HYBRID BOUNDS FOR QUADRATIC WEYL SUMS AND ARITHMETIC APPLICATIONS

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ABSTRACT. Let $D < 0$ be an odd fundamental discriminant and q be a prime number which splits in $\mathbb{Q}(\sqrt{D})$. Given a suitable smooth function f supported on $[X, 2X]$ for $X \geq 1$, we establish a uniform bound in X, D and q for

$$\sum_{c \equiv 0 \ (\mathrm{mod}\ q)} W_h(D; c) f(c),$$

where

$$W_h(D; c) := \sum_{\substack{b \ (\mathrm{mod}\ 2c) \\ b^2 \equiv D \ (\mathrm{mod}\ 4c)}} e\left(\frac{hb}{2c}\right), \quad h \in \mathbb{Z}, \quad e(z) := e^{2\pi i z}$$

is the Weyl sum for roots of the quadratic congruence $x^2 \equiv D \pmod{4c}$. We use this result to study several arithmetic problems, including analogs for quadratic roots of Linnik's theorem on the least prime in an arithmetic progression.

1. Introduction and statement of results

Let $D < 0$ be a discriminant and define the quadratic Weyl sum

$$W_h(D; c) := \sum_{\substack{b \ (\mathrm{mod}\ 2c) \\ b^2 \equiv D \ (\mathrm{mod}\ 4c)}} e\left(\frac{hb}{2c}\right), \quad h \in \mathbb{Z}, \quad e(z) := e^{2\pi i z}.$$

In many arithmetic problems, one needs a bound for these sums as the modulus c varies in some range, for example

$$W_h(f, D, q) := \sum_{c \equiv 0 \ (\mathrm{mod}\ q)} W_h(D; c) f(c), \tag{1.1}$$

where $q \geq 1$ is an integer and f is a smooth function supported on $[X, 2X]$ for $X \geq 1$. The main result of this paper is the following uniform bound for (1.1).

Theorem 1.1. *Let $D < 0$ be an odd fundamental discriminant and q be a prime number which splits in $\mathbb{Q}(\sqrt{D})$. Let $f : \mathbb{R} \to \mathbb{C}$ be a C^∞ function supported on $[X, 2X]$ for $X \geq 1$ which satisfies*

$$f^{(j)} \ll X^{-j}, \quad j = 0, 1, \ldots$$

For $X \gg |D|^{1/2}$, we have

$$W_h(f, D, q) \ll_\varepsilon |h|(q|D|X|h|)^\varepsilon \min\{A(X, D), B(X, D, q), C(X, D, q)\},$$

where

$$A(X, D) := X^{3/4}|D|^{1/16}, \quad B(X, D, q) := X^{1/2}\left(1 + \frac{X^{1/4}}{q^{1/4}|D|^{1/8}}\right)\left(1 + \frac{|D|^{1/4}}{q^{1/2}}\right),$$

$$C(X, D, q) := X^{1/2}\left(1 + \frac{X^{1/4}}{q^{1/2}|D|^{1/8}}\right)\left(1 + \frac{|D|^{1/4}}{q^{1/8}}\right).$$

A bound for (1.1) with a power saving in X was first established by Hooley [H]. This result was later improved by Bykovski [B] using spectral methods. In these papers, D is fixed and $q = 1$; however, for many arithmetic problems, it is crucial to have a wide range of uniformity in at least two of the variables X, D and q. Examples of this occur in the groundbreaking work of Duke, Friedlander and Iwaniec [DFI] on the equidistribution of quadratic roots to prime moduli, and the work of Tóth [T] which gives an analog of this result for positive discriminants. Other examples where such uniformity is required occur in Duke, Friedlander and Iwaniec's recent paper [DFI2], where they established a strong uniform bound for (1.1) (their method works for both positive and negative discriminants). A central role was played by their important work on bilinear forms with Kloosterman fractions [DFI3].

In this paper, we study (1.1) from the perspective of period formulas and mean-values of L–functions. Our approach is influenced by the papers [B], [DFI], and is related to our joint work with Matt Young [LMY], [LMY2].

To prove Theorem 1.1, we first express $W_h(f, D, q)$ as the trace of a certain weight zero, smooth Poincaré series for $\Gamma_0(q)$ over the Heegner points of discriminant D. After spectrally decomposing the Poincaré series and calculating the spectral coefficients, we are led to estimating (up to a very small error term) an expression of the form

$$\sum_{|t_g| \ll (qX|D|^{1/2}|h|)^{\varepsilon}} \overline{\rho_g(-h)}\check{\phi}(t_g) W_{D,g} + \text{“continuous spectrum contribution”},$$

where g runs over an orthonormal basis of Maass cusp forms for $\Gamma_0(q)$ with spectral parameter t_g. Here $\rho_g(h)$ is the h-th Fourier coefficient of g, $\check{\phi}(t)$ is the integral transform

$$\check{\phi}(t) := \int_0^\infty \phi(u) K_{it}(u) u^{-3/2} du$$

where

$$\phi(u) := f\left(\frac{\pi|h|\sqrt{|D|}}{u}\right),$$

and $W_{D,g}$ is the trace of g over the Heegner points of discriminant D on $\Gamma_0(q)\backslash\mathbb{H}$.

Formulas of Waldspurger [W] and Zhang [Zh] relate $|W_{D,g}|^2$ to $L(g, 1/2)L(g \times \chi_D, 1/2)$, where χ_D is the Kronecker symbol associated to $\mathbb{Q}(\sqrt{D})$. Various applications of Hölder's inequality are possible here (see Section 7 for further discussion), and we are led to estimating mean-values of different families of L–functions. For example, with the choice of exponents $4, 2, 4$ in Hölder's inequality, we are led to estimating the mean-values

$$M_1 := \sum_{|t_g| \le (qX|D|^{1/2}|h|)^{\varepsilon}} |\rho_g(h)|^4 |\check{\phi}(t_g)|^4, \quad M_2 := \sum_{|t_g| \le (qX|D|^{1/2}|h|)^{\varepsilon}} \frac{L(g \times \chi_D, \frac{1}{2})}{L(\text{sym}^2 g, 1)},$$

$$M_3 := \sum_{|t_g| \le (qX|D|^{1/2}|h|)^{\varepsilon}} L(g, \tfrac{1}{2})^2,$$

where the sums are over Maass newforms g for $\Gamma_0(q)$. We estimate M_1 using the Kuznetsov formula, and we estimate M_3 using the spectral large sieve inequality [DI]. We estimate M_2 in two different ways, first using a hybrid subconvexity bound of Blomer-Harcos [BH], and second using the following estimate proved in Liu-Masri-Young [LMY, Theorem 1.5] for q a prime with $(q, D) = 1$ and $M \ge 1$,

$$\sum_{|t_g| \le M} \frac{L(g \times \chi_D, \frac{1}{2})}{L(\text{sym}^2 g, 1)} \ll_\varepsilon (qM^2 + |D|^{1/2})(|D|Mq)^{\varepsilon}.$$

As mentioned earlier, Duke, Friedlander and Iwaniec [DFI2] established a strong uniform bound for (1.1) using methods quite different from those in this paper, and gave many interesting arithmetic applications. One application they gave was to the equidistribution modulo 1 of roots of the quadratic congruence $x^2 \equiv D \pmod{c}$ as $X, |D|, q \to \infty$ (see [DFI2, Theorem 1.3]). Here we give some examples of results of this type which can be deduced from the bounds in Theorem 1.1. In particular, these bounds yield a wide range of uniformity in q, which has significance for some analogs of Linnik's problem on the least prime in an arithmetic progression.

Let $I \subset [0,1]$ be a fixed subinterval of length $\ell(I) > 0$, and define

$$N_I(X, D, q) := |\{b \,(\mathrm{mod}\ 2c) :\ 1 \le c \le X,\ c \equiv 0 \,(\mathrm{mod}\ q),\ b^2 \equiv D \,(\mathrm{mod}\ 4c),\ \frac{b}{2c} \in I\}|.$$

If $X = |D|^{1/2}$, so that the minimum in Theorem 1.1 is $A(X, D)$, we obtain the following

Corollary 1.2. *For D and q as in Theorem 1.1, we have*

$$N_I(|D|^{1/2}, D, q) \sim \frac{12}{\pi^2} \frac{L(\chi_D, 1)}{q+1} |D|^{1/2} \ell(I)$$

as $q, |D| \to \infty$ subject to the condition $q \le |D|^{1/16-\varepsilon}$. In particular, given q and I, for any $|D|^{1/2} \gg q^{8+\varepsilon}$ we have $N_I(|D|^{1/2}, D, q) > 0$.

Remark 1.3. Corollary 1.2 can be viewed as an analog for quadratic roots of Linnik's problem on the least prime in an arithmetic progression: given q and a residue class $a \pmod{q}$ with $(a, q) = 1$, for any $X \gg q^L$ one has $\pi(X, a, q) > 0$ where

$$\pi(X, a, q) := |\{p \le X :\ p \text{ a prime},\ p \equiv a \,(\mathrm{mod}\ q)\}|$$

and L is the famous Linnik "constant" (see [IK, Chapter 18]).

On the other hand, if $X = |D|$, so that the minimum in Theorem 1.1 is $B(X, D, q)$, we obtain the following

Corollary 1.4. *For D and q as in Theorem 1.1, we have*

$$N_I(|D|, D, q) \sim \frac{12}{\pi^2} \frac{L(\chi_D, 1)}{q+1} |D| \ell(I)$$

as $q, |D| \to \infty$ subject to the condition $|D|^{1/12} \le q \le |D|^{1/2-\varepsilon}$.

One can also prove "sparse" equidistribution analogs of these results in which the subinterval I is allowed to shrink as a function of X, D or q. For example, if $X = |D|^{1/2}$ and q is fixed, we obtain the following

Corollary 1.5. *Let D and q be as in Theorem 1.1. If $I_D \subset [0,1]$ is a subinterval of length $\ell(I_D) = |D|^{-\eta}$ with $0 \le \eta < 1/48$, we have*

$$N_{I_D}(|D|^{1/2}, D, q) \sim \frac{12}{\pi^2} \frac{L(\chi_D, 1)}{q+1} |D|^{1/2-\eta}$$

as $|D| \to \infty$. In particular, given I_D with $\ell(I_D) \gg |D|^{-\frac{1}{48}+\varepsilon}$, we have $N_{I_D}(|D|^{1/2}, D, q) > 0$.

In a different direction, Theorem 1.1 can be used to study the distribution of traces of singular values of weakly holomorphic modular functions of level q. Let $\mathcal{Q}_{D,q}$ be the set of positive definite, integral binary quadratic forms

$$Q(X, Y) = aX^2 + bXY + cY^2$$

of discriminant $b^2 - 4ac = D < 0$ with $a \equiv 0 \pmod{q}$. The group $\Gamma_0^*(q)$ generated by $\Gamma_0(q)$ and the Fricke involution acts on $\mathcal{Q}_{D,q}$ with finite quotient. Let $M_0^!(\Gamma_0^*(q))$ be the space of weakly holomorphic modular forms of weight zero for $\Gamma_0^*(q)$. Such a form f is holomorphic on the complex upper half-plane $\mathbb{H}$, meromorphic in the cusps of $\Gamma_0^*(q)$, and has a Fourier expansion in the cusp at ∞ of the form

$$f(z) = \sum_{n=1}^{N_f} a_f(-n)e(-nz) + \sum_{n=0}^{\infty} a_f(n)e(nz)$$

for some integer $N_f \geq 1$. Define the trace

$$\mathrm{Tr}^*_{D,q}(f) := \sum_{Q \in \mathcal{Q}_{D,q}/\Gamma_0^*(q)} \frac{f(\tau_Q)}{\#\Gamma_0^*(q)_Q},$$

where

$$\tau_Q = \frac{-b+\sqrt{D}}{2a}$$

is the root of $Q(X,1)$ in $\mathbb{H}$ and $\Gamma_0^*(q)_Q$ is the stabilizer of Q in $\Gamma_0^*(q)$.

The classical modular j-function

$$j(z) := e(-z) + 744 + 196884 \cdot e(z) + \cdots$$

is a weakly holomorphic modular form of weight zero for $\Gamma_0(1)$ whose values $j(\tau_Q)$ are algebraic integers called “singular moduli”. Let $J := j - 744$. Zagier [Z2] proved that the generating function for traces of singular moduli,

$$e(-z) - 2 - \sum_{D<0} \mathrm{Tr}_{D,1}(J)e(|D|z),$$

is a weight 3/2 weakly holomorphic modular form for $\Gamma_0(4)$ satisfying Kohnen's plus space condition.

Bruinier and Funke [BF] generalized this result to forms $f \in M_0^!(\Gamma_0^*(q))$. In particular, if q is a prime (or $q = 1$) and f has constant term $a_f(0) = 0$, they proved that the generating function

$$\sum_{D<0} \mathrm{Tr}^*_{D,q}(f)e(|D|z) + \sum_{n\geq 1}(\sigma_1(n) + q\sigma_1(n/q))a_f(-n) - \sum_{m\geq 1}\sum_{n\geq 1} m a_f(-mn)e(-m^2 z)$$

is a weight 3/2 weakly holomorphic modular form for $\Gamma_0(4q)$ satisfying Kohnen's plus space condition. Here $\sigma_1(0) := -1/24$, $\sigma_1(n) := \sum_{t|n} t$ for $n \in \mathbb{Z}_{\geq 0}$, and $\sigma_1(x) = 0$ for $x \notin \mathbb{Z}_{\geq 0}$. In particular, if $q = 1$ and $f = J$, they recovered Zagier's result.

We are interested in the asymptotic distribution of the traces $\mathrm{Tr}^*_{D,q}(f)$. For traces of singular moduli, Bruinier, Jenkins and Ono [BJO] established the Rademacher type exact formula

$$\mathrm{Tr}_{D,1}(J) = -24H(D) + 2\sum_{c\in\mathbb{Z}^+} W_1(D;c)\sinh\left(\frac{\pi\sqrt{|D|}}{c}\right),$$

where

$$H(D) := \sum_{Q\in\mathcal{Q}_{D,1}/\Gamma_0(1)} \frac{1}{\#\Gamma_0(1)_Q}$$

is the Hurwitz class number. Based on this exact formula, they conjectured that

$$\mathrm{Tr}_{D,1}(J) = -24H(D) + 2\sum_{1\leq c<\sqrt{|D|/3}} W_1(D;c)\sinh\left(\frac{\pi\sqrt{|D|}}{c}\right) + o(H(D))$$

as $|D| \to \infty$ through fundamental discriminants. This conjecture was proved by Duke [D].

For traces of weakly holomorphic forms $f \in M_0^!(\Gamma_0^*(q))$, Choi, Jeon, Kang and Kim [CJKK] established the exact formula

$$\mathrm{Tr}^*_{D,q}(f) = -24 H_q^*(D) \sum_{h>0} a_f(-h) c_q(h) + 2 \sum_{h>0} a_f(-h) \sum_{c \equiv 0 \,(\mathrm{mod}\ q)} W_h(D;c) \sinh\left(\frac{\pi h \sqrt{|D|}}{c}\right),$$

where

$$H_q^*(D) := \sum_{Q \in \mathcal{Q}_{D,q}/\Gamma_0^*(q)} \frac{1}{\#\Gamma_0^*(q)_Q}$$

is the "level q" Hurwitz class number and

$$c_q(h) := -\frac{q^{\alpha+1}}{q+1} \sigma_1(h/q^\alpha) + \sigma_1(h), \quad q^\alpha \| h.$$

We will use Theorem 1.1 to establish the following asymptotic formula for $\mathrm{Tr}^*_{D,q}(f)$ with a power saving in D.

Corollary 1.6. *For D and q as in Theorem 1.1 and $f \in M_0^!(\Gamma_0^*(q))$, we have*

$$\mathrm{Tr}^*_{D,q}(f) = -24 H_q^*(D) \sum_{h>0} a_f(-h) c_q(h) + 2 \sum_{h>0} a_f(-h) \sum_{\substack{1 \le c < \sqrt{|D|} \\ c \equiv 0 \,(\mathrm{mod}\ q)}} W_h(D;c) \sinh\left(\frac{\pi h \sqrt{|D|}}{c}\right) + O_{f,\varepsilon}(|D|^{7/16+\varepsilon})$$

as $|D| \to \infty$.

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2. Heegner points

Let $D < 0$ be an odd fundamental discriminant and q be a prime number which splits in $\mathbb{Q}(\sqrt{D})$. Let $\mathcal{Q}_{D,q}$ be the set of positive definite, integral binary quadratic forms

$$Q(X,Y) = [a_Q, b_Q, c_Q](X,Y) = a_Q X^2 + b_Q XY + c_Q Y^2$$

of discriminant $b_Q^2 - 4a_Q c_Q = D$ with $a_Q \equiv 0 \pmod{q}$. There is a (right) action of $\Gamma_0(q)$ on $\mathcal{Q}_{D,q}$ defined by

$$Q = [a_Q, b_Q, c_Q] \mapsto Q^\sigma = [a_Q^\sigma, b_Q^\sigma, c_Q^\sigma],$$

where for $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(q)$ we have

$$\begin{aligned} a_Q^\sigma &= a_Q \alpha^2 + b_Q \alpha\gamma + c_Q \gamma^2, \\ b_Q^\sigma &= 2a_Q \alpha\beta + b_Q(\alpha\delta + \beta\gamma) + 2c_Q \gamma\delta, \\ c_Q^\sigma &= a_Q \beta^2 + b_Q \beta\delta + c_Q \delta^2. \end{aligned}$$

Given a solution $r \pmod{2q}$ of $r^2 \equiv D \pmod{4q}$ (there are 2 such solutions since q is a prime), we define the subset of forms

$$\mathcal{Q}_{D,q,r} := \{Q = [a_Q, b_Q, c_Q] \in \mathcal{Q}_{D,q} : \ b_Q \equiv r \,(\mathrm{mod}\ 2q)\}.$$

Then $\Gamma_0(q)$ also acts on $\mathcal{Q}_{D,q,r}$, and we have the decomposition (see [GKZ, p. 507])

$$\mathcal{Q}_{D,q}/\Gamma_0(q) = \bigcup_{\substack{r \,(\mathrm{mod}\, 2q) \\ r^2 \equiv D \,(\mathrm{mod}\, 4q)}} \mathcal{Q}_{D,q,r}/\Gamma_0(q). \tag{2.1}$$

To each form $Q \in \mathcal{Q}_{D,q}$, we associate the root

$$\tau_Q = \frac{-b_Q + \sqrt{D}}{2a_Q} \in \mathbb{H}.$$

Fix a set of representatives for the $\Gamma_0(q)$-equivalence classes of forms in $\mathcal{Q}_{D,q}$, and define the set of *Heegner points* of discriminant D by

$$\Lambda_{D,q} := \{\tau_Q : \ Q \in \mathcal{Q}_{D,q}/\Gamma_0(q)\}.$$

The Heegner points are compatible with the action of $\Gamma_0(q)$ in the sense if $\sigma \in \Gamma_0(q)$, then

$$\sigma\tau_Q = \tau_{Q^{\sigma^{-1}}}.$$

Given r and $\mathcal{Q}_{D,q,r}$ as above, each set

$$\Lambda_{D,q,r} := \{\tau_Q : \ Q \in \mathcal{Q}_{D,q,r}/\Gamma_0(q)\}$$

is a $\mathrm{Gal}(H/K)$-orbit of Heegner points of discriminant D, where H is the Hilbert class field of $K = \mathbb{Q}(\sqrt{D})$ (see [GZ, pp. 235-236]).

3. Preliminaries on Maass forms

In this section we review some facts we will need concerning Maass forms (see e.g. [I] and [B, Section 2]). Let $f_1, f_2 : \mathbb{H} \to \mathbb{C}$ be $\Gamma_0(q)$-invariant functions and $Y_0(q)$ be a fundamental domain for $\Gamma_0(q)$. Define the Petersson inner product

$$\langle f_1, f_2\rangle_q := \int_{Y_0(q)} f_1(z)\overline{f_2(z)}\frac{dxdy}{y^2}, \tag{3.1}$$

provided the integral converges.

The hyperbolic Laplacian is defined by

$$\Delta := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Given a Maass form g with Laplace eigenvalue λ_g, let

$$t_g := \sqrt{\lambda_g - 1/4} \in \mathbb{R} \cup (-1/2, 1/2)i$$

be the spectral parameter. Let $\mathcal{B}_q$ be an orthonormal basis of Hecke-Maass newforms of weight 0 for $\Gamma_0(q)$, and $\mathcal{B}_1$ be a basis of Hecke-Maass cusp forms of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$ which is orthonormal with respect to the inner product (3.1). One has the following upper bounds (see [I, Section 11.1])

$$\#\{g \in \mathcal{B}_q : \ |t_g| \le T\} \ll qT^2 \quad \text{and} \quad \#\{g \in \mathcal{B}_1 : \ t_g \le T\} \ll T^2.$$

Given $g \in \mathcal{B}_q \cup \mathcal{B}_1$, let $\lambda_g(n)$ be the n-th Hecke eigenvalue. Then $\lambda_g(-n) = \pm\lambda_g(n)$ depending on whether g is even or odd, and $\lambda_g(n)$ satisfies

$$\lambda_g(n) \ll n^{\frac{1}{2}-\delta} \tag{3.2}$$

for some $\delta > 0$. A newform $g \in \mathcal{B}_q$ is an eigenfunction for the Fricke involution $z \mapsto -1/qz$, and one has

$$\lambda_g(q) = \varepsilon(g)q^{-1/2} \tag{3.3}$$

where $\varepsilon(g) = \pm 1$ is the Fricke eigenvalue.

Given $g \in \mathcal{B}_1$, define (see [ILS, Proposition 2.6])

$$g_q(z) := \left(1 - \frac{q\lambda_g^2(q)}{(q+1)^2}\right)^{-1/2} \left(g(qz) - \frac{q^{1/2}\lambda_g(q)}{q+1} g(z)\right), \tag{3.4}$$

and let $\mathcal{B}_1^* := \{g_q :\ g \in \mathcal{B}_1\}$. Then an orthonormal basis for the subspace of cusp forms in $L^2(Y_0(q))$ is given by

$$\mathcal{B} := \mathcal{B}_q \cup \mathcal{B}_1 \cup \mathcal{B}_1^*.$$

A form $g \in \mathcal{B}_q \cup \mathcal{B}_1$ has the Fourier expansion

$$g(z) = \sqrt{y} \sum_{n \neq 0} \rho_g(n) K_{it_g}(2\pi n y) e(nx),$$

where $\rho_g(n) = \rho_g(1)\lambda_g(n)$ and (see [B, eq. (2.9)])

$$|\rho_g(1)| = \left(\frac{2\cosh(\pi t_g)}{L(\mathrm{sym}^2 g, 1)}\right)^{1/2} \times \begin{cases} \left(\frac{q^2}{q+1}\right)^{-1/2}, & g \in \mathcal{B}_q \\ (q+1)^{-1/2}, & g \in \mathcal{B}_1. \end{cases} \tag{3.5}$$

Similarly, $g_q \in \mathcal{B}_1^*$ has the Fourier expansion

$$g_q(z) = \sqrt{y} \sum_{n \neq 0} \rho_{g_q}(n) K_{it_{g_q}}(2\pi n y) e(nx),$$

where $\rho_{g_q}(n) = \rho_g(1)\lambda_{g_q}(n)$ and

$$\lambda_{g_q}(n) := \left(1 - \frac{q\lambda_g^2(q)}{(q+1)^2}\right)^{-1/2} \left(q^{1/2}\lambda_g(n/q) - \frac{q^{1/2}\lambda_g(q)}{q+1}\lambda_g(n)\right) \tag{3.6}$$

for $g \in \mathcal{B}_1$, with the convention $\lambda_g(x) = 0$ for $x \in \mathbb{Q}\backslash\mathbb{Z}$.

Using (3.2)–(3.6) and the Hoffstein-Lockhart [HL] bound

$$L(\mathrm{sym}^2 g, 1) \gg_\varepsilon (|t_g|q)^{-\varepsilon}, \tag{3.7}$$

we obtain

$$\rho_g(n) \ll_\varepsilon |n|^{1/2} |t_g|^\varepsilon q^{-\frac{1}{2}+\varepsilon} e^{\frac{\pi}{2}|t_g|}, \qquad g \in \mathcal{B}. \tag{3.8}$$

For the Eisenstein series $E_{\mathfrak{a}}(z,s)$ associated to a cusp $\mathfrak{a}$ of $\Gamma_0(q)$, one has the Fourier expansion (see [I, Section 3.4])

$$E_{\mathfrak{a}}(z, \tfrac{1}{2}+it) = \delta_{\mathfrak{a}=\infty} y^{1/2+it} + \phi_{\mathfrak{a}}(\tfrac{1}{2}+it) y^{1/2-it} + \sqrt{y}\sum_{n\neq 0} \tau_{\mathfrak{a}}(n,t) K_{it}(2\pi n y) e(nx),$$

where $\phi_{\mathfrak{a}}(s)$ is a certain meromorphic function and $\tau_{\mathfrak{a}}(n,t) = \rho_{\mathfrak{a}}(1,t)\eta_{\mathfrak{a}}(n,t)$, where (see [B, eq. (2.13)])

$$|\rho_{\mathfrak{a}}(1,t)| = \left(\frac{4\cosh(\pi t)}{q|\zeta^{(q)}(1+2it)|}\right)^{1/2}, \tag{3.9}$$

$$\eta_\infty(n,t) := \frac{\eta(n,t)}{q^{1/2+it}} - q^{1/2}\eta(n/q,t), \quad \eta_0(n,t) := \eta(n,t) - q^{-it}\eta(n/q,t), \tag{3.10}$$

$$\eta(n,t) := \sum_{ad=|n|} (a/d)^{it},$$

with the convention $\eta(x,t) = 0$ for $x \in \mathbb{Q}\backslash\mathbb{Z}$.

Using (3.9) and (3.10), we obtain

$$\tau_{\mathfrak{a}}(n,t) \ll_\varepsilon |tn|^\varepsilon e^{\frac{\pi}{2}|t|}. \tag{3.11}$$

Finally, recall that if $F \in L^2(Y_0(q))$, one has the spectral expansion

$$F(z) = \frac{\langle F, 1\rangle_q}{\mathrm{vol}(Y_0(q))} + \sum_{g\in\mathcal{B}} \langle F, g\rangle_q g(z) + \sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \langle F, E_{\mathfrak{a}}(\cdot, \tfrac{1}{2}+it)\rangle_q E_{\mathfrak{a}}(z, \tfrac{1}{2}+it)\frac{dt}{4\pi}, \tag{3.12}$$

which converges in the norm topology (see [I, Section 7.3]). If for example F is smooth and compactly supported, then (3.12) converges pointwise absolutely and uniformly on compact sets.

4. Traces of Poincaré series

Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{C}$ be a C^∞ function with compact support and define the Poincaré series

$$\mathscr{P}_{h,\phi}(z) := \sum_{\sigma\in\Gamma_\infty\backslash\Gamma_0(q)} \phi(2\pi|h|\mathrm{Im}(\sigma z)) e(-h\mathrm{Re}(\sigma z)), \quad z \in \mathbb{H},$$

which is absolutely convergent. We will need the following well-known identity (see [By, Lemma 5] and [DFI, Section 2]).

Proposition 4.1. *Let $D < -4$ be an odd fundamental discriminant. Then*

$$\sum_{c\equiv 0 \,(\mathrm{mod}\ q)} W_h(D;c)\phi\left(\frac{\pi|h|\sqrt{|D|}}{c}\right) = \sum_{Q\in\mathcal{Q}_{D,q}/\Gamma_0(q)} \mathscr{P}_{h,\phi}(\tau_Q).$$

Proof. For $c \equiv 0 \pmod q$ we have

$$\begin{aligned}\sum_{Q\in\mathcal{Q}_{D,q}/\Gamma_0(q)} \sum_{\substack{\sigma\in\Gamma_\infty\backslash\Gamma_0(q)\\ \mathrm{Im}(\sigma\tau_Q)=\frac{\sqrt{|D|}}{2c}}} e(-h\mathrm{Re}(\sigma\tau_Q)) &= \sum_{\substack{Q\in\mathcal{Q}_{D,q}/\Gamma_\infty\\ \mathrm{Im}(\tau_Q)=\frac{\sqrt{|D|}}{2c}}} e(-h\mathrm{Re}(\tau_Q))\\ &= \sum_{\substack{b_Q \,(\mathrm{mod}\ 2c)\\ b_Q^2\equiv D\,(\mathrm{mod}\ 4c)}} e\left(\frac{hb_Q}{2c}\right)\\ &= W_h(D;c),\end{aligned}$$

where we used that the stabilizer of Q in $\Gamma_0(q)$ is $\{\pm I\}$ for $D < -4$. It follows that

$$\begin{aligned}\sum_{Q\in\mathcal{Q}_{D,q}/\Gamma_0(q)} \mathscr{P}_{h,\phi}(\tau_Q) &= \sum_{Q\in\mathcal{Q}_{D,q}/\Gamma_0(q)} \sum_{\sigma\in\Gamma_\infty\backslash\Gamma_0(q)} \phi(2\pi|h|\mathrm{Im}(\sigma\tau_Q))e(-h\mathrm{Re}(\sigma\tau_Q))\\ &= \sum_{c\equiv 0\,(\mathrm{mod}\ q)} \phi\left(\frac{\pi|h|\sqrt{|D|}}{c}\right) \sum_{Q\in\mathcal{Q}_{D,q}/\Gamma_0(q)} \sum_{\substack{\sigma\in\Gamma_\infty\backslash\Gamma_0(q)\\ \mathrm{Im}(\sigma\tau_Q)=\frac{\sqrt{|D|}}{2c}}} e(-h\mathrm{Re}(\sigma\tau_Q))\\ &= \sum_{c\equiv 0\,(\mathrm{mod}\ q)} W_h(D;c)\phi\left(\frac{\pi|h|\sqrt{|D|}}{c}\right).\end{aligned}$$

□

5. Spectral expansion of $W_h(f,D,q)$

Let $f:\mathbb{R}\to\mathbb{C}$ be a C^∞ function supported on $[X,2X]$ for $X\geq 1$ which satisfies

$$f^{(j)} \ll X^{-j}, \quad j=0,1,\dots$$

and define

$$\phi(u) := f\left(\frac{\pi|h|\sqrt{|D|}}{u}\right). \tag{5.1}$$

Then ϕ is a C^∞ function supported on $[Y^{-1}, 2Y^{-1}]$ for

$$Y^{-1} = \frac{\pi|h|\sqrt{|D|}}{2X},$$

and one obtains the bound

$$\phi^{(j)} \ll Y^j, \quad j=0,1,\dots$$

By Proposition 4.1 with ϕ defined as in (5.1), we have

$$W_h(f,D,q) := \sum_{c\equiv 0 \,(\mathrm{mod}\; q)} W_h(D;c)f(c) = \sum_{\tau\in\Lambda_{D,q}} \mathscr{P}_{h,\phi}(\tau). \tag{5.2}$$

Then spectrally expanding $\mathscr{P}_{h,\phi}(z)$ as in (3.12) and substituting in (5.2) yields

$$\begin{aligned} W_h(f,D,q) = h(D)\frac{\langle \mathscr{P}_{h,\phi},1\rangle_q}{\mathrm{vol}(Y_0(q))} + \sum_{g\in\mathcal{B}} \langle \mathscr{P}_{h,\phi}, g\rangle_q W_{D,g} \\ + \sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \langle \mathscr{P}_{h,\phi}, E_{\mathfrak{a}}(\cdot, \tfrac12 + it)\rangle_q W_{D,\mathfrak{a}}(t)\frac{dt}{4\pi}, \end{aligned}$$

where $h(D)$ is the class number of K and the hyperbolic Weyl sums are defined by

$$W_{D,g} := \sum_{\tau\in\Lambda_{D,q}} g(\tau) \qquad \text{and} \qquad W_{D,\mathfrak{a}}(t) := \sum_{\tau\in\Lambda_{D,q}} E_{\mathfrak{a}}(\tau, \frac12 + it).$$

Unfolding the Poincaré series gives

$$\langle \mathscr{P}_{h,\phi}, 1\rangle_q = 0.$$

Similarly, unfolding gives (see for example [IK, Chapter 16])

$$\langle \mathscr{P}_{h,\phi}, g\rangle_q = (2\pi|h|)^{\frac12}\overline{\rho_g(-h)}\check{\phi}(t_g)$$

and

$$\langle \mathscr{P}_{h,\phi}, E_{\mathfrak{a}}(\cdot, \frac12 + it)\rangle_q = (2\pi|h|)^{\frac12}\overline{\tau_{\mathfrak{a}}(-h,t)}\check{\phi}(t),$$

where $\check{\phi}$ is the integral transform

$$\check{\phi}(t) := \int_0^\infty \phi(u)K_{it}(u)u^{-3/2}du. \tag{5.3}$$

Finally, combining the preceding calculations yields

$$\begin{aligned} W_h(f,D,q) = (2\pi|h|)^{\frac12}\sum_{g\in\mathcal{B}} \overline{\rho_g(-h)}\check{\phi}(t_g)W_{D,g} \\ + (2\pi|h|)^{\frac12}\sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \overline{\tau_{\mathfrak{a}}(-h,t)}\check{\phi}(t)W_{D,\mathfrak{a}}(t)\frac{dt}{4\pi}. \end{aligned} \tag{5.4}$$

6. Period integral formulas

Given $g \in \mathcal{B}$, define the hyperbolic Weyl sum

$$W_{D,g,r} := \sum_{\tau \in \Lambda_{D,q,r}} g(\tau).$$

Since $\Lambda_{D,q,r}$ is a Gal(H/K)-orbit of Heegner points of discriminant D, it follows from a formula of Waldspurger [W] and Zhang [Zh] that

$$W_{D,g,r} = \theta_{g,D} \frac{|D|^{1/4}}{q^{1/2}} \frac{L(g \times \chi_D, \frac{1}{2})^{1/2} L(g, \frac{1}{2})^{1/2}}{L(\mathrm{sym}^2 g, 1)^{1/2}} \quad \text{if} \quad g \in \mathcal{B}_q \cup \mathcal{B}_1, \tag{6.1}$$

where $\theta_{g,D}$ is some complex number satisfying $|\theta_{g,D}| \leq 10$. Similarly,

$$W_{D,g_q,r} = \theta_{g,D} \frac{|D|^{1/4}}{q^{1/2}} \frac{L(g \times \chi_D, \frac{1}{2})^{1/2} L(g, \frac{1}{2})^{1/2}}{L(\mathrm{sym}^2 g, 1)^{1/2}} \quad \text{if} \quad g_q \in \mathcal{B}_1^*, \tag{6.2}$$

where on the right hand side of (6.2), $g \in \mathcal{B}_1$ is the Maass form in the definition of g_q (see (3.4)). The deduction of these formulas from Waldsburger/Zhang can be found in [LMY, Lemma 5.1], for example.

A similar formula for the Eisenstein series was established by Duke, Friedlander and Iwaniec [DFI4, eq. (10.30)],

$$W_{D,\mathfrak{a}}(t) = \theta_{D,q,t} \frac{|D|^{1/4}}{q^{1/2}} \frac{|\zeta(\frac{1}{2}+it)||L(\chi_D, \frac{1}{2}+it)|}{|\zeta(1+2it)|}, \tag{6.3}$$

where $\theta_{D,q,t}$ is some complex number satisfying $|\theta_{D,q,t}| \leq 10$.

7. Contribution of the discrete spectrum

In this section we estimate the sum over the Maass forms $\mathcal{B}$ in (5.4).

The estimates in the following lemma will be used repeatedly in our analysis. These estimates are implicit in [DFI, eq. (18)]. Since we need each of the individual estimates, we have included a proof for the convenience of the reader.

Lemma 7.1. *Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{C}$ be a C^∞ function supported on $[Y^{-1}, 2Y^{-1}]$ which satisfies*

$$\phi^{(j)} \ll Y^j, \quad j = 0, 1, \ldots$$

The following estimates hold for the integral transform $\check{\phi}(t)$ defined in (5.3).

(1) *For $t \in \mathbb{R}$, we have*

$$\check{\phi}(t) \ll Y^{1/2} \log(Y+6).$$

(2) *For $t \in i\mathbb{R}$ with $0 < |t| < 1/2$, we have*

$$\check{\phi}(t) \ll Y^{1/2} \log(Y+6)(Y^{it} + Y^{-it}).$$

(3) *For $t \in \mathbb{R}$ with $|t| \geq 1$ and $Y \gg 1$, we have*

$$\check{\phi}(t) \ll Y^{1/2}(1+|t|)^{-A} e^{-\frac{\pi}{2}|t|}, \quad A = 0, 1, \ldots$$

Proof. (1) Suppose that $t \in \mathbb{R}$. Substituting the integral representation

$$K_{it}(u) = \int_0^\infty e^{-u\cosh(\xi)} \cos(t\xi) d\xi$$

into $\check{\phi}$ yields

$$\check{\phi}(t) = \int_0^\infty \cos(t\xi) F_\phi(\xi) d\xi,$$

where

$$F_{\phi}(\xi) := \int_0^{\infty} \phi(u) e^{-u\cosh(\xi)} u^{-3/2} du.$$

Estimating trivially yields $F_\phi(\xi) \ll Y^{1/2}$, while integrating by parts once in u and estimating yields $F_\phi(\xi) \ll Y^{3/2}e^{-\xi}$. Applying these bounds in the ranges $\xi < \log(Y+6)$ and $\xi > \log(Y+6)$, respectively, and using $\cos(t\xi) \ll 1$, we obtain

$$\check{\phi}(t) \ll Y^{1/2} \log(Y+6).$$

(2) Suppose that $t \in i\mathbb{R}$ with $0 < |t| < 1/2$. Then arguing as in (1), but instead using the bound $\cos(t\xi) \ll e^{it\xi} + e^{-it\xi}$, we deduce that

$$\check{\phi}(t) \ll Y^{1/2} \log(Y+6)(Y^{it} + Y^{-it}).$$

(3) Suppose that $t \in \mathbb{R}$ with $|t| \geq 1$. Substitute

$$K_\nu(u) = \frac{\pi}{2} \frac{1}{\sin(\pi\nu)} (I_{-\nu}(u) - I_\nu(u))$$

into $\check{\phi}$, where

$$I_\nu(u) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+1+\nu)} \left(\frac{u}{2}\right)^{\nu+2k}.$$

This yields the identity

$$\check{\phi}(t) = \Phi(t) + \Phi(-t),$$

where

$$\Phi(t) := \frac{\pi}{2} \frac{1}{\sin(\pi i t)} \sum_{k=0}^{\infty} \frac{\tilde{\phi}(-\frac{1}{2} - it + 2k)}{k!\Gamma(k+1-it)2^{-it+2k}}$$

and

$$\tilde{\phi}(s) := \int_0^{\infty} \phi(u) u^{s-1} du$$

is the Mellin transform of ϕ. Integrating by parts A-times yields

$$\tilde{\phi}(s) \ll Y^{-\mathrm{Re}(s)} (1+|s|)^{-A}. \tag{7.1}$$

Since $\check{\phi}(t) = \check{\phi}(-t)$, we may assume that $t \geq 1$. By Stirling's formula, if $k+1 \leq t$ then $\Gamma(k+1-it) \gg |t|^{k+\frac{1}{2}} e^{-\pi|t|/2} e^{-k}$, while if $k+1 \geq t$ then $\Gamma(k+1-it) \gg k^{1/2}(k+2)^k e^{-\pi|t|/4} e^{-k}$. Applying these bounds in the ranges $k+1 \leq t$ and $k+1 \geq t$, respectively, and using (7.1) and $\sin(\pi i t) \asymp e^{\pi|t|}$, we obtain

$$\Phi(t) \ll Y^{1/2}(1+|t|)^{-A} e^{-\frac{\pi}{2}|t|}.$$

A similar argument shows that $\Phi(-t)$ satisfies the same bound.

□

First we estimate the contribution of the oldforms.

Lemma 7.2. *For $X \gg |D|^{1/2}$ we have*

$$\sum_{g \in \mathcal{B}_1 \cup \mathcal{B}_1^*} \overline{\rho_g(-h)} \check{\phi}(t_g) W_{D,g} \ll_\varepsilon |h|^{1/2} (q|D|X|h|)^\varepsilon q^{-1} X^{1/2} |D|^{1/6}.$$

Proof. Since $W_{D,g}$ grows polynomially in t_g, by (3.8) and Lemma 7.1 we may impose the truncation $t_g \le (q|D|Y|h|)^\varepsilon$ with an error term which is $O((q|D|Y|h|)^{-1000})$ to obtain

$$\sum_{g\in\mathcal{B}_1\cup\mathcal{B}_1^*} \overline{\rho_g(-h)}\check{\phi}(t_g)W_{D,g} \ll_\varepsilon |h|^{1/2}(q|D|Y|h|)^\varepsilon Y^{1/2}\log(Y+6)q^{-\frac{1}{2}} \sum_{t_g\le(q|D|Y|h|)^\varepsilon} |W_{D,g}|$$

for $Y \gg 1$. By (6.1) and (6.2), and the Conrey-Iwaniec [CI] bound

$$L(g\times\chi_D, 1/2) \ll_\varepsilon |D|^{1/3}(q|D|Y|h|)^\varepsilon$$

for $t_g \le (q|D|Y|h|)^\varepsilon$, we obtain

$$W_{D,g} \ll_\varepsilon q^{-1/2}|D|^{5/12}(q|D|Y|h|)^\varepsilon.$$

Since there are $\ll (q|D|Y|h|)^{2\varepsilon}$ oldforms with $t_g \le (q|D|Y|h|)^\varepsilon$, upon substituting the bound $Y \asymp X|D|^{-1/2}$, we complete the proof. □

Next we estimate the contribution of the newforms.

Lemma 7.3. *For $X \gg |D|^{1/2}$ we have*

$$\sum_{g\in\mathcal{B}_q} \overline{\rho_g(-h)}\check{\phi}(t_g)W_{D,g} \ll_\varepsilon |h|^{1/2}(q|D|X|h|)^\varepsilon \min\{A(X,D), B(X,D,q), C(X,D,q)\}, \tag{7.2}$$

where

$$A(X,D) := X^{3/4}|D|^{1/16}, \quad B(X,D,q) := X^{1/2}\left(1+\frac{X^{1/4}}{q^{1/4}|D|^{1/8}}\right)\left(1+\frac{|D|^{1/4}}{q^{1/2}}\right),$$

$$C(X,D,q) := X^{1/2}\left(1+\frac{X^{1/4}}{q^{1/2}|D|^{1/8}}\right)\left(1+\frac{|D|^{1/4}}{q^{1/8}}\right).$$

Proof. Since $W_{D,g}$ grows polynomially in t_g, by (3.8) and Lemma 7.1 we may impose the truncation $|t_g| \le (q|D|Y|h|)^\varepsilon$ with an error term which is $O((q|D|Y|h|)^{-1000})$ to obtain

$$\sum_{g\in\mathcal{B}_q} \overline{\rho_g(-h)}\check{\phi}(t_g)W_{D,g} \ll_\varepsilon \sum_{|t_g|\le(q|D|Y|h|)^\varepsilon} |\rho_g(h)||\check{\phi}(t_g)||W_{D,g}|$$

for $Y \gg 1$. By (2.1) we have

$$W_{D,g} \ll |W_{D,g,r}|.$$

Then from (6.1) we obtain

$$\sum_{g\in\mathcal{B}_q} \overline{\rho_g(-h)}\check{\phi}(t_g)W_{D,g} \ll_\varepsilon \frac{|D|^{1/4}}{q^{1/2}} \sum_{|t_g|\le(q|D|Y|h|)^\varepsilon} |\rho_g(h)||\check{\phi}(t_g)|\frac{L(g\times\chi_D,\frac{1}{2})^{1/2}L(g,\frac{1}{2})^{1/2}}{L(\mathrm{sym}^2 g,1)^{1/2}}.$$

Various applications of Hölder's inequality are possible here. For example, applying Hölder's inequality with exponents $4, 2, 4$ yields

$$\sum_{g\in\mathcal{B}_q} \overline{\rho_g(-h)}\check{\phi}(t_g)W_{D,g} \ll_\varepsilon \frac{|D|^{1/4}}{q^{1/2}} M_1^{1/4}M_2^{1/2}M_3^{1/4},$$

where

$$M_1 := \sum_{|t_g|\le(q|D|Y|h|)^\varepsilon} |\rho_g(h)|^4|\check{\phi}(t_g)|^4, \quad M_2 := \sum_{|t_g|\le(q|D|Y|h|)^\varepsilon} \frac{L(g\times\chi_D,\frac{1}{2})}{L(\mathrm{sym}^2 g,1)},$$

$$M_3 := \sum_{|t_g| \leq (q|D|Y|h|)^{\varepsilon}} L(g, \tfrac{1}{2})^2.$$

We first estimate M_3 using a variant of the approximate functional equation in Li-Young [LY, Lemma 2.4] and the following spectral large sieve inequality which was first derived by Deshouillers and Iwaniec [DI] in a slightly weaker form.

Theorem 7.4 ([IK], Theorem 7.24)**.** *Let $T \geq 1$ and $N \geq 1$. For any sequence of complex numbers $\{a_n\}_{n=1}^N$, we have*

$$\sum_{\substack{g \in \mathcal{B} \\ |t_g| \leq T}} \frac{1}{L(\mathrm{sym}^2 g, 1)} \left| \sum_{n=1}^{N} a_n \lambda_g(n) \right|^2 \ll \left(qT^2 + N \log(N)\right) \sum_{n=1}^{N} |a_n|^2. \tag{7.3}$$

By [LY, Lemma 2.4], there exists a function $W(x)$, depending on $Q := q(q|D|Y|h|)^{\varepsilon}$ and ε only, such that $W(x)$ is supported on $x \leq Q^{1/2+\varepsilon}$ and satisfies the bound

$$x^j W^{(j)} \ll 1,$$

where the implied constant depends on j and ε only (not on Q), and

$$\left|L(g, \tfrac{1}{2})\right|^2 \ll Q^{\varepsilon} \int_{-\log(Q)}^{\log(Q)} \left| \sum_{n \geq 1} \frac{\lambda_g(n)}{n^{\frac{1}{2}+iv}} W(n) \right|^2 dv + O(Q^{-100}), \tag{7.4}$$

where the implied constant depends on ε, W, and the degree of $L(g, s)$ only.

We insert (7.4) into the average over g with $|t_g| \ll (q|D|Y|h|)^{\varepsilon}$, then apply the spectral large sieve inequality (7.3) and the bound (see [I2])

$$L(\mathrm{sym}^2 g, 1) \ll_{\varepsilon} (|t_g| q)^{\varepsilon}$$

to obtain

$$M_3 \ll_{\varepsilon} q(q|D|Y|h|)^{\varepsilon}.$$

We next estimate M_2 in two different ways. If we apply the Blomer-Harcos [BH] bound

$$L(g \times \chi_D, 1/2) \ll_{\varepsilon} q^{1/2} |D|^{3/8} (q|D|Y|h|)^{\varepsilon}$$

for $|t_g| \leq (q|D|Y|h|)^{\varepsilon}$, the bound (3.7), and multiply by the number of newforms which is $\ll q(q|D|Y|h|)^{2\varepsilon}$, we obtain

$$M_2 \ll_{\varepsilon} (q|D|Y|h|)^{\varepsilon} q^{3/2} |D|^{3/8}.$$

On the other hand, if we use the following estimate proved in Liu-Masri-Young [LMY, Theorem 1.5] for q a prime with $(q, D) = 1$ and $M \geq 1$,

$$\sum_{|t_g| \leq M} \frac{L(g \times \chi_D, \frac{1}{2})}{L(\mathrm{sym}^2 g, 1)} \ll_{\varepsilon} (qM^2 + |D|^{1/2})(|D|Mq)^{\varepsilon}, \tag{7.5}$$

we obtain

$$M_2 \ll_{\varepsilon} (q|D|Y|h|)^{\varepsilon} (q + |D|^{1/2}).$$

Taken together, these estimates yield

$$M_2 \ll_{\varepsilon} (q|D|Y|h|)^{\varepsilon} \min(q^{3/2}|D|^{3/8}, q + |D|^{1/2}).$$

In Proposition 8.2, we will prove that

$$M_1 \ll_{\varepsilon} |h|^{2+\varepsilon} \left(q^{-1+\varepsilon} Y^2 \log^4(Y+6) + q^{-2+\varepsilon} Y^2 (Y + Y^{-1} + \log^4(Y+6)) \log^2(Y+6) \right).$$

Now, if we instead apply Hölder's inequality with exponents $2, 4, 4$, we get

$$\sum_{g\in\mathcal{B}_q} \overline{\rho_g(-h)}\check{\phi}(t_g)W_{D,g} \ll_\varepsilon \frac{|D|^{1/4}}{q^{1/2}} N_1^{1/2}N_2^{1/4}N_3^{1/4},$$

where

$$N_1 := \sum_{|t_g|\le(q|D|Y|h|)^\varepsilon} |\rho_g(h)|^2|\check{\phi}(t_g)|^2, \quad N_2 := \sum_{|t_g|\le(q|D|Y|h|)^\varepsilon} \frac{L(g\times\chi_D,\frac{1}{2})^2}{L(\mathrm{sym}^2 g,1)},$$

$$N_3 := \sum_{|t_g|\le(q|D|Y|h|)^\varepsilon} \frac{L(g,\frac{1}{2})^2}{L(\mathrm{sym}^2 g,1)}.$$

We estimate N_3 and N_2 using the spectral large sieve inequality as above to obtain

$$N_3 \ll_\varepsilon q(q|D|Y|h|)^\varepsilon$$

and

$$N_2 \ll_\varepsilon (q+q^{1/2}|D|)(q|D|Y|h|)^\varepsilon.$$

Note that to estimate N_2, we take $Q := q|D|^2(q|D|Y|h|)^\varepsilon$.

In Proposition 8.1, we will prove that

$$N_1 \ll_\varepsilon Y\log^2(Y+6) + |h|^{1+\varepsilon}q^{-1+\varepsilon}Y\log^2(Y+6)(Y^{1/2}+Y^{-1/2}).$$

Finally, by combining the estimates for M_1, M_2 and M_3 (resp. N_1, N_2 and N_3) and substituting the bounds $Y \asymp X|D|^{-1/2}$ and $q < X$, we obtain (7.2) after a tedious but straightforward calculation.

□

8. Application of the Kuznetsov formula

We begin by showing how to quickly deduce an estimate for N_1 from [DFI, eq. (23)].

Proposition 8.1. *For $Y \gg 1$ we have*

$$N_1 \ll_\varepsilon Y\log^2(Y+6) + |h|^{1+\varepsilon}q^{-1+\varepsilon}Y\log^2(Y+6)(Y^{1/2}+Y^{-1/2}). \tag{8.1}$$

Proof. Let

$$H(t) := \frac{1}{\cosh(\pi t)}h(t)(Y^{2it}+Y^{-2it}+L^2),$$

where $h(t) = 3/(1+t^2)(4+t^2)$ and $L = \log(Y+Y^{-1}+6)$. Define the sum

$$S := \sum_{g\in\mathcal{B}} H(t_g)\,|\rho_g(h)|^2 + \sum_{\mathfrak{a}} \int_{-\infty}^{\infty} H(t)\,|\tau_{\mathfrak{a}}(h,t)|^2\,\frac{dt}{4\pi}.$$

Duke, Friedlander and Iwaniec [DFI, eq. (23)] used the Kuznetsov formula to prove that

$$S \ll L^2 + L^2(Y+Y^{-1})^{1/2}h^{1/2}q^{-1}(h,q)^{1/2}\tau(hq). \tag{8.2}$$

By Lemma 7.1 and positivity, we have

$$N_1 := \sum_{\substack{g\in\mathcal{B}_q\\ |t_g|\le(q|D|Y|h|)^\varepsilon}} |\rho_g(h)|^2|\check{\phi}(t_g)|^2 \ll YS$$

for $Y \gg 1$. The estimate (8.1) now follows immediately from (8.2). □

In the following proposition we adapt the argument in [DFI, pp. 430-431] to estimate the fourth moment M_1.

Proposition 8.2. *For $Y \gg 1$ we have*

$$M_1 \ll_\varepsilon |h|^{2+\varepsilon}\left(q^{-1+\varepsilon}Y^2\log^4(Y+6)+q^{-2+\varepsilon}Y^2(Y+Y^{-1}+\log^4(Y+6))\log^2(Y+6)\right). \qquad (8.3)$$

Proof. First assume that $(h,q)=1$. Using the Hecke relations ([I, eq. (8.39)])

$$\lambda_g(m)\lambda_g(n)=\sum_{\ell|(m,n)}\lambda_g(mn/\ell^2),$$

a short calculation yields

$$|\rho_g(h)|^4=|\rho_g(1)|^2\sum_{d|h}\sum_{k|h}\rho_g(h^2/d^2)\overline{\rho_g(h^2/k^2)}.$$

Since (see (3.5))

$$|\rho_g(1)|^2 \asymp q^{-1+\varepsilon}e^{\pi|t_g|},$$

by Lemma 7.1 and positivity we have

$$M_1 := \sum_{\substack{g\in\mathcal{B}_q\\ |t_g|\le(q|D|Y|h|)^\varepsilon}}|\rho_g(h)|^4|\check{\phi}(t_g)|^4 \ll_\varepsilon q^{-1+\varepsilon}Y^2\sum_{d|h}\sum_{k|h}\sum_{g\in\mathcal{B}}\rho_g(h^2/d^2)\overline{\rho_g(h^2/k^2)}H^*(t_g)$$

for $Y \gg 1$, where

$$H^*(t) := \frac{1}{\cosh(\pi t)}(1+t^4)^{-1}(Y^{4it}+Y^{-4it}+\log^4(Y+6)).$$

The function $H^*(t)$ clearly satisfies the conditions in [IK, eq. (15.19)]. Therefore by the Kuznetsov formula [IK, Theorem 16.3], we have

$$\sum_{g\in\mathcal{B}}\rho_g(h^2/d^2)\overline{\rho_g(h^2/k^2)}H^*(t_g)+\sum_{\mathfrak{a}}\int_{-\infty}^{\infty}\tau_{\mathfrak{a}}(h^2/d^2,t)\overline{\tau_{\mathfrak{a}}(h^2/k^2,t)}H^*(t)\frac{dt}{4\pi}$$
$$=\delta(h^2/d^2,h^2/k^2)H_0^*+\sum_{c\equiv 0\ (\mathrm{mod}\ q)}c^{-1}S(h^2/d^2,h^2/k^2;c)\widetilde{H^*}\left(\frac{4\pi h^2}{dkc}\right),$$

where $S(m,n;c)$ is the Kloosterman sum,

$$H_0^* := \frac{1}{\pi^2}\int_{-\infty}^{\infty}\sinh(\pi t)H^*(t)t\,dt$$

and

$$\widetilde{H^*}(x) := \frac{2i}{\pi}\int_{-\infty}^{\infty}J_{2it}(x)H^*(t)t\,dt.$$

Estimating trivially yields

$$H_0^* \ll \log^4(Y+6).$$

We next estimate the sum of Kloosterman sums. Make the change of variables $s=1/2+it$ and shift the line of integration to $\sigma+it$ with $1/2<\sigma<1$ to obtain

$$\widetilde{H^*}(x)=\frac{2i}{\pi}\int_{(\sigma)}J_{2s-1}(x)H^*\left((s-1/2)i\right)(s-1/2)\,ds.$$

By [GR, eq. 8.411.6] and Stirling's formula,

$$J_{2s-1}(x) \ll e^{\pi|s|}x^{2\sigma-1}.$$

We also have

$$H^*((s-1/2)i) \ll |s|^{-4}e^{-\pi|s|}\left(Y^{2-4\sigma}+Y^{4\sigma-2}+\log^4(Y+6)\right).$$

These estimates then yield

$$\widetilde{H^*}(x) \ll x^{2\sigma-1}\left(Y^{2-4\sigma}+Y^{4\sigma-2}+\log^4(Y+6)\right). \tag{8.4}$$

By [IK, eq. (16.50)], we have

$$\sum_{c\equiv 0\,(\mathrm{mod}\ q)} c^{-1-\omega}|S(m,n;c)| \ll (\omega-1/2)^{-2}\tau((m,n))(m,n,q)^{1/2}\tau(q)q^{-\frac{1}{2}-\omega} \tag{8.5}$$

for $1/2<\omega\le 1$. Then from (8.4) and (8.5), we obtain

$$\begin{aligned}&\sum_{c\equiv 0\,(\mathrm{mod}\ q)} c^{-1}S(h^2/d^2,h^2/k^2;c)\widetilde{H^*}\left(\frac{4\pi h^2}{dkc}\right)\\ &\ll \left(\frac{h^2}{dk}\right)^{2\sigma-1}\left(Y^{4\sigma-2}+Y^{2-4\sigma}+\log^4(Y+6)\right)\\ &\qquad\qquad \times(\sigma-3/4)^{-2}\tau((h^2/d^2,h^2/k^2))(h^2/d^2,h^2/k^2,q)^{1/2}\tau(q)q^{\frac{1}{2}-2\sigma}\\ &\ll |h|^{2+\epsilon}q^{-1+\varepsilon}\left(Y+Y^{-1}+\log^4(Y+6)\right)\log^2(Y+6),\end{aligned}$$

where for the last inequality we substituted

$$\sigma=\frac{3}{4}+\frac{1}{2\log(Y+6)}.$$

By (3.11),

$$\tau_{\mathfrak{a}}(h^2/\ell^2,t)\ll_\varepsilon |ht|^\varepsilon e^{\frac{\pi}{2}|t|},$$

hence we obtain

$$\sum_{\mathfrak{a}}\int_{-\infty}^{\infty}\tau_{\mathfrak{a}}(h^2/d^2,t)\overline{\tau_{\mathfrak{a}}(h^2/k^2,t)}H^*(t)\frac{dt}{4\pi}\ll_\varepsilon |h|^\varepsilon\log^4(Y+6).$$

Combining the preceding estimates yields (8.3) for $(h,q)=1$.

If $(h,q)>1$, write $h=q^\alpha n$ for some integer $\alpha\ge 1$ and integer n coprime to q. Then by the Hecke relations ([I, (8.39)]) $\lambda_g(q^\alpha n)=\lambda_g(q^\alpha)\lambda_g(n)=\lambda_g(q)^\alpha\lambda_g(n)$ and $\lambda_g(q)=\pm q^{-1/2}$ (see (3.3)), we have

$$|\rho_g(h)|^4=q^{-2\alpha}|\rho_g(n)|^4,$$

and we reduce to the case already considered (with a sharper bound in q). □

9. Contribution of the continuous spectrum

In this section we estimate the sum over the cusps $\mathfrak{a}$ in (5.4).

Lemma 9.1. *For $X\gg|D|^{1/2}$ we have*

$$\sum_{\mathfrak{a}}\int_{-\infty}^{\infty}\overline{\tau_{\mathfrak{a}}(-h,t)}\check{\phi}(t)W_{D,\mathfrak{a}}(t)\frac{dt}{4\pi}\ll_\varepsilon (q|D|X|h|)^\varepsilon q^{-1/2}X^{1/2}|D|^{1/6}.$$

Proof. Since $W_{D,\mathfrak{a}}(t)$ grows polynomially in t, by (3.11) and Lemma 7.1 we may impose the truncation $|t|\le(q|D|Y|h|)^\varepsilon$ with an error term which is $O((q|D|Y|h|)^{-1000})$ to obtain

$$\int_{-\infty}^{\infty}\overline{\tau_{\mathfrak{a}}(-h,t)}\check{\phi}(t)W_{D,\mathfrak{a}}(t)\frac{dt}{4\pi}\ll_\varepsilon (q|D|Y|h|)^\varepsilon Y^{1/2}\log(Y+6)\int_{|t|\le(q|D|Y|h|)^\varepsilon}|W_{D,\mathfrak{a}}(t)|dt$$

for $Y\gg 1$. By (6.3), the convexity bound for $\zeta(1/2+it)$, the Conrey-Iwaniec [CI] bound

$$L(\chi_D,1/2+it)\ll_\varepsilon |D|^{1/6}(q|D|Y|h|)^\varepsilon$$

for $|t| \le (q|D|Y|h|)^{\varepsilon}$, and a standard lower bound for $|\zeta(1+2it)|$, we obtain

$$W_{D,\mathfrak{a}}(t) \ll_{\varepsilon} q^{-1/2}|D|^{5/12}(q|D|Y|h|)^{\varepsilon}.$$

Thus

$$\int_{-\infty}^{\infty} \overline{\tau_{\mathfrak{a}}(-h,t)}\check{\phi}(t)W_{D,\mathfrak{a}}(t)\frac{dt}{4\pi} \ll_{\varepsilon} Y^{1/2}\log(Y+6)q^{-1/2}|D|^{5/12}(q|D|Y|h|)^{\varepsilon}.$$

Upon substituting the bound $Y \asymp X|D|^{-1/2}$, we complete the proof. □

10. Proof of Theorem 1.1

Theorem 1.1 follows by combining the spectral decomposition (5.4) with the estimates in Lemmas 7.2, 7.3 and 9.1.

11. Proofs of the Corollaries

The corollaries can be proved by adapting the arguments in [DFI2, Sections 12 and 14].

Proof of Corollaries 1.2 and 1.4. The following result implies Corollaries 1.2 and 1.4.

Proposition 11.1. *Let D, q and f be as in Theorem 1.1. Let $\phi : \mathbb{R}_{\ge 0} \to \mathbb{C}$ be a C^{∞} function supported on a fixed subinterval $I \subset [0,1]$ of length $\ell(I) > 0$ and define*

$$N_{\phi}(f,D,q) := \sum_{c \equiv 0 \,(\mathrm{mod}\ q)} f(c) \sum_{\substack{b \,(\mathrm{mod}\ 2c) \\ b^2 \equiv D \,(\mathrm{mod}\ 4c)}} \phi\left(\frac{b}{2c}\right).$$

For $X \gg |D|^{1/2}$, we have

$$N_{\phi}(f,D,q) = \hat{f}(0)\hat{\phi}(0)\frac{12}{\pi^2}\frac{L(\chi_D,1)}{q+1} + O_{\varepsilon}\left((q|D|X)^{\varepsilon}\min\{A(X,D),B(X,D,q),C(X,D,q)\}\right),$$

where

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(u)e(\xi u)du$$

(resp. $\hat{\phi}$) is the Fourier transform of f (resp. ϕ).

Proof. By Poisson summation, we have

$$\phi(x) = \sum_{h \in \mathbb{Z}} \hat{\phi}(h)e(hx).$$

Integrating by parts A-times yields

$$\hat{\phi}(h) = \int_{-\infty}^{\infty} \phi(u)e(hu)du \ll |h|^{-A}.$$

Then by Poisson summation and Theorem 1.1, we have

$$\begin{aligned} N_{\phi}(f,D,q) &= \sum_{h \in \mathbb{Z}} \hat{\phi}(h)W_h(f,D,q) \\ &= \hat{\phi}(0) \sum_{c \equiv 0 \,(\mathrm{mod}\ q)} f(c)\rho(c) + O_{\varepsilon}\left((q|D|X)^{\varepsilon}\min\{A(X,D),B(X,D,q),C(X,D,q)\}\right), \end{aligned}$$

where

$$\rho(c) := \#\{b \,(\mathrm{mod}\ 2c) : \ b^2 \equiv D \,(\mathrm{mod}\ 4c)\}.$$

Define the L–series

$$L(s) := \sum_{c\equiv 0 \,(\mathrm{mod}\ q)} \rho(c)c^{-s}.$$

By Lemma 11.2, we have

$$L(s) = \frac{2}{1+q^s}\frac{\zeta(s)L(\chi_D,s)}{\zeta(2s)}.$$

The function $L(s)$ has a simple pole at $s=1$ with residue

$$R := \frac{12}{\pi^2}\frac{L(\chi_D,1)}{q+1}.$$

By Mellin inversion,

$$\sum_{c\equiv 0 \,(\mathrm{mod}\ q)} f(c)\rho(c) = \frac{1}{2\pi i}\int_{(\sigma)} \tilde{f}(s)L(s)ds \quad \text{for} \quad \sigma>1,$$

where

$$\tilde{f}(s) := \int_0^\infty f(u)u^{s-1}du$$

is the Mellin transform of f. Shifting the contour to (1/2), we pick up the residue R to obtain

$$\sum_{c\equiv 0 \,(\mathrm{mod}\ q)} f(c)\rho(c) = \hat{f}(0)\frac{12}{\pi^2}\frac{L(\chi_D,1)}{q+1} + \frac{1}{2\pi i}\int_{(1/2)} \tilde{f}(s)\frac{\zeta(s)L(\chi_D,s)}{\zeta(2s)}\frac{2}{1+q^s}ds.$$

Integrating by parts A-times yields

$$\tilde{f}(s) \ll X^{1/2}|s|^{-A} \quad \text{for} \quad \sigma = 1/2.$$

Then using the Conrey-Iwaniec [CI] bound

$$L(\chi_D, 1/2+it) \ll_\varepsilon (1+|t|)^B|D|^{1/6+\varepsilon},$$

we obtain

$$\sum_{c\equiv 0 \,(\mathrm{mod}\ q)} f(c)\rho(c) = \hat{f}(0)\frac{12}{\pi^2}\frac{L(\chi_D,1)}{q+1} + O_\varepsilon(q^{-1/2}X^{1/2}|D|^{1/6+\varepsilon}).$$

□

The following lemma is a minor variant of the classical formula

$$\sum_{c=1}^{\infty}\rho(c)c^{-s} = \frac{\zeta(s)L(\chi_D,s)}{\zeta(2s)}$$

due to Dirichlet (see e.g. [Z, Proposition 3, (i)]).

Lemma 11.2. *For D and q as in Theorem 1.1, we have*

$$L(s) = \frac{2}{1+q^s}\frac{\zeta(s)L(\chi_D,s)}{\zeta(2s)}.$$

Proof. Observe that

$$L(s) = \sum_{c\equiv 0 \,(\mathrm{mod}\ q)} \rho(c)c^{-s} = L_q(s)\prod_{p\nmid qD}L_p(s)\prod_{p\mid D}L_p(s),$$

where

$$L_q(s) := \frac{\rho(q)}{q^s} + \frac{\rho(q^2)}{q^{2s}} + \cdots$$

and

$$L_p(s) := 1 + \frac{\rho(p)}{p^s} + \frac{\rho(p^2)}{p^{2s}} + \cdots .$$

A calculation yields

$$L_q(s) = q^{-s}(1+\chi_D(q))(1-q^{-s})^{-1}$$

and

$$L_p(s) = \begin{cases} (1-p^{-s})^{-1}(1+\chi_D(p)p^{-s}), & \text{if} \quad p \nmid qD \\ (1-p^{-s})^{-1}(1-p^{-2s}), & \text{if} \quad p|D. \end{cases}$$

The result now follows. □

Proof of Corollary 1.5. The following result implies Corollary 1.5.

Proposition 11.3. *Let D, q and f be as in Theorem 1.1. Let $I_D \subset [0,1]$ be a subinterval of length $\ell(I_D) = |D|^{-\eta}$ for some $\eta > 0$ and let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{C}$ be a C^∞ function supported on I_D which satisfies*

$$\phi^{(j)} \ll |D|^{\eta j}, \quad j = 0, 1, \ldots$$

For $X \gg |D|^{1/2}$, we have

$$\begin{aligned} N_\phi(f,D,q) = \hat{f}(0)\hat{\phi}(0)\frac{12}{\pi^2}\frac{L(\chi_D,1)}{q+1} \\ + O_\varepsilon\left(|D|^{2\eta}(q|D|X)^\varepsilon \min\{A(X,D), B(X,D,q), C(X,D,q)\}\right). \end{aligned}$$

Proof. Integrating by parts A-times yields

$$\hat{\phi}(h) \ll \left(\frac{|D|^\eta}{|h|}\right)^A .$$

Then proceeding as in the proof of Proposition 11.1, we have

$$\begin{aligned} N_\phi(f,D,q) &= \sum_{h\in\mathbb{Z}} \hat{\phi}(h) W_h(f,D,q) \\ &= \hat{\phi}(0) \sum_{c\equiv 0 \pmod q} f(c)\rho(c) + \sum_{|h|\leq |D|^{\eta+\varepsilon}} \hat{\phi}(h)W_h(f,D,q) + O(|D|^{-1000}) \\ &= \hat{f}(0)\hat{\phi}(0)\frac{12}{\pi^2}\frac{L(\chi_D,1)}{q+1} \\ &\quad + O_\varepsilon\left(|D|^{2\eta}(q|D|X)^\varepsilon \min\{A(X,D), B(X,D,q), C(X,D,q)\}\right). \end{aligned}$$

□

Proof of Corollary 1.6. Let $\{\omega_\ell\}_{\ell=0}^\infty$ be a smooth partition of unity such that each constituent ω_ℓ is supported on $[Y_\ell, 2Y_\ell]$ with $Y_\ell := 2^{\ell/2}\sqrt{|D|}$ and satisfies

$$\omega_\ell^{(j)}(y) \ll y^{-j}, \quad j = 0, 1, \ldots$$

Then the function

$$f_\ell(y) := \sinh\left(\frac{\pi h\sqrt{|D|}}{y}\right)\omega_\ell(y)$$

satisfies

$$f_\ell^{(j)}(y) \ll \frac{\sqrt{|D|}}{y} y^{-j}, \quad j = 0, 1, \ldots$$

Let $f = f_\ell$ and $X = Y_\ell$ in Theorem 1.1, then bound the minimum by $A(Y_\ell, D)$ to obtain

$$\sum_{\substack{c \geq \sqrt{|D|} \\ c \equiv 0 \,(\mathrm{mod}\, q)}} W_h(D; c) \sinh\left(\frac{\pi h\sqrt{|D|}}{c}\right) \ll \sum_{\ell=0}^{\infty} |W_h(f_\ell, D, q)| \ll_\varepsilon \sum_{\ell=0}^{\infty} \frac{\sqrt{|D|}}{Y_\ell} Y_\ell^{3/4} |D|^{1/16} (q|D|Y_\ell)^\varepsilon$$

$$\ll_\varepsilon |D|^{7/16+\varepsilon}.$$

□

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