A KRONECKER LIMIT FORMULA FOR TOTALLY REAL FIELDS AND ARITHMETIC APPLICATIONS

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Abstract. We establish a Kronecker limit formula for the zeta function $\zeta_F(s, A)$ of a wide ideal class A of a totally real number field F of degree n. This formula relates the constant term in the Laurent expansion of $\zeta_F(s, A)$ at $s = 1$ to a toric integral of a $SL_n(\mathbb{Z})$-invariant function $\log G(Z)$ along a Heegner cycle in the symmetric space of $GL_n(\mathbb{R})$. We give several applications of this formula to algebraic number theory, including a relative class number formula for H/F where H is the Hilbert class field of F, and an analog of Kronecker's solution of Pell's equation for totally real multiquadratic fields. We also use a well-known conjecture from transcendence theory on algebraic independence of logarithms of algebraic numbers to study the transcendence of the toric integral of $\log G(Z)$. Explicit examples are given for each of these results.

1. Introduction and statement of results

The celebrated Kronecker limit formula expresses the constant term in the Laurent expansion at $s = 1$ of the Dedekind zeta function $\zeta_K(s, A)$ of an ideal class A of an imaginary quadratic field K in terms of the value of $\log|\eta(z)|$ at a Heegner point τ_A in the complex upper half-plane $\mathbb{H}$ where $\eta(z)$ is the Dedekind eta function

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad q := e^{2\pi i z}.$$

There are many interesting applications of this formula to algebraic number theory, including relative class number formulae and Kronecker's "solution" to Pell's equation (see e.g. [S, Chapter II]). Roughly speaking, to prove the Kronecker limit formula, one computes the constant term in the Laurent expansion at $s = 1$ of the SL_2-Eisenstein series

$$E(z, s) := \sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} \operatorname{Im}(\gamma z)^s, \quad z \in \mathbb{H}, \quad \operatorname{Re}(s) > 1$$

then appeals to a classical identity of Dirichlet/Hecke relating $\zeta_K(s, A)$ to the value of $E(z, s)$ at the Heegner point τ_A. Note that Hecke proved a similar limit formula for real quadratic fields by relating the ideal class zeta function to an integral of $E(z, s)$ over a Heegner cycle in $\mathbb{H}$.

In the early 1980's, Bump and Goldfeld [BG] proved a Kronecker limit formula for real cubic fields. This was based on an intriguing identity relating the integral of a minimal parabolic SL_3-Eisenstein series over a Heegner cycle to the Rankin/Selberg integral of a Hilbert modular Eisenstein series. Kudla [K] showed this identity was an instance of the so-called "basic identity" associated to a see-saw dual reductive pair. Efrat [E] later gave a Kronecker limit formula for non-real cubic fields by instead using the maximal parabolic

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SL_3-Eisenstein series. These results are just the tip of the iceberg. Indeed, as Bump and Goldfeld [BG] remarked,

> "*Much work remains to be done in this direction, and one can only begin to see a whole new world of limit formulae emerging into view*".

In this paper we will prove a Kronecker limit formula for the zeta function $\zeta_F(s, A)$ of a wide ideal class A of a totally real number field F of degree $n \geq 2$, thereby extending the limit formulae of Hecke and Bump/Goldfeld (see Theorem 1.6). This formula relates the constant term in the Laurent expansion of $\zeta_F(s, A)$ at $s = 1$ to a toric integral of a $SL_n(\mathbb{Z})$-invariant function $\log G(Z)$ along a Heegner cycle in the symmetric space of $GL_n(\mathbb{R})$. We will give some applications of the limit formula to algebraic number theory, including a relative class number formula for H/F where H is the Hilbert class field of F, and an analog of Kronecker's solution of Pell's equation for totally real multiquadratic fields (see Theorems 1.8 and 1.9). We will also use a well-known conjecture from transcendence theory on algebraic independence of logarithms of algebraic numbers to study the transcendence of the toric integral of $\log G(Z)$ (see Corollary 1.10). Explicit examples of these results are given in Section 2.

To prove the limit formula we will generalize the method of Efrat [E]. New difficulties arise when working with the maximal parabolic SL_n-Eisenstein series for arbitrary $n \geq 2$, though many of these may be overcome by appealing to work of Friedberg [F], Goldfeld [G] and Terras [T]. A key step in the proof is an identity relating $\zeta_F(s, A)$ to a toric integral of the maximal parabolic SL_n-Eisenstein series along a Heegner cycle in the symmetric space of $GL_n(\mathbb{R})$.

In order to state our results we fix the following notation. Let $\mathcal{H}^n = GL_n(\mathbb{R})/O_n(\mathbb{R})\mathbb{R}^*$ be the symmetric space of $GL_n(\mathbb{R})$. By the Iwasawa decomposition, each coset $Z \in \mathcal{H}^n$ has a unique representative of the form

$$Z = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix} \tag{1.1}$$

where $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y_i \in \mathbb{R}_+$ for $1 \leq i \leq n-1$. Left matrix multiplication induces an action of $GL_n(\mathbb{Z})$ on $\mathcal{H}^n$. For more details concerning these facts, see [G, Section 1.2].

Let P be the maximal parabolic subgroup of $SL_n(\mathbb{Z})$, which consists of those matrices with bottom row $(0, \ldots, 0, 1)$. Define the maximal parabolic Eisenstein series

$$E_n(Z, s) := \sum_{\gamma \in P \backslash SL_n(\mathbb{Z})} \mathrm{Det}(\gamma \cdot Z)^s, \quad \mathrm{Re}(s) > 1$$

where $\mathrm{Det}(\gamma \cdot Z)$ is the determinant of the unique representative of the coset $\gamma \cdot Z \in \mathcal{H}^n$ of the form (1.1) and $s \in \mathbb{C}$. Note that $E_n(Z, s)$ is well-defined since $\mathrm{Det}(p \cdot Z) = \mathrm{Det}(Z)$ for all $p \in P$.

The completed Eisenstein series

$$E_n^*(Z, s) := \pi^{-ns/2} \Gamma(ns/2) \zeta(ns) E_n(Z, s)$$

satisfies the functional equation

$$E_n^*(Z,s) = E_n^*((Z^T)^{-1}, 1-s)$$

and extends to a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0, 1$.

Given $Z \in \mathcal{H}^n$, we may write

$$ZZ^T = \begin{pmatrix} m & \mathbf{r}^T \\ \mathbf{r} & S \end{pmatrix}$$

where

$$m = (y_1 y_2 + \cdots y_{n-1})^2 + (x_{1,2} y_1 y_2 \cdots y_{n-2})^2 + (x_{1,3} y_1 y_2 \cdots y_{n-3})^2 + \cdots + x_{1,n}^2,$$

$$\mathbf{r} = Z_1 \begin{pmatrix} x_{1,2} y_1 y_2 \cdots y_{n-2} \\ x_{1,3} y_1 y_2 \cdots y_{n-3} \\ \vdots \\ x_{1,n} \end{pmatrix}$$

with

$$Z_1 = \begin{pmatrix} 1 & x_{2,3} & \dots & x_{2,n} \\ & \ddots & & \vdots \\ & & 1 & x_{n-1,n} \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \cdots y_{n-2} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix},$$

and $S = Z_1 Z_1^T$. Let

$$\mathbf{q} = S^{-1}\mathbf{r}.$$

We will prove the following result.

Proposition 1.1. *The Laurent expansion of $E_n^*(Z,s)$ at $s = 1$ is given by*

$$E_n^*(Z,s) = \frac{2/n}{s-1} + \gamma - \log(4\pi) - \frac{2}{n}\log(y_1 y_2^2 \cdots y_{n-1}^{n-1}) - 4\log g(Z) + O(|s-1|),$$

where γ is Euler's constant and

$$g(Z) := \exp\left(-\frac{(y_1 y_2^2 \cdots y_{n-1}^{n-1})^{1/(n-1)} E_{n-1}^*(Z_1, n/(n-1))}{4}\right) \times \prod_{\substack{\mathbf{b} \in \mathbb{Z}^{n-1} \\ \mathbf{b} \pmod{\pm 1} \\ \mathbf{b} \neq \mathbf{0}}} \left|1 - \exp\left(-2\pi w^{1/2} (\mathbf{b}^T S^{-1} \mathbf{b})^{1/2} + 2\pi i \mathbf{b}^T \mathbf{q}\right)\right|.$$

Remark 1.2. The function $g(Z)$ is a GL_n analog of $|\eta(z)|$ which generalizes the GL_3 analog defined in [E, p. 175].

Let F be a totally real number field of degree n and U be the group of units of F. Let A be a wide ideal class of F and define the ideal class zeta function

$$\zeta_F(s,A) := \sum_{\substack{\mathfrak{A} \in A \\ \mathfrak{A} \neq 0}} \frac{1}{N(\mathfrak{A})^s}, \quad \mathrm{Re}(s) > 1$$

where $N(\mathfrak{A})$ is the norm. The completed zeta function is defined by

$$\zeta_F^*(s,A) := \pi^{-ns/2}\Gamma(s/2)^n D_F^{s/2}\zeta_F(s,A),$$

where D_F is the discriminant of F. The function $\zeta_F^*(s,A)$ satisfies the functional equation

$$\zeta_F^*(s,A) = \zeta_F^*(1-s,A)$$

and extends to meromorphic function on $\mathbb{C}$ with a simple pole at $s=1$.

We will calculate the constant term in the Laurent expansion of $\zeta_F^*(s,A)$ at $s=1$ by relating $\zeta_F^*(s,A)$ to a toric integral of $E_n^*(Z,s)$ along a Heegner cycle in $\mathcal{H}^n$ associated to A and appealing to Proposition 1.1.

Fix an ideal $\mathfrak{B}\in A^{-1}$. Let $\alpha_1,\alpha_2,\dots,\alpha_n$ be a $\mathbb{Z}$-basis for $\mathfrak{B}$ and $\alpha_1^{(i)},\alpha_2^{(i)},\dots,\alpha_n^{(i)}$ for $i=1,2,\dots,n$ denote their images under the real embeddings of F. Define the matrix

$$M_{\mathfrak{B}}(\mathbf{t}) := \begin{pmatrix} \alpha_1^{(1)}t_1 & \alpha_1^{(2)}t_2 & \cdots & \alpha_1^{(n-1)}t_{n-1} & \alpha_1^{(n)}(t_1t_2\cdots t_{n-1})^{-1} \\ \alpha_2^{(1)}t_1 & \alpha_2^{(2)}t_2 & \cdots & \alpha_2^{(n-1)}t_{n-1} & \alpha_2^{(n)}(t_1t_2\cdots t_{n-1})^{-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_n^{(1)}t_1 & \alpha_n^{(2)}t_2 & \cdots & \alpha_n^{(n-1)}t_{n-1} & \alpha_n^{(n)}(t_1t_2\cdots t_{n-1})^{-1} \end{pmatrix}$$

where $\mathbf{t}=(t_1,t_2,\dots,t_{n-1})\in\mathbb{R}_+^{n-1}$, and let

$$Q_{\mathfrak{B}}(\mathbf{t}) := M_{\mathfrak{B}}(\mathbf{t})M_{\mathfrak{B}}(\mathbf{t})^T.$$

The positive definite, symmetric matrix $Q_{\mathfrak{B}}(\mathbf{t})$ may be written as

$$Q_{\mathfrak{B}}(\mathbf{t}) = \mathrm{Det}(Q_{\mathfrak{B}}(\mathbf{t}))^{1/n}(y_1(\mathbf{t})^{n-1}y_2(\mathbf{t})^{n-2}\cdots y_{n-1}(\mathbf{t}))^{-2/n}\tau_{\mathfrak{B}}(\mathbf{t})\tau_{\mathfrak{B}}(\mathbf{t})^T,$$

where

$$\tau_{\mathfrak{B}}(\mathbf{t}) = \begin{pmatrix} 1 & x_{1,2}(\mathbf{t}) & x_{1,3}(\mathbf{t}) & \dots & x_{1,n}(\mathbf{t}) \\ & 1 & x_{2,3}(\mathbf{t}) & \dots & x_{2,n}(\mathbf{t}) \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n}(\mathbf{t}) \\ & & & & 1 \end{pmatrix}\begin{pmatrix} y_1(\mathbf{t})y_2(\mathbf{t})\cdots y_{n-1}(\mathbf{t}) & & & & \\ & y_1(\mathbf{t})y_2(\mathbf{t})\cdots y_{n-2}(\mathbf{t}) & & & \\ & & \ddots & & \\ & & & y_1(\mathbf{t}) & \\ & & & & 1 \end{pmatrix}$$

is in $\mathcal{H}^n$. Here we have suppressed the dependence of the variables $x_{i,j}(\mathbf{t})$ and $y_i(\mathbf{t})$ on $\mathfrak{B}$ and the $\mathbb{Z}$-basis $\alpha_1,\alpha_2,\dots,\alpha_n$.

Given a unit $\varepsilon\in U$, let $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n$ denote the images of ε under the real embeddings of F. There is an action of the unit group U on $\mathbb{R}_+^{n-1}$ given by

$$\begin{aligned} \varepsilon : \mathbb{R}_+^{n-1} &\longrightarrow \mathbb{R}_+^{n-1}, \\ (t_1,t_2,\dots,t_{n-1}) &\longmapsto (|\varepsilon_1|t_1,|\varepsilon_2|t_2,\dots,|\varepsilon_{n-1}|t_{n-1}). \end{aligned}$$

Let $\mathbb{R}_+^{n-1}/U$ denote a fundamental domain for this action. Then $\{\tau_{\mathfrak{B}}(\mathbf{t}): \ \mathbf{t}\in\mathbb{R}_+^{n-1}/U\}$ defines a Heegner cycle in $\mathcal{H}^n$.

We will establish the following identity.

Proposition 1.3. *Let F be a totally real number field of degree n and A be a wide ideal class of F. Then*

$$\zeta_F^*(s,A) = n2^{n-1}\int\!\!\cdots\!\!\int_{\mathbb{R}_+^{n-1}/U} E_n^*(\tau_{\mathfrak{B}}(\mathbf{t}),s)\frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t_{n-1}} \tag{1.2}$$

where $\{\tau_{\mathfrak{B}}(\mathbf{t}): \ \mathbf{t}\in\mathbb{R}_+^{n-1}/U\}$ is the Heegner cycle associated to $\mathfrak{B}\in A^{-1}$.

Remark 1.4. Although the Heegner cycle $\{\tau_{\mathfrak{B}}(\mathbf{t}) : \mathbf{t} \in \mathbb{R}_+^{n-1}/U\}$ depends on the ideal $\mathfrak{B} \in A^{-1}$ and the choice of integral basis for $\mathfrak{B}$, the integral on the right hand side of (1.2) depends only on the ideal class A.

Remark 1.5. One can identify $\mathbb{R}_+^{n-1}$ with

$$T^n = \{(t_1, t_2, \cdots, t_{n-1}, t_n) \in \mathbb{R}_+^n \ : \ \prod_{i=1}^{n} t_i = 1\}.$$

Using this identification, (1.2) can be written as an integral over T^n/U with respect to the U-invariant measure on T^n.

By combining Propositions 1.1 and 1.3, we will obtain the following Kronecker limit formula for the zeta function of a wide ideal class of a totally real field.

Theorem 1.6. *Let F be a totally real number field of degree n and A be a wide ideal class of F. Then*

$$\lim_{s\to 1}\left[\zeta_F^*(s,A) - \frac{2^n \mathrm{vol}(\mathbb{R}_+^{n-1}/U)}{s-1}\right] = \\ n2^{n-1}\gamma \mathrm{vol}(\mathbb{R}_+^{n-1}/U) - n2^{n-1}\int\cdots\int_{\mathbb{R}_+^{n-1}/U} \log G(\tau_{\mathfrak{B}}(\mathbf{t}))\frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t_{n-1}}$$

where

$$G(Z) := 4\pi(y_1 y_2^2 \cdots y_{n-1}^{n-1})^{2/n} g(Z)^4$$

and $\{\tau_{\mathfrak{B}}(\mathbf{t}) : \mathbf{t} \in \mathbb{R}_+^{n-1}/U\}$ is the Heegner cycle associated to $\mathfrak{B} \in A^{-1}$.

Proof. We may write the Laurent expansion in Proposition 1.1 as

$$E_n^*(Z,s) = \frac{2/n}{s-1} + \gamma - \log G(Z) + O(|s-1|) \tag{1.3}$$

where $G(Z)$ is defined as in the statement of the theorem. In particular, this shows that $G(Z)$ is $SL_n(\mathbb{Z})$-invariant. Inserting (1.3) into the integral on the right hand side of (1.2) immediately yields the result. □

Remark 1.7. By Remark 1.4, the integral appearing in Theorem 1.6 depends only on the ideal class A of F. In some of the applications which follow, it will be convenient to denote this integral by

$$\rho_n(A) := \int\cdots\int_{\mathbb{R}_+^{n-1}/U} \log G(\tau_{\mathfrak{B}}(\mathbf{t}))\frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t_{n-1}}.$$

We now give some applications of Theorem 1.6 to algebraic number theory in the spirit of Siegel [S, Chapter II]. Given a number field K, let $\mathrm{CL}(K)$ be the wide ideal class group, h_K be the class number, R_K be the regulator, w_K be the number of roots of unity, and D_K be the absolute value of the discriminant.

We will prove the following relative class number formula for H/F where H is the Hilbert class field of F.

Theorem 1.8. *Let F be a totally real number field of degree n and H be the Hilbert class field of F. Write the ideal class group of F as*

$$\mathrm{CL}(F) = \{[\mathfrak{A}_1] = [\mathcal{O}_F], [\mathfrak{A}_2], \dots, [\mathfrak{A}_{h_F}]\}.$$

Then

$$\frac{(-1)^{h_F-1}2^{h_F-1}}{n^{h_F-1}}\frac{h_H}{h_F}\frac{R_H}{R_F} = \mathrm{Det}\left(\int_{\mathbb{R}_+^{n-1}/U}\!\!\cdots\int \log\left(\frac{G(\tau_{\mathfrak{A}_\ell^{-1}\mathfrak{A}_k}(\mathbf{t}))}{G(\tau_{\mathfrak{A}_\ell}(\mathbf{t}))}\right)\frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t_{n-1}}\right)_{k,\ell}$$

where $1 \le k, \ell \le h_F - 1$.

We will also prove the following analog of Kronecker's solution of Pell's equation for totally real multiquadratic fields. A result of this type for real quadratic fields is given in [S, p. 97, Proposition 13].

Theorem 1.9. *Let F be a totally real abelian number field with $\mathrm{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^\ell$. Let E be an unramified real quadratic extension of F with $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^{\ell+1}$ and $\chi_{E/F}$ be the genus character of F associated to E/F by class field theory. Then*

$$\sum_{A\in\mathrm{CL}(F)} \chi_{E/F}(A)\rho_{2^\ell}(A) = -\frac{D_F^{1/2}}{2^{\ell-1}}\prod_{i=1}^{2^\ell}\frac{\log(\varepsilon_i)h_i}{\sqrt{\Delta_i}},$$

where $\Delta_i > 0$ for $1 \le i \le 2^\ell$ are the discriminants of the quadratic subfields K_i of E which are not contained in F, ε_i is the fundamental unit of K_i, and h_i is the class number of K_i.

Next, let

$$\mathcal{L} := \{\log(\alpha) : \ \alpha \in \overline{\mathbb{Q}}^*\}$$

be the set of logarithms of algebraic numbers. The following is a well-known conjecture on algebraic independence from transcendence theory (see e.g. [W, Conjecture 1.15]).

Conjecture 1.10 (Algebraic Independence of Logarithms)**.** *If $\lambda_1, \lambda_2, \dots, \lambda_k$ are $\mathbb{Q}$-linearly independent elements of $\mathcal{L}$, then $\lambda_1, \lambda_2, \dots, \lambda_k$ are algebraically independent over $\mathbb{Q}$.*

Assuming Conjecture 1.10, we will prove the following result.

Corollary 1.11. *Let notation and assumptions be as in Theorem 1.9. If Conjecture 1.10 is true, then*

$$\sum_{A\in\mathrm{CL}(F)} \chi_{E/F}(A)\rho_{2^\ell}(A)$$

is transcendental.

Organization. The paper is organized as follows. In Section 2 we give some explicit examples of Theorems 1.8 and 1.9. In Sections 3 and 4 we prove Propositions 1.1 and 1.3, resp. In Sections 5, 6 and 7, we prove Theorems 1.8, 1.9 and Corollary 1.11, resp.

2. Examples

In this section we give some explicit examples of Theorems 1.8 and 1.9.

Example 2.1. Consider the tower of fields

$$\mathbb{Q} \subset F \subset F_{\text{gen}} \subset H$$

where $F = \mathbb{Q}(\sqrt{5}, \sqrt{3\cdot 29})$, $F_{\text{gen}} = \mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{29}, \sqrt{7+2\sqrt{5}})$ is the genus field of F, i.e., the maximal unramified extension of F which is abelian over $\mathbb{Q}$, and H is the Hilbert class field of F. The genus field F_{gen} was determined in [Y, Example 2.2 (1)]. Note that because F has class number $h_F = 4$ and F_{gen}/F is an unramified abelian extension of degree $4 = h_F = [H : F]$, the genus field F_{gen} is actually the Hilbert class field H. Now, the ideal class group of F is $\mathrm{CL}(F) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A set of representatives of the (nontrivial) ideal classes $\mathrm{CL}(F) = \{[\mathfrak{A}_1] = [\mathcal{O}_F], [\mathfrak{A}_2], [\mathfrak{A}_3], [\mathfrak{A}_4]\}$ is given by

$$\begin{aligned}
\mathfrak{A}_2 = \mathfrak{A}_2^{-1} &= \left\langle 19, \left(-\frac{9}{2}\sqrt{3\cdot 29} + 3\right)\sqrt{5} - \frac{5}{2}\sqrt{3\cdot 29} + 198 \right\rangle \\
\mathfrak{A}_3 = \mathfrak{A}_3^{-1} &= \left\langle 31, \left(-\frac{1}{2}\sqrt{3\cdot 29} + 13\right)\sqrt{5} - \frac{25}{2}\sqrt{3\cdot 29} + 15 \right\rangle \\
\mathfrak{A}_4 = \mathfrak{A}_4^{-1} &= \left\langle 3, \left(-\frac{1}{2}\sqrt{3\cdot 29} - \frac{3}{2}\right)\sqrt{5} + \frac{3}{2}\sqrt{3\cdot 29} + \frac{45}{2} \right\rangle.
\end{aligned}$$

Applying Theorem 1.8 to these particular fields yields the following formula for the class number of $H = \mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{29}, \sqrt{7+2\sqrt{5}})$,

$$h_H = -32\frac{R_F}{R_H}\mathrm{Det}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \tag{2.1}$$

where

$$u_{k\ell} = \iiint\limits_{\mathbb{R}_+^3/U} \log\left(\frac{G(\tau_{\mathfrak{A}_\ell^{-1}\mathfrak{A}_k}(\mathbf{t}))}{G(\tau_{\mathfrak{A}_\ell}(\mathbf{t}))}\right)\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}, \quad 1 \le k, \ell \le 3.$$

In particular, if Δ denotes the determinant on the right hand side of (2.1), we have

$$\frac{R_F}{R_H}\Delta \in \mathbb{Q}.$$

Remark 2.2. A formula for the ratio of regulators R_H/R_F may be deduced from [CF, Theorem 1].

Example 2.3. Consider the tower of fields

$$\mathbb{Q} \subset F \subset E \subset F_{\text{gen}} = H$$

where F and $F_{\text{gen}} = H$ are as in Example 2.1 and $E = \mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{29})$. Then F is a real biquadratic field and E is an unramified real quadratic extension of F with $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$. Recall the set of representatives $\mathrm{CL}(F) = \{[\mathfrak{A}_1] = [\mathcal{O}_F], [\mathfrak{A}_2], [\mathfrak{A}_3], [\mathfrak{A}_4]\}$ for the ideal class group of F that were given in Example 2.1. Since the ideal $\mathfrak{A}_4$ becomes principal in E (it is generated by $\sqrt{3}$), the genus character $\chi_{E/F} : \mathrm{CL}(F) \to \{\pm 1\}$ is given by $\chi_{E/F}([\mathfrak{A}_1]) = \chi_{E/F}([\mathfrak{A}_4]) = 1$ and $\chi_{E/F}([\mathfrak{A}_2]) = \chi_{E/F}([\mathfrak{A}_3]) = -1$. The discriminant of F

is $D_F = 3027600 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 29^2$. The quadratic subfields of E not contained in F are $K_1 = \mathbb{Q}(\sqrt{3})$, $K_2 = \mathbb{Q}(\sqrt{29})$, $K_3 = \mathbb{Q}(\sqrt{3 \cdot 5})$ and $K_4 = \mathbb{Q}(\sqrt{5 \cdot 29})$. The corresponding class numbers and discriminants are $h_1 = 1$, $h_2 = 1$, $h_3 = 2$, $h_4 = 4$, and $\Delta_1 = 12$, $\Delta_2 = 29$, $\Delta_3 = 60$, $\Delta_4 = 145$, resp. The fundamental units are $\varepsilon_1 = 2 + \sqrt{3}$, $\varepsilon_2 = (5 + \sqrt{29})/2$, $\varepsilon_3 = 4 + \sqrt{15}$ and $\varepsilon_4 = 12 + \sqrt{145}$. Applying Theorem 1.9 to these particular fields yields the identity

$$\iiint_{\mathbb{R}^3_+/U} \log\left(\frac{G(\tau_{\mathfrak{A}_1^{-1}}(\mathbf{t}))G(\tau_{\mathfrak{A}_4^{-1}}(\mathbf{t}))}{G(\tau_{\mathfrak{A}_2^{-1}}(\mathbf{t}))G(\tau_{\mathfrak{A}_3^{-1}}(\mathbf{t}))}\right)\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} = \\ -4\log(2+\sqrt{3})\log((5+\sqrt{29})/2)\log(4+\sqrt{15})\log(12+\sqrt{145}).$$

Moreover, by Corollary 1.11 (which assumes Conjecture 1.10) the number

$$\iiint_{\mathbb{R}^3_+/U} \log\left(\frac{G(\tau_{\mathfrak{A}_1^{-1}}(\mathbf{t}))G(\tau_{\mathfrak{A}_4^{-1}}(\mathbf{t}))}{G(\tau_{\mathfrak{A}_2^{-1}}(\mathbf{t}))G(\tau_{\mathfrak{A}_3^{-1}}(\mathbf{t}))}\right)\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}$$

is transcendental.

3. Maximal parabolic Eisenstein series on $SL_n(\mathbb{Z})$

In this section we compute the Laurent expansion at $s = 1$ of the maximal parabolic Eisenstein series on $SL_n(\mathbb{Z})$ and thus prove Proposition 1.1. We follow closely the work of Efrat [E], Friedberg [F], Goldfeld [G] and Terras [T]. For convenience, we recall the setup from Section 1. Let $\mathcal{H}^n = GL_n(\mathbb{R})/O_n(\mathbb{R})\mathbb{R}^*$ be the symmetric space of $GL_n(\mathbb{R})$. By the Iwasawa decomposition, each coset $Z \in \mathcal{H}^n$ has a unique representative of the form

$$Z = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ & 1 & x_{2,3} & \dots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix} \tag{3.1}$$

where $x_{i,j} \in \mathbb{R}$ for $1 \le i < j \le n$ and $y_i \in \mathbb{R}_+$ for $1 \le i \le n-1$. Left matrix multiplication induces an action of $GL_n(\mathbb{Z})$ on $\mathcal{H}^n$.

Let P be the maximal parabolic subgroup of $SL_n(\mathbb{Z})$, which consists of those matrices with bottom row $(0, \dots, 0, 1)$. Define the maximal parabolic Eisenstein series

$$E_n(Z, s) := \sum_{\gamma \in P \backslash SL_n(\mathbb{Z})} \mathrm{Det}(\gamma \cdot Z)^s, \quad \mathrm{Re}(s) > 1$$

where $\mathrm{Det}(\gamma \cdot Z)$ is the determinant of the unique representative of the coset $\gamma \cdot Z \in \mathcal{H}^n$ of the form (3.1) and $s \in \mathbb{C}$.

The completed Eisenstein series

$$E_n^*(Z, s) := \pi^{-ns/2}\Gamma(ns/2)\zeta(ns)E_n(Z, s)$$

satisfies the functional equation

$$E_n^*(Z, s) = E_n^*((Z^T)^{-1}, 1 - s)$$

and extends to a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0, 1$.

Define

$$Q = Q_Z := ZZ^T$$

and let

$$Q[\mathbf{a}] := \mathbf{a}^T Q \mathbf{a} \quad \text{for} \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

One has the identity (see e.g. [G, p. 308-309, eq. (10.7.4)])

$$\zeta(ns)E_n(Z,s) = \mathrm{Det}(Z)^s \zeta(ns/2, Q), \tag{3.2}$$

where

$$\zeta(s,Q) := \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n \\ \mathbf{a}\neq\mathbf{0}}} Q[\mathbf{a}]^{-s}, \quad \mathrm{Re}(s) > \frac{n}{2}$$

is the Epstein zeta function of Q. In particular, since

$$\mathrm{Det}(Z) = y_1^{n-1} y_2^{n-2} \cdots y_{n-1}$$

we have

$$E_n^*(Z,s) = \pi^{-ns/2}\Gamma(ns/2)(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^s \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n \\ \mathbf{a}\neq\mathbf{0}}} Q[\mathbf{a}]^{-ns/2}. \tag{3.3}$$

We now compute the Laurent expansion of $E_n^*(Z,s)$ at $s=1$ by splitting the sum in (3.3) into terms with $a_1 = 0$ and terms with $a_1 \neq 0$.

Write

$$Q = \begin{pmatrix} m & \mathbf{r}^T \\ \mathbf{r} & S \end{pmatrix}$$

where

$$m = (y_1 y_2 + \cdots y_{n-1})^2 + (x_{1,2} y_1 y_2 \cdots y_{n-2})^2 + (x_{1,3} y_1 y_2 \cdots y_{n-3})^2 + \cdots + x_{1,n}^2,$$

$$\mathbf{r} = Z_1 \begin{pmatrix} x_{1,2} y_1 y_2 \cdots y_{n-2} \\ x_{1,3} y_1 y_2 \cdots y_{n-3} \\ \vdots \\ x_{1,n} \end{pmatrix}$$

with

$$Z_1 = \begin{pmatrix} 1 & x_{2,3} & \ldots & x_{2,n} \\ & \ddots & & \vdots \\ & & 1 & x_{n-1,n} \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \cdots y_{n-2} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix},$$

and $S = Z_1 Z_1^T$. Also, write $\mathbf{a} = \begin{pmatrix} a_1 \\ \mathbf{b} \end{pmatrix}$ where

$$\mathbf{b} = \begin{pmatrix} a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}.$$

If $a_1 = 0$ then $Q[\mathbf{a}] = S[\mathbf{b}]$, hence the contribution of the terms with $a_1 = 0$ in (3.3) is

$$\begin{aligned}
\pi^{-ns/2}\Gamma(ns/2)(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^s &\sum_{\substack{\mathbf{b}\in\mathbb{Z}^{n-1} \\ \mathbf{b}\neq \mathbf{0}}} S[\mathbf{b}]^{-ns/2} \\
&= (y_1 y_2^2 \cdots y_{n-1}^{n-1})^{s/(n-1)} E^*_{n-1}(Z_1, n/(n-1)s).
\end{aligned}$$

Let $s = 1$ to get

$$(y_1 y_2^2 \cdots y_{n-1}^{n-1})^{1/(n-1)} E^*_{n-1}(Z_1, n/(n-1)). \tag{3.4}$$

Next, suppose that $a_1 \neq 0$. We need to analyze

$$\pi^{-ns/2}\Gamma(ns/2)(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^s \sum_{\substack{\mathbf{a}\in\mathbb{Z}^{n} \\ a_1\neq 0}} Q[\mathbf{a}]^{-ns/2}. \tag{3.5}$$

Let $\mathbf{q} = S^{-1}\mathbf{r}$ and $w = m - \mathbf{q}^T S \mathbf{q}$. Then

$$Q = \begin{pmatrix} m & \mathbf{r}^T \\ \mathbf{r} & S \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{q}^T \\ & \mathrm{I}_{n-1} \end{pmatrix} \begin{pmatrix} w & \\ & S \end{pmatrix} \begin{pmatrix} 1 & \\ \mathbf{q} & \mathrm{I}_{n-1} \end{pmatrix},$$

so that

$$Q[\mathbf{a}] = w[a_1] + S[\mathbf{q}a_1 + \mathbf{b}].$$

Hence (3.5) may be written as

$$\pi^{-ns/2}\Gamma(ns/2)(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^s \sum_{\substack{a_1\in\mathbb{Z} \\ a_1\neq 0}} \sum_{\mathbf{b}\in\mathbb{Z}^{n-1}} \left(w[a_1] + S[\mathbf{q}a_1 + \mathbf{b}]\right)^{-ns/2}. \tag{3.6}$$

We wish to apply the Poisson summation formula to the sum over $\mathbf{b} \in \mathbb{Z}^{n-1}$ in (3.6). Define

$$f(\mathbf{x}, s) := \left(w[a_1] + S[\mathbf{q}a_1 + \mathbf{x}]\right)^{-ns/2}, \quad \mathbf{x} \in \mathbb{R}^{n-1}.$$

The Fourier transform is given by

$$\widehat{f}(\mathbf{y}, s) = \int_{\mathbb{R}^{n-1}} \left(w[a_1] + S[\mathbf{q}a_1 + \mathbf{x}]\right)^{-ns/2} \exp(-2\pi i \mathbf{y}^T \mathbf{x}) d\mathbf{x}.$$

Write $S = W^T W$ and make the change of variables $\mathbf{u} = (w[a_1])^{-1/2} W(\mathbf{q}a_1 + \mathbf{x})$. Then

$$\mathbf{x} = (w[a_1])^{1/2} W^{-1}\mathbf{u} - \mathbf{q}a_1$$

and

$$d\mathbf{x} = (w[a_1])^{(n-1)/2}\mathrm{Det}(W)^{-1} d\mathbf{u} = (w[a_1])^{(n-1)/2}\mathrm{Det}(S)^{-1/2} d\mathbf{u},$$

so that

$$\widehat{f}(\mathbf{y}, s) = (w[a_1])^{\frac{n-1}{2} - \frac{ns}{2}} \mathrm{Det}(S)^{-1/2} \exp(2\pi i \mathbf{y}^T \mathbf{q} a_1) I\left(2\pi (w[a_1])^{1/2} (\mathbf{y}^T W^{-1})^T, ns/2\right)$$

where

$$I(\mathbf{y},s):=\int_{\mathbb{R}^{n-1}}(1+\mathbf{x}^T\mathbf{x})^{-s}\exp(-i\mathbf{y}^T\mathbf{x})d\mathbf{x},\quad \mathrm{Re}(s)>\frac{n-1}{2}.$$

We now evaluate $I(\mathbf{y},s)$. For $\mathbf{y}=\mathbf{0}$, we have (see [T, p. 480-481])

$$I(\mathbf{0},s)=\pi^{(n-1)/2}\frac{\Gamma(s-(n-1)/2)}{\Gamma(s)}. \tag{3.7}$$

For $\mathbf{y}\neq\mathbf{0}$, we follow [T, p. 481]. By [SW, Theorem 3.3, p. 155], we have

$$I(\mathbf{y},s)=(2\pi)^{(n-1)/2}\int_0^\infty(1+x^2)^{-s}x^{n-2}J_{\frac{n}{2}-\frac{3}{2}}(||\mathbf{y}||x)(||\mathbf{y}||x)^{\frac{3}{2}-\frac{n}{2}}dx,$$

where

$$J_\nu(x):=(1/2x)^\nu\pi^{-1/2}\Gamma(\mu+1/2)^{-1}\int_0^\pi\exp(-ix\cos(t))\sin(t)^{2\nu}dt,\quad \mathrm{Re}(\nu)>-\frac{1}{2}$$

is the J-Bessel function of the first kind and $||\mathbf{y}||=(\mathbf{y}^T\mathbf{y})^{1/2}$. Then by [AS, p. 488, eq. (11.4.44)], we have

$$I(\mathbf{y},s)=\frac{(2\pi)^{(n-1)/2}(||\mathbf{y}||/2)^{s-(n-1)/2}}{\Gamma(s)}K_{\frac{n-1}{2}-s}(||\mathbf{y}||), \tag{3.8}$$

where

$$K_\nu(z):=\frac{1}{2}\int_0^\infty\exp(-z(u+\frac{1}{u})/2)u^{\nu-1}du,\quad |\arg(z)|<\frac{\pi}{2}$$

is the modified K-Bessel function.

Apply the Poisson summation formula to the sum over $\mathbf{b}\in\mathbb{Z}^{n-1}$ in (3.6) to get

$$\pi^{-ns/2}\Gamma(ns/2)(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^s\sum_{\substack{a_1\in\mathbb{Z}\\ a_1\neq0}}\sum_{\mathbf{b}\in\mathbb{Z}^{n-1}}\widehat{f}(\mathbf{b},s). \tag{3.9}$$

To analyze this expression we separate the cases $\mathbf{b}=\mathbf{0}$ and $\mathbf{b}\neq\mathbf{0}$.

If $\mathbf{b}=\mathbf{0}$, then (3.7) yields

$$\begin{aligned}\widehat{f}(\mathbf{0},s)&=(w[a_1])^{\frac{n-1}{2}-\frac{ns}{2}}\mathrm{Det}(S)^{-1/2}I\left(\mathbf{0},ns/2\right)\\&=\mathrm{Det}(S)^{-1/2}\pi^{(n-1)/2}\frac{\Gamma(\frac{ns}{2}-\frac{n-1}{2})}{\Gamma(ns/2)}w^{\frac{n-1}{2}-\frac{ns}{2}}|a_1|^{(n-1)-ns}.\end{aligned}$$

Hence the contribution of $\mathbf{b}=\mathbf{0}$ to (3.9) is

$$2\mathrm{Det}(S)^{-1/2}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^s\pi^{-\frac{ns}{2}+\frac{(n-1)}{2}}\Gamma(\frac{ns}{2}-\frac{n-1}{2})w^{\frac{n-1}{2}-\frac{ns}{2}}\zeta(ns-(n-1)), \tag{3.10}$$

where we used

$$\zeta(ns-(n-1))=\frac{1}{2}\sum_{\substack{a_1\in\mathbb{Z}\\ a_1\neq0}}|a_1|^{(n-1)-ns}.$$

To compute the Laurent expansion of (3.10) at $s=1$, note that

$$\begin{aligned}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^s&=y_1^{n-1}y_2^{n-2}\cdots y_{n-1}+(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})\log(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})(s-1)\\&\quad+O(|s-1|^2),\end{aligned}$$

$$\pi^{-\frac{ns}{2}+\frac{(n-1)}{2}}\Gamma(\frac{ns}{2}-\frac{n-1}{2}) = 1+\frac{n\gamma_0}{2}(s-1)+O(|s-1|^2)$$

where $\gamma_0 = -\gamma - \log(\pi) - 2\log(2)$ (here γ is Euler's constant),

$$w^{\frac{n-1}{2}-\frac{ns}{2}} = w^{-1/2} - \frac{n}{2}\log(w)w^{-1/2}(s-1)+O(|s-1|^2),$$

and

$$\zeta(ns-(n-1)) = \frac{1/n}{s-1}+\gamma+O(|s-1|).$$

Then multiplying terms yields the Laurent expansion

$$\begin{aligned}
&\frac{\frac{2}{n}(\mathrm{Det}(S)w)^{-1/2}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})}{s-1}\\
&\quad + 2\gamma(\mathrm{Det}(S)w)^{-1/2}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})\\
&\quad + \frac{2}{n}(\mathrm{Det}(S)w)^{-1/2}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})\left(\log(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})+\frac{n\gamma_0}{2}-\frac{n}{2}\log(w)\right)\\
&\quad + O(|s-1|)\\
&= \frac{2/n}{s-1}+2\gamma+\gamma_0+\frac{2}{n}\log(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})-\log(w)+O(|s-1|)\\
&= \frac{2/n}{s-1}+\gamma-\log(4\pi)+\frac{2}{n}\log(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})-\log(w)+O(|s-1|), \qquad (3.11)
\end{aligned}$$

where we used

$$\mathrm{Det}(S)w = \mathrm{Det}(Q) = \mathrm{Det}(Z)^2 = (y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^2. \qquad (3.12)$$

We now calculate $w = m - \mathbf{q}^T S\mathbf{q}$. We have

$$\mathbf{q} = S^{-1}\mathbf{r} = \left((Z_1^T)^{-1}Z_1^{-1}\right)Z_1\begin{pmatrix}x_{1,2}y_1y_2\cdots y_{n-2}\\ x_{1,3}y_1y_2\cdots y_{n-3}\\ \vdots\\ x_{1,n}\end{pmatrix} = (Z_1^T)^{-1}\begin{pmatrix}x_{1,2}y_1y_2\cdots y_{n-2}\\ x_{1,3}y_1y_2\cdots y_{n-3}\\ \vdots\\ x_{1,n}\end{pmatrix}.$$

Therefore

$$\begin{aligned}
\mathbf{q}^T S\mathbf{q} &= (x_{1,2}y_1y_2\cdots y_{n-2}, x_{1,3}y_1y_2\cdots y_{n-3},\cdots,x_{1,n})Z_1^{-1}\left(Z_1Z_1^T\right)(Z_1^T)^{-1}\begin{pmatrix}x_{1,2}y_1y_2\cdots y_{n-2}\\ x_{1,3}y_1y_2\cdots y_{n-3}\\ \vdots\\ x_{1,n}\end{pmatrix}\\
&= (x_{1,2}y_1y_2\cdots y_{n-2})^2+(x_{1,3}y_1y_2\cdots y_{n-3})^2+\cdots+x_{1,n}^2,
\end{aligned}$$

so that

$$\begin{aligned}
w &= m-\mathbf{q}^TS\mathbf{q}\\
&= (y_1y_2+\cdots y_{n-1})^2+(x_{1,2}y_1y_2\cdots y_{n-2})^2+(x_{1,3}y_1y_2\cdots y_{n-3})^2+\cdots+x_{1,n}^2\\
&\qquad -\left((x_{1,2}y_1y_2\cdots y_{n-2})^2+(x_{1,3}y_1y_2\cdots y_{n-3})^2+\cdots+x_{1,n}^2\right)\\
&= (y_1y_2+\cdots y_{n-1})^2.
\end{aligned}$$

Substituting this formula for w into (3.11) and simplifying yields the Laurent expansion

$$\frac{2/n}{s-1} + \gamma - \log(4\pi) - \frac{2}{n}\log(y_1 y_2^2 \cdots y_{n-1}^{n-1}) + O(|s-1|). \tag{3.13}$$

If $\mathbf{b} \neq \mathbf{0}$, then (3.8) yields

$$\begin{aligned}
\widehat{f}(\mathbf{b}, s) &= (w[a_1])^{\frac{n-1}{2}-\frac{ns}{2}} \mathrm{Det}(S)^{-1/2} \exp(2\pi i \mathbf{b}^T \mathbf{q} a_1) I\left(2\pi (w[a_1])^{1/2} (\mathbf{b}^T W^{-1})^T, ns/2\right) \\
&= (w[a_1])^{\frac{n-1}{2}-\frac{ns}{2}} \mathrm{Det}(S)^{-1/2} \exp(2\pi i \mathbf{b}^T \mathbf{q} a_1) \\
&\qquad \times \frac{(2\pi)^{(n-1)/2} (||2\pi (w[a_1])^{1/2} (\mathbf{b}^T W^{-1})^T||/2)^{\frac{ns}{2}-\frac{n-1}{2}}}{\Gamma(ns/2)} \\
&\qquad\quad \times K_{\frac{n-1}{2}-\frac{ns}{2}}(||2\pi (w[a_1])^{1/2} (\mathbf{b}^T W^{-1})^T||) \\
&= (w[a_1])^{\frac{n-1}{2}-\frac{ns}{2}} \mathrm{Det}(S)^{-1/2} \exp(2\pi i \mathbf{b}^T \mathbf{q} a_1) \\
&\qquad \times \frac{(2\pi)^{(n-1)/2} (\frac{1}{2} \cdot 2\pi w^{1/2} (S^{-1}[\mathbf{b}])^{1/2} |a_1|)^{\frac{ns}{2}-\frac{n-1}{2}}}{\Gamma(ns/2)} \\
&\qquad\quad \times K_{\frac{n-1}{2}-\frac{ns}{2}}(2\pi w^{1/2} (S^{-1}[\mathbf{b}])^{1/2} |a_1|),
\end{aligned}$$

where we used

$$||2\pi (w[a_1])^{1/2} (\mathbf{b}^T W^{-1})^T|| = 2\pi \left(w[a_1] S^{-1}[\mathbf{b}]\right)^{1/2} = 2\pi w^{1/2} (S^{-1}[\mathbf{b}])^{1/2} |a_1|.$$

Now, using the functional equation $K_{-\nu}(z) = K_\nu(z)$ and the identity

$$K_{1/2}(z) = (\pi/2z)^{1/2} e^{-z},$$

we have

$$\begin{aligned}
\widehat{f}(\mathbf{b}, 1) &= (w[a_1])^{-1/2} \mathrm{Det}(S)^{-1/2} \exp(2\pi i \mathbf{b}^T \mathbf{q} a_1) \\
&\qquad \times (2\pi)^{(n-1)/2} \frac{(\pi/4)^{1/2}}{\Gamma(n/2)} \exp\left(-2\pi w^{1/2} (S^{-1}[\mathbf{b}])^{1/2} |a_1|\right) \\
&= (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{-1} |a_1|^{-1} \frac{\pi^{n/2}}{\Gamma(n/2)} \exp\left(2\pi i \left(\mathbf{b}^T \mathbf{q} a_1 + i w^{1/2} (S^{-1}[\mathbf{b}])^{1/2} |a_1|\right)\right),
\end{aligned}$$

where we again used (3.12). Hence if $s = 1$, the contribution of the terms with $\mathbf{b} \neq \mathbf{0}$ to (3.9) is

$$\begin{aligned}
&\sum_{\substack{a_1 \in \mathbb{Z} \\ a_1 \neq 0}} \sum_{\substack{\mathbf{b} \in \mathbb{Z}^{n-1} \\ \mathbf{b} \neq \mathbf{0}}} |a_1|^{-1} \exp\left(2\pi i \left(\mathbf{b}^T \mathbf{q} a_1 + i w^{1/2} (S^{-1}[\mathbf{b}])^{1/2} |a_1|\right)\right) \\
&\quad = 4 \sum_{a_1=1}^{\infty} \sum_{\substack{\mathbf{b} \in \mathbb{Z}^{n-1} \\ \mathbf{b} \ (\mathrm{mod}\ \pm 1) \\ \mathbf{b} \neq \mathbf{0}}} \frac{1}{a_1} \mathrm{Re}\left(\exp\left(\left(-2\pi w^{1/2} (S^{-1}[\mathbf{b}])^{1/2} + 2\pi i \mathbf{b}^T \mathbf{q}\right) a_1\right)\right) \\
&\quad = 4 \sum_{\substack{\mathbf{b} \in \mathbb{Z}^{n-1} \\ \mathbf{b} \ (\mathrm{mod}\ \pm 1) \\ \mathbf{b} \neq \mathbf{0}}} - \log\left|1 - \exp\left(-2\pi w^{1/2} (S^{-1}[\mathbf{b}])^{1/2} + 2\pi i \mathbf{b}^T \mathbf{q}\right)\right| \\
&\quad = -4 \log \prod_{\substack{\mathbf{b} \in \mathbb{Z}^{n-1} \\ \mathbf{b} \ (\mathrm{mod}\ \pm 1) \\ \mathbf{b} \neq \mathbf{0}}} \left|1 - \exp\left(-2\pi w^{1/2} (S^{-1}[\mathbf{b}])^{1/2} + 2\pi i \mathbf{b}^T \mathbf{q}\right)\right|, \qquad (3.14)
\end{aligned}$$

where we used

$$\mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{z^k}{k}\right) = -\log|1 - z|.$$

Finally, by combining (3.4), (3.13) and (3.14), we get the Laurent expansion

$$E_n^*(Z, s) = \frac{2/n}{s-1} + \gamma - \log(4\pi) - \frac{2}{n} \log(y_1 y_2^2 \cdots y_{n-1}^{n-1}) - 4 \log g(Z) + O(|s-1|),$$

where

$$\begin{aligned}
g(Z) := \exp&\left(-\frac{(y_1 y_2^2 \cdots y_{n-1}^{n-1})^{1/(n-1)} E_{n-1}^*(Z_1, n/(n-1))}{4}\right) \\
&\times \prod_{\substack{\mathbf{b} \in \mathbb{Z}^{n-1} \\ \mathbf{b} \ (\mathrm{mod}\ \pm 1) \\ \mathbf{b} \neq \mathbf{0}}} \left|1 - \exp\left(-2\pi w^{1/2} (S^{-1}[\mathbf{b}])^{1/2} + 2\pi i \mathbf{b}^T \mathbf{q}\right)\right|.
\end{aligned}$$

This proves Proposition 1.1.

4. Dedekind zeta functions of totally real fields

In this section we relate the zeta function of a wide ideal class of a totally real number field of degree n to the integral of the maximal parabolic Eisenstein series $E_n^*(Z, s)$ along a Heegner cycle in $\mathcal{H}^n$ and thus prove Proposition 1.3. Let F be a totally real number field of degree n and U be the group of units of F. Let A be a wide ideal class of F and fix $\mathfrak{B} \in A^{-1}$. Then the ideal class zeta function may be written as

$$\zeta_F(s, A) := \sum_{\substack{\mathfrak{A} \in A \\ \mathfrak{A} \neq 0}} \frac{1}{N(\mathfrak{A})^s} = N(\mathfrak{B})^s \sum_{\lambda \in \mathfrak{B}^*/U} \frac{1}{|N(\lambda)|^s}, \quad \mathrm{Re}(s) > 1 \qquad (4.1)$$

where $N(\mathfrak{A})$ is the norm and $\mathfrak{B}^* = \mathfrak{B}\backslash\{0\}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the images of $\lambda \in \mathfrak{B}$ under the real embeddings of F. Then

$$|N(\lambda)|^s = |\lambda_1\lambda_2\cdots\lambda_n|^s.$$

Note that for $x > 0$,

$$x^{-s} = \frac{1}{\Gamma(s/2)}\int_0^\infty e^{-x^2 t} t^{s/2}\frac{dt}{t}.$$

Then for $a_1, a_2, \ldots, a_n > 0$,

$$(a_1a_2\cdots a_n)^{-s}\Gamma(s/2)^n = \int_0^\infty\int_0^\infty\cdots\int_0^\infty \exp\left(-\left(a_1^2t_1 + a_2^2t_2 + \cdots a_n^2t_n\right)\right)(t_1t_2\cdots t_n)^{s/2}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\cdots\frac{dt_n}{t_n}. \tag{4.2}$$

Consider the change of variables

$$\begin{aligned} t_1 &= x_1^2 w \\ t_2 &= x_2^2 w \\ &\vdots \\ t_{n-1} &= x_{n-1}^2 w \\ t_n &= (x_1x_2\cdots x_{n-1})^{-2}w \end{aligned}$$

and the corresponding Jacobian

$$J = n2^{n-1}(x_1x_2\cdots x_{n-1})^{-1}w^{n-1}.$$

Then making this change of variables in (4.2) yields

$$\begin{aligned} &(a_1a_2\cdots a_n)^{-s}\Gamma(s/2)^n = \\ &n2^{n-1}\int_0^\infty\int_0^\infty\cdots\int_0^\infty\left(\int_0^\infty \exp\left(-\left(\sum_{k=1}^{n-1}a_k^2x_k^2 + a_n^2(x_1\cdots x_{n-1})^{-2}\right)w\right)w^{ns/2}\frac{dw}{w}\right) \\ &\qquad\times\frac{dx_1}{x_1}\frac{dx_2}{x_2}\cdots\frac{dx_{n-1}}{x_{n-1}} \\ &= n2^{n-1}\Gamma(ns/2)\int_0^\infty\int_0^\infty\cdots\int_0^\infty\left(\sum_{k=1}^{n-1}a_k^2x_k^2 + a_n^2(x_1\cdots x_{n-1})^{-2}\right)^{-ns/2} \\ &\qquad\times\frac{dx_1}{x_1}\frac{dx_2}{x_2}\cdots\frac{dx_{n-1}}{x_{n-1}}. \end{aligned} \tag{4.3}$$

We now apply the identity (4.3) in (4.1) to get

$$\begin{aligned} &\zeta_F(s,A)\Gamma(s/2)^n = n2^{n-1}\Gamma(ns/2)N(\mathfrak{B})^s \\ &\quad\times\sum_{\lambda\in\mathfrak{B}^*/U}\int_0^\infty\int_0^\infty\cdots\int_0^\infty\left(\sum_{k=1}^{n-1}\lambda_k^2t_k^2 + \lambda_n^2(t_1\cdots t_{n-1})^{-2}\right)^{-ns/2}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{n-1}}{t_{n-1}}. \end{aligned} \tag{4.4}$$

Given a unit $\varepsilon \in U$, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ denote the images of ε under the real embeddings of F. There is an action of the unit group U on $\mathbb{R}_+^{n-1}$ given by

$$\begin{aligned} \varepsilon : \mathbb{R}_+^{n-1} &\longrightarrow \mathbb{R}_+^{n-1}, \\ (t_1, t_2, \ldots, t_{n-1}) &\longmapsto (|\varepsilon_1| t_1, |\varepsilon_2| t_2, \ldots, |\varepsilon_{n-1}| t_{n-1}). \end{aligned}$$

Let $\mathbb{R}_+^{n-1}/U$ denote a fundamental domain for this action. Then using this action, (4.4) becomes

$$\begin{aligned} \zeta_F(s, A)\Gamma(s/2)^n &= n 2^{n-1} \Gamma(ns/2) N(\mathfrak{B})^s \\ &\times \sum_{\lambda \in \mathfrak{B}^*} \int \cdots \int_{\mathbb{R}_+^{n-1}/U} \left(\sum_{k=1}^{n-1} \lambda_k^2 t_k^2 + \lambda_n^2 (t_1 \cdots t_{n-1})^{-2} \right)^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_{n-1}}{t_{n-1}}. \end{aligned} \tag{4.5}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a $\mathbb{Z}$-basis for $\mathfrak{B}$ and $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_n^{(i)}$ for $i = 1, 2, \ldots, n$ denote their images under the real embeddings of F. Given $\lambda \in \mathfrak{B}$, write

$$\lambda = m_1 \alpha_1 + m_2 \alpha_2 + \cdots + m_n \alpha_n$$

where $m_1, m_2, \ldots, m_n \in \mathbb{Z}$. Then

$$\sum_{k=1}^{n-1} \lambda_k^2 t_k^2 + \lambda_n^2 (t_1 \cdots t_{n-1})^{-2} = \mathbf{m}^T \left(M_{\mathfrak{B}}(\mathbf{t}) M_{\mathfrak{B}}(\mathbf{t})^T \right) \mathbf{m}$$

where

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix}$$

and

$$M_{\mathfrak{B}}(\mathbf{t}) := \begin{pmatrix} \alpha_1^{(1)} t_1 & \alpha_1^{(2)} t_2 & \cdots & \alpha_1^{(n-1)} t_{n-1} & \alpha_1^{(n)} (t_1 t_2 \cdots t_{n-1})^{-1} \\ \alpha_2^{(1)} t_1 & \alpha_2^{(2)} t_2 & \cdots & \alpha_2^{(n-1)} t_{n-1} & \alpha_2^{(n)} (t_1 t_2 \cdots t_{n-1})^{-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_n^{(1)} t_1 & \alpha_n^{(2)} t_2 & \cdots & \alpha_n^{(n-1)} t_{n-1} & \alpha_n^{(n)} (t_1 t_2 \cdots t_{n-1})^{-1} \end{pmatrix}$$

where $\mathbf{t} = (t_1, t_2, \ldots, t_{n-1}) \in \mathbb{R}_+^{n-1}$. Define

$$Q_{\mathfrak{B}}(\mathbf{t}) := M_{\mathfrak{B}}(\mathbf{t}) M_{\mathfrak{B}}(\mathbf{t})^T,$$

and let

$$Q_{\mathfrak{B}}(\mathbf{t})[\mathbf{m}] := \mathbf{m}^T \cdot Q_{\mathfrak{B}}(\mathbf{t}) \cdot \mathbf{m}$$

be the quadratic form associated to $Q_{\mathfrak{B}}(\mathbf{t})$. Then the identity (4.5) becomes

$$\zeta_F(s, A)\Gamma(s/2)^n = n 2^{n-1} \Gamma(ns/2) N(\mathfrak{B})^s \int \cdots \int_{\mathbb{R}_+^{n-1}/U} \zeta(ns/2, Q_{\mathfrak{B}}(\mathbf{t})) \frac{dt_1}{t_1} \cdots \frac{dt_{n-1}}{t_{n-1}}, \tag{4.6}$$

where

$$\zeta(s, Q_{\mathfrak{B}}(\mathbf{t})) := \sum_{\substack{\mathbf{m}\in\mathbb{Z}^n \\ \mathbf{m}\neq\mathbf{0}}} Q_{\mathfrak{B}}(\mathbf{t})[\mathbf{m}]^{-s}, \quad \mathrm{Re}(s) > \frac{n}{2}$$

is the Epstein zeta function of $Q_{\mathfrak{B}}(\mathbf{t})$.

The positive definite, symmetric matrix $Q_{\mathfrak{B}}(\mathbf{t})$ may be written as

$$Q_{\mathfrak{B}}(\mathbf{t}) = \mathrm{Det}(Q_{\mathfrak{B}}(\mathbf{t}))^{1/n}(y_1^{n-1}(\mathbf{t})y_2^{n-2}(\mathbf{t})\cdots y_{n-1}(\mathbf{t}))^{-2/n}\tau_{\mathfrak{B}}(\mathbf{t})\tau_{\mathfrak{B}}(\mathbf{t})^T$$

where

$$\tau_{\mathfrak{B}}(\mathbf{t}) = \begin{pmatrix} 1 & x_{1,2}(\mathbf{t}) & x_{1,3}(\mathbf{t}) & \dots & x_{1,n}(\mathbf{t}) \\ & 1 & x_{2,3}(\mathbf{t}) & \dots & x_{2,n}(\mathbf{t}) \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n}(\mathbf{t}) \\ & & & & 1 \end{pmatrix}\begin{pmatrix} y_1(\mathbf{t})y_2(\mathbf{t})\cdots y_{n-1}(\mathbf{t}) & & & & \\ & y_1(\mathbf{t})y_2(\mathbf{t})\cdots y_{n-2}(\mathbf{t}) & & & \\ & & \ddots & & \\ & & & y_1(\mathbf{t}) & \\ & & & & 1 \end{pmatrix}$$

is in $\mathcal{H}^n$. Here we have suppressed the dependence of the variables $x_{i,j}(\mathbf{t})$ and $y_i(\mathbf{t})$ on $\mathfrak{B}$ and the $\mathbb{Z}$-basis $\alpha_1, \alpha_2, \dots, \alpha_n$. Then $\{\tau_{\mathfrak{B}}(\mathbf{t}) : \ \mathbf{t} \in \mathbb{R}_+^{n-1}/U\}$ defines a Heegner cycle in $\mathcal{H}^n$.

Now, by (3.2) we have the identity

$$\zeta(ns/2, Q_{\mathfrak{B}}(\mathbf{t})) = \mathrm{Det}(Q_{\mathfrak{B}}(\mathbf{t}))^{-s/2}\zeta(ns)E_n(\tau_{\mathfrak{B}}(\mathbf{t}), s).$$

Moreover,

$$\begin{aligned}\mathrm{Det}(Q_{\mathfrak{B}}(\mathbf{t})) &= \mathrm{Det}\begin{pmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} & \cdots & \alpha_1^{(n)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} & \cdots & \alpha_2^{(n)} \\ \vdots & \vdots & & \vdots \\ \alpha_n^{(1)} & \alpha_n^{(2)} & \cdots & \alpha_n^{(n)} \end{pmatrix}^2 \\ &= \mathrm{disc}(\mathfrak{B}) \\ &= N(\mathfrak{B})^2 D_F\end{aligned}$$

where D_F is the discriminant of F. Then if

$$\zeta_F^*(s, A) := \pi^{-ns/2}\Gamma(s/2)^n D_F^{s/2}\zeta_F(s, A)$$

denotes the completed ideal class zeta function, (4.6) yields

$$\zeta_F^*(s, A) = n2^{n-1}\int_{\mathbb{R}_+^{n-1}/U}\cdots\int E_n^*(\tau_{\mathfrak{B}}(\mathbf{t}), s)\frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t_{n-1}}.$$

This proves Proposition 1.3.

5. Proof of Theorem 1.8

Given a number field K, let $\mathrm{CL}(K)$ be the wide ideal class group, h_K be the class number, R_K be the regulator, w_K be the number of roots of unity, and D_K be the absolute value of the discriminant. Given an ideal class group character χ of K, the class group L–function is defined by

$$L_K(\chi, s) := \sum_{A\in\mathrm{CL}(K)} \chi(A)\zeta_K(s, A), \quad \mathrm{Re}(s) > 1$$

where $\zeta_K(s, A)$ denotes the ideal class zeta function of $A \in \mathrm{CL}(K)$. If χ is trivial, then $L(\chi, s) = \zeta_K(s)$ is the Dedekind zeta function of K. The Dedekind zeta function $\zeta_K(s)$ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ with residue

$$\mathrm{Res}_{s=1}\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{D_K}}, \tag{5.1}$$

where r_1 (resp. $2r_2$) is the number of real (resp. complex) embeddings of K.

Suppose now that F is a totally real number field of degree n and H is the Hilbert class field of F. By class field theory, one has the factorization

$$\frac{\zeta_H(s)}{\zeta_F(s)} = \prod_{\substack{\chi\in\widehat{\mathrm{CL}(F)}\\ \chi\neq 1}} L_F(\chi, s).$$

Since F is totally real of degree n and H is unramified at the infinite primes, H is totally real with $n \cdot h_F$ real embeddings. It follows from (5.1) that

$$\lim_{s\to 1}\frac{(s-1)}{(s-1)}\frac{\zeta_H(s)}{\zeta_F(s)} = 2^{n(h_F-1)}\frac{h_H}{h_F}\frac{R_H}{R_F}\sqrt{\frac{D_F}{D_H}}. \tag{5.2}$$

Here we used $w_F = w_H = 2$, since these fields are totally real and hence have only the roots of unity ± 1. On the other hand, by Theorem 1.6 and orthogonality, for $\chi \neq 1$ we have

$$L_F(\chi, 1) = -\frac{n2^{n-1}}{D_F^{1/2}}\sum_{A\in\mathrm{CL}(F)}\chi(A)\rho_n(A) \tag{5.3}$$

where we recall that

$$\rho_n(A) := \int\cdots\int_{\mathbb{R}_+^{n-1}/U}\log G(\tau_{\mathfrak{B}}(\mathbf{t}))\frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t_{n-1}}$$

and $\mathfrak{B} \in A^{-1}$. Then combining (5.2) and (5.3) yields the identity

$$\frac{(-1)^{h_F-1}2^{h_F-1}}{n^{h_F-1}}\frac{h_H}{h_F}\frac{R_H}{R_F}\frac{D_F^{h_F/2}}{D_H^{1/2}} = \prod_{\substack{\chi\in\widehat{\mathrm{CL}(F)}\\ \chi\neq 1}}\sum_{A\in\mathrm{CL}(F)}\chi(A)\rho_n(A).$$

Since $D_H = D_F^{h_F}$, we have $D_F^{h_F/2}/D_H^{1/2} = 1$. Moreover, by a well-known result of Frobenius on group determinants (see e.g. [S, p. 78]), we have

$$\prod_{\substack{\chi\in\widehat{\mathrm{CL}(F)}\\ \chi\neq 1}}\sum_{A\in\mathrm{CL}(F)}\chi(A)\rho_n(A) = \mathrm{Det}\left(\rho_n(A_k^{-1}A_\ell) - \rho_n(A_k^{-1})\right)_{k,\ell}$$

where $1 \le k, \ell \le h_F - 1$. It follows that

$$\frac{(-1)^{h_F-1}2^{h_F-1}}{n^{h_F-1}}\frac{h_H}{h_F}\frac{R_H}{R_F} = \mathrm{Det}\left(\rho_n(A_k^{-1}A_\ell) - \rho_n(A_k^{-1})\right)_{k,\ell}.$$

Finally, if we write the ideal class group of F as

$$\mathrm{CL}(F) = \{A_1 = [\mathfrak{A}_1] = [\mathcal{O}_F], A_2 = [\mathfrak{A}_2], \ldots, A_{h_F} = [\mathfrak{A}_{h_F}]\},$$

then

$$\rho_n(A_k^{-1}A_\ell) - \rho_n(A_k^{-1}) = \int_{\mathbb{R}_+^{n-1}/U} \cdots \int \log\left(\frac{G(\tau_{\mathfrak{A}_\ell^{-1}\mathfrak{A}_k}(\mathbf{t}))}{G(\tau_{\mathfrak{A}_\ell}(\mathbf{t}))}\right)\frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t_{n-1}}.$$

This proves Theorem 1.8.

6. Proof of Theorem 1.9

Let F be a totally real abelian number field with $\mathrm{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^\ell$, and let E be an unramified real quadratic extension of F with $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^{\ell+1}$. Then the zeta function $\zeta_E(s)$ (resp. $\zeta_F(s)$) factors as $\zeta(s)$ times the product of the Dirichlet L–functions associated to the quadratic subfields of E (resp. F). Note that there are $2^\ell - 1$ quadratic subfields of F, $2^{\ell+1}-1$ quadratic subfields of E, and 2^ℓ quadratic subfields of E that are not contained in F. By class field theory, the unramified quadratic extension E/F gives rise to a real ideal class group character $\chi_{E/F}$ of F (a genus character) whose L-function factors as

$$L_F(\chi_{E/F}, s) = \frac{\zeta_E(s)}{\zeta_F(s)}.$$

Then by the preceding facts we obtain the factorization

$$L_F(\chi_{E/F}, s) = \prod_{i=1}^{2^\ell} L(\chi_i, s),$$

where χ_i for $1 \le i \le 2^\ell$ are the Kronecker symbols associated to the quadratic subfields K_i of E which are not contained in F.

By Dirichlet's class number formula, we have

$$L(\chi_i, 1) = \frac{2\log(\varepsilon_i)h_i}{\sqrt{\Delta_i}},$$

where $\Delta_i > 0$, ε_i and h_i are the discriminant, fundamental unit, and class number of K_i, resp. Therefore

$$L_F(\chi_{E/F}, 1) = 2^{2^\ell}\prod_{i=1}^{2^\ell}\frac{\log(\varepsilon_i)h_i}{\sqrt{\Delta_i}}. \tag{6.1}$$

Let $n = 2^\ell$ in (5.3) and equate this with (6.1) to get

$$\sum_{A\in \mathrm{CL}(F)} \chi_{E/F}(A)\rho_{2^\ell}(A) = -\frac{D_F^{1/2}}{2^{\ell-1}}\prod_{i=1}^{2^\ell}\frac{\log(\varepsilon_i)h_i}{\sqrt{\Delta_i}}.$$

This proves Theorem 1.9

7. Proof of Corollary 1.11

Conjecture 1.10 asserts that if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are $\mathbb{Q}$-linearly independent elements of

$$\mathcal{L} := \{\log(\alpha):\ \alpha \in \overline{\mathbb{Q}}^*\},$$

then $\lambda_1, \lambda_2, \ldots, \lambda_k$ are algebraically independent over $\mathbb{Q}$ (recall that algebraic independence over $\mathbb{Q}$ means that if $P(X_1, X_2, \ldots, X_k)$ is a nonzero polynomial with coefficients in $\mathbb{Q}$, then

$P(\lambda_1, \lambda_2, \ldots, \lambda_k) \neq 0)$. It follows that if $R(X_1, X_2, \ldots, X_k)$ is a nonconstant polynomial with coefficients in $\overline{\mathbb{Q}}$, then the number $R(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is transcendental.

Now, because the units ε_i, $1 \leq i \leq 2^\ell$, are multiplicatively independent, the numbers $\log(\varepsilon_i)$, $1 \leq i \leq 2^\ell$, are $\mathbb{Q}$-linearly independent. Define the (nonconstant) polynomial

$$R(X_1, X_2, \ldots, X_{2^\ell}) := \frac{D_F^{1/2}}{2^{\ell-1}} \prod_{i=1}^{2^\ell} \alpha_i X_i \in \overline{\mathbb{Q}}[X_1, X_2, \ldots, X_{2^\ell}]$$

where

$$\alpha_i := \frac{h_i}{\sqrt{\Delta_i}}.$$

Then assuming Conjecture 1.10, the number

$$R\left(\log(\varepsilon_1), \log(\varepsilon_2), \ldots, \log(\varepsilon_{2^\ell})\right)$$

is transcendental. However, by Theorem 1.9 we have

$$R\left(\log(\varepsilon_1), \log(\varepsilon_2), \ldots, \log(\varepsilon_{2^\ell})\right) = \sum_{A \in \mathrm{CL}(F)} \chi_{E/F}(A)\rho_{2^\ell}(A).$$

This proves Corollary 1.11.

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