\mathrm{Weyl}}(X,G)}{N_{2d}^{\mathrm{cm}}(X,G)} = 1 + O_{d,G,\epsilon}(X^{-C_1(\delta_d,M(G))+\epsilon}),$$

where

$$C_1(\delta_d, M(G)) := \begin{cases} 1/2, & \text{if } \delta_d + M(G) \leq 1 \\ 1 - \dfrac{\delta_d + M(G)}{2}, & \text{if } 1 < \delta_d + M(G) < 2. \end{cases}$$

This proves Theorem 1.2.

2.4. **Proof of Theorem 1.9.** As above we have

$$\frac{N_{2d}^{\mathrm{Weyl}}(X,G)}{N_{2d}^{\mathrm{cm}}(X)} = \frac{N_{2d}^{\mathrm{cm}}(X,G) - N_{2d}^{\neg\,\mathrm{Weyl}}(X,G)}{N_{2d}^{\mathrm{cm}}(X)} = \frac{N_{2d}^{\mathrm{cm}}(X,G)}{N_{2d}^{\mathrm{cm}}(X)} + O_{d,G,\epsilon}(X^{-C_1(\delta_d,M(G))+\epsilon}).$$

Also by Theorem 2.4 we have

$$\frac{N_{2d}^{\mathrm{cm}}(X,G)}{N_{2d}^{\mathrm{cm}}(X)} = \frac{r_d(G)}{\sum_{G\leq S_d} r_d(G)} + O_{d,\epsilon}(X^{-1+\beta(\delta_d,\delta')})$$

so that

$$\frac{N_{2d}^{\mathrm{Weyl}}(X,G)}{N_{2d}^{\mathrm{cm}}(X)} = \frac{r_d(G)}{\sum_{G\leq S_d} r_d(G)} + O_{d,G,\epsilon}(X^{-C_2(\delta_d,M(G),\delta')+\epsilon}), \tag{2.19}$$

where

$$C_2(\delta_d, M(G), \delta') := \min\{C_1(\delta_d, M(G)), 1 - \beta(\delta_d, \delta')\} > 0. \tag{2.20}$$

This proves (1.9). To prove (1.10), we sum over all $G \leq S_d$ in (2.19) to get

$$\frac{N_{2d}^{\mathrm{Weyl}}(X)}{N_{2d}^{\mathrm{cm}}(X)} = 1 + O_{d,\epsilon}(X^{-C_3(\delta_d,\delta')+\epsilon}),$$

where

$$C_3(\delta_d, \delta') := \min_{G\leq S_d} C_2(\delta_d, M(G), \delta') > 0. \tag{2.21}$$

This proves Theorem 1.9. ☐

3. Proof of Theorem 1.13

In this section we review some basic facts about wreath products of groups, discuss the structure of Galois groups of CM fields, and prove Theorem 1.13.

3.1. **Wreath products.** We begin by reviewing some basic facts about wreath products of groups (see e.g. [DM96]). Let H and K be groups, and suppose that $\theta : H \to \mathrm{Aut}(K)$ is a homomorphism, where we write $\theta(h) = \theta_h$. This gives a (left) group action of H on K defined by $(h, k) \mapsto \theta_h(k)$. Recall that the semidirect product of K and H with respect to θ is the group

$$K \rtimes_\theta H := \{(k, h) \mid k \in K, h \in H\},$$

where the group operation is defined by

$$(k_1, h_1)(k_2, h_2) := (k_1\theta_{h_1}(k_2), h_1h_2).$$

When understood, we suppress θ in our notation for the semidirect product.

Now, let Ω be an arbitrary set, and let K^Ω denote the set of all functions $f : \Omega \to K$. Pointwise multiplication of functions gives K^Ω the structure of a group. A (left) group action of H on Ω gives a homomorphism

$$\begin{aligned} \theta : H &\longrightarrow \mathrm{Aut}(K^\Omega) \\ h &\longmapsto \theta_h \end{aligned}$$

defined by $\theta_h(f)(\omega) := f(h^{-1} \cdot \omega)$ for every $\omega \in \Omega$ and every $f \in K^\Omega$. In turn, this gives a (left) group action of H on K^Ω defined by $(h, f) \mapsto \theta_h(f)$. The *wreath product* of K and H with respect to θ is defined by

$$K \wr_\Omega H := K^\Omega \rtimes_\theta H.$$

When the set $\Omega = \{\omega_1, \ldots \omega_n\}$ is finite, it is customary to identify K^Ω with the direct product K^n via the isomorphism $f \mapsto (f(\omega_1), \ldots, f(\omega_n))$. In particular, if $\Omega = \{1, \ldots, n\}$ and $H \leq S_n$ is a group of permutations, then we have a (left) group action of H on Ω in the usual way, and the corresponding action of H on K^n is by permutation of the components, i.e., if $\sigma \in H \leq S_n$ and $x = (x_1, \ldots, x_n) \in K^n$, then

$$\sigma \circ x := (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}),$$

and in this case we write $K \wr H$ instead of $K \wr_{\{1,\ldots,n\}} H$.

With this notation, if $G \leq S_d$ is a transitive subgroup, then the wreath product $C_2 \wr G$ from the introduction is given by

$$C_2 \wr G = C_2 \wr_{\{1,\ldots,d\}} G = C_2^d \rtimes G.$$

The wreath product determines a short exact sequence

$$1 \longrightarrow C_2^d \longrightarrow C_2 \wr G \longrightarrow G \longrightarrow 1.$$

3.2. **Galois groups of CM fields.** We next discuss the structure of Galois groups of CM fields.

Proposition 3.1. *Let K be a number field of degree d with* $\mathrm{Gal}(K^c/\mathbb{Q}) \cong G \leq S_d$*, and let L be a quadratic extension of K. Then* $\mathrm{Gal}(L^c/\mathbb{Q})$ *embeds as a subgroup of the wreath product $C_2 \wr G$.*

Proof. Choose a primitive element α_1 with $K = \mathbb{Q}(\alpha_1)$ and let $\alpha_1, \ldots, \alpha_d$ be its conjugates, so that $K^c = \mathbb{Q}(\alpha_1, \ldots, \alpha_d)$ and $L^c = \mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_d})$. For each $g \in \mathrm{Gal}(L^c/\mathbb{Q})$ and $i \in \{1, \ldots, d\}$, we have

$$g(\alpha_i) = \alpha_j, \quad g(\sqrt{\alpha_i}) = \pm\sqrt{\alpha_j} \tag{3.1}$$

for some $j \in \{1, \ldots, d\}$ and choice of sign $\pm$. We define a function

$$\begin{aligned} \phi : \mathrm{Gal}(L^c/\mathbb{Q}) &\longrightarrow C_2 \wr G = \{\pm 1\}^d \rtimes G \\ g &\longmapsto (x_g, \sigma_g), \end{aligned}$$

where (matching (3.1)) $\sigma_g(i) = j$ and $x_g := (x_{g,1}, \ldots, x_{g,d}) \in \{\pm 1\}^d$ is the vector whose j-th component is given by

$$x_{g,j} := \frac{g(\sqrt{\alpha_i})}{\sqrt{\alpha_j}} = \frac{g\big(\sqrt{\alpha_{\sigma_g^{-1}(j)}}\big)}{\sqrt{\alpha_j}}.$$

In particular, we have

$$g(\sqrt{\alpha_i}) = x_{g,\sigma_g(i)}\sqrt{\alpha_{\sigma_g(i)}}$$

for $i \in \{1, \ldots, d\}$.

The data of x_g and σ_g determines $g(\sqrt{\alpha_i})$ for each i, and hence it determines g, so that ϕ is injective.

We next prove that ϕ is a homomorphism. Let $g, h \in \mathrm{Gal}(L^c/\mathbb{Q})$. Then by definition of the wreath product, we have

$$\phi(gh) = \phi(g)\phi(h) \quad \Longleftrightarrow \quad (x_{gh}, \sigma_{gh}) = (x_g(\sigma_g \circ x_h), \sigma_g\sigma_h)$$

where

$$\sigma_g \circ x_h := (x_{h,\sigma_g^{-1}(1)}, \ldots, x_{h,\sigma_g^{-1}(d)}).$$

By the isomorphism $\mathrm{Gal}(K^c/\mathbb{Q}) \cong G$, we have $\sigma_{gh} = \sigma_g\sigma_h$. Thus, it remains to prove that

$$x_{gh} = x_g(\sigma_g \circ x_h). \tag{3.2}$$

Since the $\sigma_{gh}(i)$-th component of $\sigma_g \circ x_h$ is given by

$$x_{h,\sigma_g^{-1}(\sigma_{gh}(i))} = x_{h,\sigma_h(i)},$$

we see that (3.2) is equivalent to

$$x_{gh,\sigma_{gh}(i)} = x_{g,\sigma_{gh}(i)}x_{h,\sigma_h(i)}$$

for $i \in \{1, \ldots, d\}$. We compute

$$\begin{aligned}(gh)(\sqrt{\alpha_i}) &= g(h(\sqrt{\alpha_i}))\\ &= g(x_{h,\sigma_h(i)}\sqrt{\alpha_{\sigma_h(i)}})\\ &= x_{h,\sigma_h(i)}g(\sqrt{\alpha_{\sigma_h(i)}})\\ &= x_{h,\sigma_h(i)}x_{g,\sigma_g(\sigma_h(i))}\sqrt{\alpha_{\sigma_g(\sigma_h(i))}}\\ &= x_{h,\sigma_h(i)}x_{g,\sigma_{gh}(i)}\sqrt{\alpha_{\sigma_{gh}(i)}},\end{aligned}$$

and thus

$$x_{gh,\sigma_{gh}(i)} := \frac{(gh)(\sqrt{\alpha_i})}{\sqrt{\alpha_{\sigma_{gh}(i)}}} = x_{h,\sigma_h(i)}x_{g,\sigma_{gh}(i)}.$$

This completes the proof. □

Proposition 3.2. *Let $d \geq 2$ and suppose that G is a transitive subgroup of S_d. Then the wreath product $C_2 \wr G$ is non-abelian. In particular, if E is a G-Weyl CM field of degree $2d \geq 4$, then $E/\mathbb{Q}$ is non-abelian.*

Proof. As we have seen, the elements of the wreath product $C_2 \wr G$ take the form (x, σ) with $x = (x_1, \ldots, x_d) \in C_2^d$ and $\sigma \in G$ a permutation of the set $\{1, \ldots, d\}$, with multiplication given by

$$(x, \sigma)(y, \tau) = (x(\sigma \circ y), \sigma\tau)$$

where

$$\sigma \circ y := (y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(d)}).$$

It now suffices to exhibit two elements which do not commute; for example, choose $x = y = (-1, 1, \ldots, 1)$, $\tau = \mathrm{id}$, and any σ such that $\sigma^{-1}(-1) = d$ (the existence of which is ensured by the transitivity of G).

Finally, if E is a G-Weyl CM field of degree $2d \geq 4$, then we have $\mathrm{Gal}(E^c/\mathbb{Q}) \cong C_2 \wr G$, so that $E/\mathbb{Q}$ is non-abelian. □

Now, let E be a CM field of degree $2d$ with maximal totally real subfield F, and let G be the transitive subgroup of S_d with $\mathrm{Gal}(F^c/\mathbb{Q}) \cong G$. Choose a primitive element α_1 with $F = \mathbb{Q}(\alpha_1)$ and let $\alpha_1, \ldots, \alpha_d$ be its conjugates, so that $F^c = \mathbb{Q}(\alpha_1, \ldots, \alpha_d)$ and $E^c = \mathbb{Q}(\sqrt{-\alpha_1}, \ldots, \sqrt{-\alpha_d})$. For each $g \in \mathrm{Gal}(E^c/F^c)$ and $i \in \{1, \ldots, n\}$, we have

$$g(\alpha_i) = \alpha_i, \quad g(\sqrt{-\alpha_i}) = \pm\sqrt{-\alpha_i}$$

for some choice of sign $\pm$. We define a function

$$\begin{aligned}\psi : \mathrm{Gal}(E^c/F^c) &\longrightarrow \{\pm 1\}^d\\ g &\longmapsto y_g,\end{aligned}$$

where $y_g := (y_{g,1}, \ldots, y_{g,d}) \in \{\pm 1\}^d$ is the vector whose i-th component is given by

$$y_{g,i} := \frac{g(\sqrt{-\alpha_i})}{\sqrt{-\alpha_i}}.$$

Then arguing as in Proposition 3.1, we see that ψ is an injective homomorphism.

By Galois theory, we have the short exact sequence

$$1 \longrightarrow \mathrm{Gal}(E^c/F^c) \longrightarrow \mathrm{Gal}(E^c/\mathbb{Q}) \longrightarrow \mathrm{Gal}(F^c/\mathbb{Q}) \longrightarrow 1. \tag{3.3}$$

The injective homomorphism ψ implies that $\mathrm{Gal}(E^c/F^c) \cong C_2^v$ for some $1 \le v \le d$. Then from (3.3) we get a short exact sequence

$$1 \longrightarrow C_2^v \longrightarrow \mathrm{Gal}(E^c/\mathbb{Q}) \longrightarrow G \longrightarrow 1. \tag{3.4}$$

The exact sequence (3.4) is called the *imprimitivity sequence* for $\mathrm{Gal}(E^c/\mathbb{Q})$ (see [Dod84, p. 4]).

Recall that if Φ is a CM type for E, then the *reflex field* associated to the CM pair (E, Φ) is the field

$$E_\Phi := \mathbb{Q}(\{\mathrm{Tr}_\Phi(a) \mid a \in E\}),$$

where

$$\mathrm{Tr}_\Phi(a) := \sum_{\phi \in \Phi} \phi(a)$$

is the *type trace* of $a \in E$. The Reflex Degree Theorem of Dodson [Dod84, p. 5] states that the reflex degree $[E_\Phi : \mathbb{Q}]$ is related to imprimitivity sequences in the following way: if $G \le S_d$ is the transitive subgroup such that $\mathrm{Gal}(F^c/\mathbb{Q}) \cong G$, and

$$1 \longrightarrow C_2^v \longrightarrow \mathrm{Gal}(E^c/\mathbb{Q}) \longrightarrow G \longrightarrow 1 \tag{3.5}$$

is the imprimitivity sequence for $\mathrm{Gal}(E^c/\mathbb{Q})$, then there is a subgroup S of G such that

$$[E_\Phi : \mathbb{Q}] = 2^v [G : S]. \tag{3.6}$$

The subgroup S is defined in [Dod84, p. 5] (the so-called *splitting subgroup*), although we do not need an explicit description for our purposes.

With these preliminaries, we now proceed to the proof of Theorem 1.13.

3.3. **Proof of Theorem 1.13.** It is known that the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of CM types $\Phi(E)$ of E. Importantly, one can prove that the size of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit of a CM type Φ equals the degree of the reflex field E_Φ over $\mathbb{Q}$; that is (see [BSM16, Proposition 6.3]),

$$[E_\Phi : \mathbb{Q}] = \#(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \Phi).$$

Since there are exactly 2^d CM types in $\Phi(E)$, this shows that

$$[E_\Phi : \mathbb{Q}] \le 2^d, \tag{3.7}$$

and moreover, that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\Phi(E)$ is transitive if and only if $[E_\Phi : \mathbb{Q}] = 2^d$.

Now, suppose that E is a G-Weyl CM field. Then

$$\mathrm{Gal}(E^c/\mathbb{Q}) \cong C_2 \wr G = C_2^d \rtimes G,$$

and in particular, we have $|\,\mathrm{Gal}(E^c/\mathbb{Q})| = 2^d |G|$. On the other hand, by the imprimitivity sequence (3.5) we have $G \cong \mathrm{Gal}(E^c/\mathbb{Q})/C_2^v$, so that $|\,\mathrm{Gal}(E^c/\mathbb{Q})| = 2^v |G|$. Hence $v = d$, and it follows from (3.6) that

$$[E_\Phi : \mathbb{Q}] = 2^d [G : S] \ge 2^d. \tag{3.8}$$

From inequalities (3.7) and (3.8), we conclude that $[E_\Phi : \mathbb{Q}] = 2^d$, and thus the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\Phi(E)$ is transitive.

Finally, in [BSM16, Proposition 5.1], it is shown as a consequence of the recently proved averaged Colmez conjecture [AGHM15, YZ15] that if the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\Phi(E)$ is transitive, then the Colmez conjecture is true for E and takes the form (1.11). This proves Theorem 1.13. □

4. Some known cases of Hypothesis 1.1

In this section, we give a table which lists some known cases of Hypothesis 1.1. We also give a table that lists cases of Hypothesis 1.1 which would follow from a sufficiently strong 2-torsion exponent δ_d.

For $d \geq 6$, the lists are extracted from the tables in [Du17]; in particular, as Dummit notes, the labeling of the transitive subgroups is the standard one originally given by Conway, Hulpke, and McKay [CHM98]. For simplicity, when summarizing results in the tables, we sometimes state upper bounds which are weaker than what is known.

For a transitive subgroup $G \leq S_d$, the following table gives a list of general pairs (d, G) for which Hypothesis 1.1 is known to hold. In each case, the upper bound in the Malle conjecture (1.2) is known, and we may take $\delta_d = 1/2$. The table does not necessarily contain a complete list of all known results, and it should be possible to obtain additional cases of Hypothesis 1.1. Among other possibilities, Wang informs us that her methods can handle additional cases such as $d = 9$, $G = S_3 \times S_3$, and Mehta [Meh17] is presently extending the results of [Klu06] to Frobenius groups.

We also note that when G satisfies Hypothesis 1.1, so does $C_2 \wr H$ by the argument of Klüners [Klu12] which we are adapting.

Table 2. General pairs (d, G) for which Hypothesis 1.1 holds.

(d, G)	Reference	Upper bound $N_d(X, G) \ll X^{M(G)}$
$d \geq 1$ and G abelian	[Mak85]	$X^{\frac{1}{\lvert G\rvert(1-1/\ell)}+\epsilon}$, ℓ the smallest prime divisor of $\lvert G\rvert$
$d = \ell$ prime, $G = D_\ell$	[Klu06, CT17]	$X^{\frac{3}{\ell-1}-\frac{1}{\ell(\ell-1)}+\epsilon}$
$d \geq 1$ and G a p-group	[KM04]	$X^{1+\epsilon}$
$d \geq 5$ and $\lvert G\rvert = d$	[EV06]	$X^{\frac{3}{8}+\epsilon}$
$d = 3$, any $G \leq S_3$ transitive	[DH71]	X^1
$d = 4$, any $G \leq S_4$ transitive	[CDO02, Bha05]	X^1
$d = 5$, any $G \leq S_5$ transitive	[Bha10]	X^1
$d = 3\lvert A\rvert$, $S_3 \times A$ with any A abelian	[Wan17]	$X^{1/\lvert A\rvert}$
$d = 4\lvert A\rvert$, $S_4 \times A$ with any A abelian	[Wan17]	$X^{1/\lvert A\rvert}$

Remark 4.1. As observed previously, the condition $|G| = d$ is equivalent to all number fields counted by $N_d(X, G)$ being Galois over $\mathbb{Q}$. This case follows from [EV06, Proposition 1.3].

The following table lists specific pairs (d, G) with $6 \leq d \leq 8$, for which an upper bound $N_d(X, G) \ll X^{M(G)}$ is known for some $M(G) < 2$, but such that $\delta_d + M(G) > 2$. The last column lists a range of 2-torsion exponents which would suffice for $\delta_d + M(G) < 2$ to hold.

The results were obtained by Dummit [Du17].

Table 3. Specific pairs (d, G) for which Hypothesis 1.1 holds for any 2-torsion exponent δ_d in the specified range.

Label #	Order of group	Isomorphic to	Upper bound $N_d(X,G) \ll X^{M(G)}$	Range of δ_d
Transitive subgroups of S_6 satisfying $N_6(X,G) \ll X^{M(G)}$ with $M(G) < 2$ $(d = 6)$				
6T5	18	F_{18}	$X^{7/4+\epsilon}$	$\delta_6 < \frac{1}{4}$
6T12	60	A_5	$X^{8/5+\epsilon}$	$\delta_6 < \frac{2}{5}$
6T14	120	S_5	$X^{19/10+\epsilon}$	$\delta_6 < \frac{1}{10}$
6T15	360	A_6	$X^{19/10+\epsilon}$	$\delta_6 < \frac{1}{10}$
Transitive subgroups of S_7 satisfying $N_7(X,G) \ll X^{M(G)}$ with $M(G) < 2$ $(d = 7)$				
7T2	14	D_7	$X^{19/12+\epsilon}$	$\delta_7 < \frac{5}{12}$
7T3	21	F_{21}	$X^{7/4+\epsilon}$	$\delta_7 < \frac{1}{4}$
7T5	168	$PSL_2(\mathbb{F}_7)$	$X^{11/6+\epsilon}$	$\delta_7 < \frac{1}{6}$
Transitive subgroups of S_8 satisfying $N_8(X,G) \ll X^{M(G)}$ with $M(G) < 2$ $(d = 8)$				
8T25	56	F_{56}	$X^{27/14+\epsilon}$	$\delta_8 < \frac{1}{14}$

5. Acknowledgments

We would like to thank Evan Dummit, Wei-Lun Tsai, Jiuya Wang, and Matt Young for helpful comments.

Barquero-Sanchez and Masri's work was partially supported by the NSF Grants DMS-1162535 and DMS-1460766, and by the Simons Foundation Grant #421991.

Thorne's work was partially supported by the National Security Agency under a Young Investigator Grant. Part of his work was done at the Mathematical Sciences Research Institute in Berkeley, CA in Spring 2017, supported by NSF Grant DMS-1440140.

References

[AGHM15] F. Andreatta, E. Z. Goren, B. Howard and K. Madapusi Pera, *Faltings heights of abelian varieties with complex multiplication.* Ann. of Math. **187** (2018), 391–531.

[BSM16] A. Barquero-Sanchez and R. Masri, *On the Colmez conjecture for non-abelian CM fields.* Research in the Mathematical Sciences, Special Collection in honor of Don Zagier's 65th birthday (2018), **5**:10.

[BSM18] A. Barquero-Sanchez and R. Masri, *The Chowla-Selberg formula for CM abelian surfaces*, submitted for publication.

[Bha05] M. Bhargava, *The density of discriminants of quartic rings and fields.* Ann. of Math. **162** (2005), 1031–1063.

[BW08] M. Bhargava and M. Matchett Wood, *The density of discriminants of S_3-sextic number fields.* Proc. Amer. Math. Soc. **136** (2008), 1581–1587.

[Bha10] M. Bhargava, *The density of discriminants of quintic rings and fields.* Ann. of Math. **172** (2010), 1559–1591.

[BSTTTZ17] M. Bhargava, A. Shankar, T. Taniguchi, F. Thorne, J. Tsimerman, and Y. Zhao, *Bounds on 2-torsion in class groups of number fields and integral points on elliptic curves.* `arXiv:1701.02458 [math.NT]` (version 1, 10 Jan 2017)

[CO12] C. Chai and F. Oort, *Abelian varieties isogenous to a Jacobian.* Ann. of Math. **176** (2012), 589–635.

[Cha97] N. Chavdarov, *The generic irreducibility of the numerator of the zeta function in a family of curves with large monodromy.* Duke Math. J. **87** (1997), 151–180.

[CDO02] H. Cohen, F. Diaz y Diaz, M. Olivier, *Enumerating quartic dihedral extensions of* $\mathbb{Q}$. Compositio Math. **133** (2002), 65–93.

[CT17] H. Cohen and F. Thorne, *On D_ℓ-extensions of odd prime degree ℓ.* `arXiv:1609.09153 [math.NT]` (version 2, 26 Jan 2017)

[Col93] P. Colmez, *Périodes des variétés abéliennes à multiplication complexe*. Ann. of Math. **138** (1993), 625–683.

[CHM98] J. H. Conway, A. Hulpke, and J. McKay, *On transitive permutation groups*. LMS Journal of Computation and Mathematics **1** (1998), 1–8.

[DH71] H. Davenport and H. Heilbronn, *On the density of discriminants of cubic fields. II.* Proc. Roy. Soc. London Ser. A **322** (1971), no. 1551, 405–420.

[DM96] J. Dixon and B. Mortimer, *Permutation groups.* Springer, Berlin-Heidelberg-New York, 1996.

[Dod84] B. Dodson, *The structure of Galois groups of CM-fields.* Trans. Amer. Math. Soc. **283** (1984), 1–32.

[Du17] E. P. Dummit, *Counting G-extensions by discriminant.* `arXiv:1704.03124v2 [math.NT]` (version 2, 17 Apr 2017)

[ELMV11] M. Einsiedler, E. Lindenstrauss, P. Michel, and A. Venkatesh, *Distribution of periodic torus orbits and Duke's theorem for cubic fields.* Ann. of Math. **173** (2011), 815–885.

[EV06] J. S. Ellenberg and A. Venkatesh, *The number of extensions of a number field with fixed degree and bounded discriminant.* Ann. of Math. **163** (2006), 723–741.

[EEHK09] J. S. Ellenberg, C. Elsholtz, C. Hall, and E. Kowalski, *Non-simple abelian varieties in a family: geometric and analytic approaches.* J. Lond. Math. Soc. **80** (2009), 135–154.

[ETW17] J. S. Ellenberg, T. Tran, and C. Westerland, *Fox-Neuwirth-Fuks cells, quantum shuffle algebras, and Malle's conjecture for function fields.* `arXiv:1701.04541 [math.NT]` (version 1, 17 Jan 2017)

[Gal73] P. X. Gallagher, *The large sieve and probabilistic Galois theory.* Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), pp. 91–101. Amer. Math. Soc., Providence, R.I., 1973.

[IK04] H. Iwaniec and E. Kowalski, *Analytic number theory.* American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004. xii+615 pp.

[KY05] A. C. Kable and A. Yukie, *On the number of quintic fields.* Invent. Math. **160** (2005), 217–259.

[KM04] J. Klüners and G. Malle, *Counting nilpotent Galois extensions.* J. Reine Angew. Math. **572** (2004), 1–26.

[Klu06] J. Klüners, *Asymptotics of number fields and the Cohen-Lenstra heuristics*, J. Théor. Nombres Bordeaux **18** (2006), no. 3, 607–615.

[Klu12] J. Klüners, *The distribution of number fields with wreath products as Galois groups.* Int. J. Number Theory **8** (2012), 845–858.

[Kow06] E. Kowalski, *Weil numbers generated by other Weil numbers and torsion fields of abelian varieties.* J. London Math. Soc. (2) **74** (2006), no. 2, 273–288.

[LMFDB] The LMFDB Collaboration, *The L-functions and Modular Forms Database*, `http://www.lmfdb.org`, 2017. [Online; accessed 2 July 2017.]

[Mak85] S. Mäki, *On the density of abelian number fields.* Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes No. 54 (1985), 104 pp.

[Mal02] G. Malle, *On the distribution of Galois groups.* J. Number Theory **92** (2002), 315–329.

[Mal04] G. Malle, *On the distribution of Galois groups. II.* Experiment. Math. **13** (2004), 129–135.

[Meh17] H. Mehta, doctoral thesis, University of South Carolina, in preparation.

[Obu13] A. Obus, *On Colmez's product formula for periods of CM-abelian varieties*. Math. Ann. **356** (2013), 401–418.

[Odl90] A.M. Odlyzko, *Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: a survey of recent results*, Sém. Théor. Nombres Bordeaux (2) **2** (1990), no. 1, 119–141.

[Oor12] F. Oort, *CM Jacobians.* Notes from a talk at the conference "Galois covers and deformations", Bordeaux, June 25–29, 2012. `http://www.staff.science.uu.nl/oort0109/Bord2-VI-12.pdf`

[PARI] The PARI Group, *PARI/GP*, version `2.8.0`. Univ. Bordeaux, 2015, `http://pari.math.u-bordeaux.fr/`.

[Sch95] W. M. Schmidt, *Number fields of given degree and bounded discriminant.* Astérisque **228** (1995), no. 4, 189–195.

[Wan17] J. Wang, *Malle's conjecture for* $S_n \times A$ *for* $n = 3, 4$. `arXiv:1705.00044 [math.NT]` (version 1, 28 Apr 2017)

[Yan10a] T. H. Yang, *An arithmetic intersection formula on Hilbert modular surfaces.* Amer. J. Math. **132** (2010), 1275–1309.

[Yan10b] T. H. Yang, *The Chowla-Selberg formula and the Colmez conjecture.* Canad. J. Math. **62** (2010), 456–472.

[Yan13] T. H. Yang, *Arithmetic intersection on a Hilbert modular surface and the Faltings height.* Asian J. Math. **17** (2013), 335–381.

[YY16] T. H. Yang and H. Yin, *CM fields of dihedral type and Colmez conjecture*. manuscripta math. (2017). https://doi.org/10.1007/s00229-017-0966-z.

[YZ15] X. Yuan and S. Zhang, *On the Averaged Colmez Conjecture.* Ann. of Math. **187** (2018), 533–638.

Escuela de Matemática, Universidad de Costa Rica, San José 11501, Costa Rica
E-mail address: `adrian.barquero_s@ucr.ac.cr`

Department of Mathematics, Mailstop 3368, Texas A&M University, College Station, TX 77843-3368
E-mail address: `masri@math.tamu.edu`

University of South Carolina, Department of Mathematics, 317-O LeConte College, 1523 Greene Street, Columbia, SC 29201
E-mail address: `thorne@math.sc.edu`