# NONVANISHING OF HECKE $L$–FUNCTIONS FOR CM FIELDS AND RANKS OF ABELIAN VARIETIES

RIAD MASRI AND TONG HAI YANG

Abstract. In this paper we prove a nonvanishing theorem for central values of $L$–functions associated to a large class of algebraic Hecke characters of CM number fields. A key ingredient in the proof is an asymptotic formula for the average of these central values. We combine the nonvanishing theorem with work of Tian and Zhang [TZ] to deduce that infinitely many of the CM abelian varieties associated to these Hecke characters have Mordell-Weil rank zero. Included among these abelian varieties are higher dimensional analogues of the elliptic $\mathbb{Q}$-curves $A(D)$ of B. Gross [Gr].

**Geometric and Functional Analysis** (GAFA), vol. **21** (2011), 648–679.

## 1. Introduction and statement of results

In [Gr], B. Gross constructed an infinite family of elliptic $\mathbb{Q}$-curves with some remarkable properties. To describe these curves, let $E = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field of discriminant $-D$ with $D > 3$ and $D \equiv 3 \mod 4$. Let $\mathcal{O}_E$ be the ring of integers of $E$, and let $h_E$ be the class number of $E$. A "canonical" Hecke character $\chi$ of $E$ is a Hecke character satisfying the condition $\chi(\alpha\mathcal{O}_E) = \pm\alpha$ for principal ideals prime to the conductor of $\chi$. There are exactly $h_E$ canonical Hecke characters of $E$. Gross proved that the Hecke character $\chi_H = \chi \circ N_{H/E}$ of the Hilbert class field $H$ of $E$ is associated to an elliptic $\mathbb{Q}$-curve $A(D)$ over $H$ which is unique up to $H$-isogeny and whose $L$–function factors as

$$L(A(D)/H, s) = \prod_{\chi} L(\chi, s)L(\bar{\chi}, s).$$

He then conjectured that the Mordell-Weil rank of $A(D)$ over $H$ is 0 if $D \equiv 7 \mod 8$ and $2h_E$ if $D \equiv 3 \mod 8$. Gross proved the rank 0 case of his conjecture for prime discriminants using descent theory. In a series important of papers, Rohrlich [R1, R2, R3] and Montgomery and Rohrlich [MR] proved the rank 0 case for all discriminants by showing that the central values $L(\chi, 1)$ are nonvanishing for all $D \equiv 7 \mod 8$ and applying results on the Birch and Swinnerton-Dyer conjecture (BSD) due to Rubin [Ru]. Similarly, Miller and Yang [MY] proved the rank $2h_E$ case by showing that the central derivatives $L'(\chi, 1)$ are nonvanishing for all $D \equiv 3 \mod 8$ and applying results of Kolyvagin and Logachev [KL].

In 1982, Rohrlich [R4] proved that canonical Hecke characters $\chi$ exist for CM number fields $E$, yet very little is known about the arithmetic of the associated CM abelian varieties $A(\chi)$ over $E$. One of the main obstacles to progress has been the lack of a nonvanishing theorem for the central values $L(\chi, 1)$ in this case. In this paper we will prove a nonvanishing theorem for these central values. Our proof relies on an asymptotic formula for the average of

R. Masri is partially supported by an NSA Young Investigator Grant. T. H. Yang is partially supported by the grants NSF DMS-0855901 and NSFC-10628103.

these central values which is of independent interest. We will then combine our nonvanishing theorem with new work of Tian and Zhang [TZ] on BSD for CM abelian varieties to deduce that infinitely many of the CM abelian varieties $A(\chi)$ have Mordell-Weil rank 0 over $E$. Included among these abelian varieties are higher dimensional analogues of Gross's elliptic $\mathbb{Q}$-curves $A(D)$.

We now describe the results of this paper in more detail. For a summary of facts concerning CM types, algebraic Hecke characters, CM abelian varieties, and reflex fields, we refer the reader to Section 9.

Let $E$ be a CM number field with maximal totally real subfield $F$ and CM type $\Phi$.

**Definition 1.1.** Let $k \geq 0$ be an integer. A *canonical* Hecke character $\chi$ of $E$ of type $(2k+1)\Phi$ is an algebraic Hecke character satisfying the following three conditions:

(1) $\chi(\bar{\mathfrak{A}}) = \overline{\chi(\mathfrak{A})}$ for an ideal $\mathfrak{A}$ of $E$ prime to the conductor of $\chi$.

(2) $\chi(\alpha\mathcal{O}_E) = \pm\prod_{\sigma\in\Phi}\sigma(\alpha)^{2k+1}$ for principal ideals $\alpha\mathcal{O}_E$ prime to the conductor of $\chi$.

(3) $\chi$ is unramified outside the relative differential $\partial_{E/F}$.

Let $X(k, E, \Phi)$ denote the set of canonical Hecke characters $\chi$ of $E$ of type $(2k+1)\Phi$.

The set $X(k, E, \Phi)$ was determined by Rohrlich [R4] when $k = 0$. The same argument works for general $k$. In particular, when every prime of $F$ above 2 is unramified in $E$ and $F$ has narrow class number 1 (the case we study in this paper), the set $X(k, E, \Phi)$ contains exactly $h_E = \#\mathrm{CL}(E)$ elements which differ from each other by ideal class characters of $E$: given any such character $\chi$, one has

$$X(k, E, \Phi) = \{\chi\xi : \ \xi \in \mathrm{CL}(E)^\wedge\}$$

where $\mathrm{CL}(E)^\wedge$ is the set of characters $\xi : \mathrm{CL}(E) \to \mathbb{C}^*$ of the ideal class group $\mathrm{CL}(E)$ of $E$.

Let $\mu$ be a fixed quadratic Hecke character of $F$. The quadratic twist of $\chi$ by $\mu$ is defined by $\chi_\mu = \chi(\mu \circ N_{E/F})$, and the $L$–function of $\chi_\mu$ is defined by

$$L(\chi_\mu, s) := \sum_{\substack{\mathfrak{A}\subset\mathcal{O}_E\\ \mathfrak{A}\neq 0}} \chi_\mu(\mathfrak{A}) N_{E/\mathbb{Q}}(\mathfrak{A})^{-s}, \quad \mathrm{Re}(s) > k + \frac{3}{2}.$$

It is known that $L(\chi_\mu, s)$ has analytic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 2(k+1) - s$ with central value $L(\chi_\mu, k+1)$.

We will prove the following nonvanishing theorem for the central values $L(\chi_\mu, k+1)$.

**Theorem 1.2.** *Let $F$ be a totally real number field of narrow ideal class number 1, and let $\mu$ be a fixed quadratic Hecke character of $F$ of conductor $\mathfrak{f} = f\mathcal{O}_F$ with $f \in \mathcal{O}_F$ totally positive and prime to 2. Let $E$ be a totally imaginary quadratic extension of $F$ with a CM type $\Phi$ such that every prime factor of $2\mathfrak{f}$ is split in $E$ and $\mathcal{O}_E^* = \mathcal{O}_F^*$. Let $k \geq 0$ be an integer with $(-1)^k\mu_\infty(-1) = 1$. Then there is an absolute constant $\delta > 0$ such that*

$$\#\{\chi \in X(k, E, \Phi) : \ L(\chi_\mu, k+1) \neq 0\} \gg d_E^\delta$$

*as the absolute discriminant $d_E \to \infty$. The implied constant in $\gg$ depends on $F, k$ and $\delta$, and is ineffective.*

By the theory of complex multiplication, a canonical Hecke character $\chi$ of type $\Phi$ is associated to a CM abelian variety $A(\chi)$ over $E$ with complex multiplication by the ring of integers of the CM number field $\mathbb{Q}(\chi)$ generated by all $\chi(\mathfrak{A})$ for ideals $\mathfrak{A}$ of $E$ prime to the conductor of $\chi$. Let $A(\chi)^\mu = A(\chi_\mu)$ denote the quadratic twist of $A(\chi)$ by $\mu$. By combining Theorem 1.2 with new work of Tian and Zhang [TZ] on BSD for CM abelian varieties, we will obtain the following result on ranks of the abelian varieties $A(\chi)$ and their quadratic twists.

**Theorem 1.3.** *Let $F$ be a totally real number field of narrow ideal class number 1, and let $\mu$ be a fixed quadratic Hecke character of $F$ of conductor $\mathfrak{f} = f\mathcal{O}_F$ with $f \in \mathcal{O}_F$ totally positive and prime to 2 and $\mu_\infty(-1) = 1$. Let $E$ be a totally imaginary quadratic extension of $F$ with a CM type $\Phi$ such that every prime factor of $2\mathfrak{f}$ is split in $E$ and $\mathcal{O}_E^* = \mathcal{O}_F^*$. Then for all sufficiently large discriminants $d_E$ there is a canonical Hecke character $\chi \in X(0, E, \Phi)$ such that the CM abelian variety $A(\chi)^\mu$ has Mordell-Weil rank 0 over $E$.*

The $L$–function of the CM abelian variety $A(\chi)$ associated to $\chi \in X(0, E, \Phi)$ factors as

$$L(A(\chi), s) = \prod_{\sigma:\mathbb{Q}(\chi)\hookrightarrow\mathbb{C}} L(\chi^\sigma, s).$$

Because two abelian varieties over $E$ are isogenous if and only if they have the same $L$–function, it follows that the abelian varieties $A(\chi^\sigma)$ for $\sigma : \mathbb{Q}(\chi) \hookrightarrow \mathbb{C}$ are isogenous. Define the set

$$S_\chi := \{\chi^\sigma : \ \sigma : \mathbb{Q}(\chi) \hookrightarrow \mathbb{C}, \ \sigma \circ \Phi = \Phi\} = \{\chi^\sigma : \ \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/E')\}$$

where $E'$ is the reflex field of $E$ and we have fixed an embedding $E' \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Then $S_\chi \subset X(0, E, \Phi)$. If $S_\chi = X(0, E, \Phi)$, the abelian varieties $A(\chi)$ for $\chi \in X(0, E, \Phi)$ are isogenous, and in this case we denote $A(\chi)$ by $A_E$.

In the following proposition we identify a large class of CM quartic fields for which the sets $S_\chi$ and $X(0, E, \Phi)$ are equal.

**Proposition 1.4.** *Let $F = \mathbb{Q}(\sqrt{p})$ be a real quadratic number field of narrow ideal class number 1 where $p \equiv 1 \mod 4$ is a fixed prime, and let $E$ be a non-biquadratic, totally imaginary quadratic extension of $F$ with $d_E = p^2q$ where $q \equiv 1 \mod 4$ is a prime.*

1. *We have $S_\chi = X(0, E, \Phi)$ for any $\chi \in X(0, E, \Phi)$. Moreover, the associated CM abelian variety $A_E$ has dimension $2h_E$.*
2. *There is a CM abelian surface $A_H$ over $H$ associated to $\chi_H = \chi \circ N_{H/E}$ which is independent of the choice of $\chi \in X(0, E, \Phi)$ and unique up to $H$-isogeny. Moreover, $A_E = \mathrm{Res}_{H/E} A_H$.*

By combining Theorem 1.2 and Proposition 1.4, we will obtain the following nonvanishing theorem.

**Theorem 1.5.** *Let $F = \mathbb{Q}(\sqrt{p})$ be a real quadratic number field of narrow ideal class number 1 where $p \equiv 1 \mod 4$ is a fixed prime, and let $\mu$ be a fixed quadratic Hecke character of $F$ of conductor $\mathfrak{f} = f\mathcal{O}_F$ with $f \in \mathcal{O}_F$ totally positive and prime to 2. Let $E$ be a non-biquadratic, totally imaginary quadratic extension of $F$ with a CM type $\Phi$ such that every prime factor of $2\mathfrak{f}$ is split in $E$, $\mathcal{O}_E^* = \mathcal{O}_F^*$, and $d_E = p^2q$ where $q \equiv 1 \mod 4$ is a prime. Let $k \geq 0$ be an integer with $(-1)^k\mu_\infty(-1) = 1$ and $\gcd(2k+1, h_E) = 1$. Then for all sufficiently large discriminants $d_E$ we have $L(\chi_\mu, k+1) \neq 0$ for all $\chi \in X(k, E, \Phi)$.*

By combining Theorem 1.5 with [TZ], we will obtain the following result on ranks of the abelian varieties $A_H$ and $A_E$ and their quadratic twists.

**Theorem 1.6.** *Let $F = \mathbb{Q}(\sqrt{p})$ be a real quadratic number field of narrow ideal class number 1 where $p \equiv 1 \mod 4$ is a fixed prime, and let $\mu$ be a fixed quadratic Hecke character of $F$ of conductor $\mathfrak{f} = f\mathcal{O}_F$ with $f \in \mathcal{O}_F$ totally positive and prime to 2 and $\mu_\infty(-1) = 1$. Let $E$ be a non-biquadratic, totally imaginary quadratic extension of $F$ with a CM type $\Phi$ such that every prime factor of $2\mathfrak{f}$ is split in $E$, $\mathcal{O}_E^* = \mathcal{O}_F^*$, and $d_E = p^2 q$ where $q \equiv 1 \mod 4$ is a prime. Then for all sufficiently large discriminants $d_E$ the CM abelian varieties $A_H^\mu$ and $A_E^\mu$ have Mordell-Weil rank 0 over $H$ and $E$, respectively.*

*Remark* 1.7. When $E$ is imaginary quadratic, one has $S_\chi = X(0, E, \Phi)$ for any $\chi \in X(0, E, \Phi)$, and $A_E = \mathrm{Res}_{H/E} A(D)$ where $A(D)$ is Gross's elliptic $\mathbb{Q}$-curve. Hence Theorem 1.6 can be viewed as a higher dimensional analogue of the rank 0 case of Gross' conjecture on ranks of the elliptic curves $A(D)$.

For imaginary quadratic fields $E$, the nonvanishing of the central values $L(\chi_\mu, k+1)$ has been studied extensively using various methods (see e.g. [R1, R2, MR, RVY, Ya2, MY, Ma1, Ma2, KMY]). In [KMY, Theorem 1.1], the authors and Kim established an asymptotic formula for the average of the central values $L(\chi_\mu, k+1)$ using an explicit formula for the central value due to the second author [Ya2], a spectral regularization, and the equidistribution of Heegner points on modular curves (see also [T]). By combining the asymptotic formula with a subconvexity bound for $L(\chi_\mu, k+1)$ due to Duke, Friedlander, and Iwaniec, the authors and Kim obtained a quantitative nonvanishing theorem for the central values (see [KMY, Theorem 1.4]).

In this paper we will use a similar method to study the nonvanishing problem over CM number fields $E$. A key ingredient is an asymptotic formula for the average of the central values $L(\chi_\mu, k+1)$ as $\chi$ varies over a certain subset of $X(k, E, \Phi)$. To state this, let $E$ be a totally imaginary quadratic extension of $F$ with relative discriminant $d_{E/F}$, and define the quotient

$$\mathrm{CL}^{\mathrm{ra}}(E) := E^* \backslash \hat{E}^* / (\hat{\mathcal{O}}_E^* \prod_{\substack{v \mid d_{E/F} \\ v \text{ prime}}} E_v^*)$$

where $\hat{E}^*$ denotes the finite ideles of $E$. For example, when $E$ is imaginary quadratic we have

$$\mathrm{CL}^{\mathrm{ra}}(E) = \mathrm{CL}(E)/\mathrm{CL}_2(E) \cong \mathrm{CL}(E)^2$$

where $\mathrm{CL}_2(E)$ is the 2-torsion subgroup of $\mathrm{CL}(E)$.

Given any fixed canonical Hecke character $\chi \in X(k, E, \Phi)$, we will obtain the following asymptotic formula for the average of $L(\chi_\mu \xi, k+1)$ as $\xi$ varies over $\mathrm{CL}^{\mathrm{ra}}(E)^\wedge$.

**Theorem 1.8.** *Let $F$ be a totally real number field of narrow ideal class number 1, and let $\mu$ be a fixed quadratic Hecke character of $F$ of conductor $\mathfrak{f} = f\mathcal{O}_F$ with $f \in \mathcal{O}_F$ totally positive and prime to 2. Let $E$ be a totally imaginary quadratic extension of $F$ with a CM type $\Phi$ such that every prime factor of $2\mathfrak{f}$ is split in $E$ and $\mathcal{O}_E^* = \mathcal{O}_F^*$. Let $k \geq 0$ be an integer with $(-1)^k \mu_\infty(-1) = 1$, and suppose that either $k$ is odd or $\mu_v(-1) = -1$ for some prime $v | \mathfrak{f}$. Then for any canonical Hecke character $\chi \in X(k, E, \Phi)$, we have*

$$\frac{1}{\#\mathrm{CL}^{\mathrm{ra}}(E)} \sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} L(\chi_\mu \xi, k+1) = c(k) L(\epsilon_{E/F}, 1) \langle \theta_{\mu,k}, \theta_{\mu,k} \rangle_{\mathrm{Pet}} + o(1)$$

*as $d_E \to \infty$. Here*

$$c(k) := \frac{2(8\pi)^{kt}}{(k!)^t}\sqrt{d_F}, \quad t := [F:\mathbb{Q}],$$

*$L(\epsilon_{E/F}, s)$ is the $L$–function of the quadratic Hecke character $\epsilon_{E/F}$ of $F$ associated to the quadratic extension $E/F$, and*

$$\langle \theta_{\mu,k}, \theta_{\mu,k} \rangle_{\mathrm{Pet}} := \int_{\Gamma_0(4\mathfrak{f}^2)\backslash \mathbb{H}^t} |\theta_{\mu,k}(z)|^2 \mathrm{Im}(z)^{k+\frac{1}{2}} d\mu, \quad z \in \mathbb{H}^t$$

*is the Peterson inner product of the real-analytic Hilbert modular theta function $\theta_{\mu,k}$ of weight $k+\frac{1}{2}$ for $\Gamma_0(4\mathfrak{f}^2)$ defined by (2.3).*

Theorem 1.8 generalizes [KMY, Theorem 1.1] in many respects. We now briefly describe the proof. First, we will establish an explicit formula for the central value $L(\chi_\mu, k+1)$ (see Theorem 2.3). Using the central value formula, we will then relate the average of the central values to a weight zero Hilbert modular function $F_{\mu,k}$ evaluated on a toric suborbit of CM points on the Hilbert modular variety $\Gamma_0(4\mathfrak{f}^2)\backslash\mathbb{H}^t$ (see Theorem 3.5 and Corollary 4.2). Finally, using a theorem of Michel and Venkatesh [MV1, MV2] which implies that the toric suborbit is equidistributed as $d_E \to \infty$, we will obtain the asymptotic formula.

The "test function" $F_{\mu,k}$ is constructed from the theta function $\theta_{\mu,k}$, which we assume to be cuspidal so that $F_{\mu,k}$ is smooth and all of its derivatives have exponential decay in the cusps of $\Gamma_0(4\mathfrak{f}^2)\backslash\mathbb{H}^t$. This allows us to apply the equidistribution theorem of Michel and Venkatesh directly to $F_{\mu,k}$. In Section 8 we use representation theory to give a criterion for the cuspidality of $\theta_{\mu,k}$ (see also [KMY, Proposition 4.3]). In order to remove the cuspidality assumption, we would need to proceed as in the proof of [KMY, Theorem 1.1], where we used a delicate spectral regularization to establish a version of equidistribution of Heegner points on modular curves for test functions which *grow moderately* in the cusps (see also [FM1, FM2]). We will return to this problem in a subsequent paper.

We do not proceed via the spectral decomposition of $F_{\mu,k}$ to prove Theorem 1.8, thus the error term is not expressed in a quantitative form. However, by adapting an ergodic method of Venkatesh [V], we will obtain a quantitative version of Theorem 1.8 under an assumption on enough small split primes in $F$. For a real number $a > 0$, define

$$\mathrm{wt}(E,a) := \#\{\mathfrak{q} \subset \mathcal{O}_F \text{ prime and split in } E: \ d_E^a \le N_{F/\mathbb{Q}}(\mathfrak{q}) \le 2d_E^a\}.$$

**Theorem 1.9.** *Let $F$ be a totally real number field of narrow ideal class number $1$, and let $\mu$ be a fixed quadratic Hecke character of $F$ of conductor $\mathfrak{f} = f\mathcal{O}_F$ with $f \in \mathcal{O}_F$ totally positive and prime to 2. Let $E$ be a totally imaginary quadratic extension of $F$ with a CM type $\Phi$ such that every prime factor of $2\mathfrak{f}$ is split in $E$ and $\mathcal{O}_E^* = \mathcal{O}_F^*$. Let $k \ge 0$ be an integer with $(-1)^k\mu_\infty(-1) = 1$, and suppose that either $k$ is odd or $\mu_v(-1) = -1$ for some prime $v|\mathfrak{f}$. Then there exist absolute constants $\delta_1, \delta_2, \delta_3 > 0$ such that for any canonical Hecke character $\chi \in X(k, E, \Phi)$, we have*

$$\frac{1}{\#\mathrm{CL}^{\mathrm{ra}}(E)} \sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} L(\chi_\mu \xi, k+1) = c(k) L(\epsilon_{E/F}, 1) \langle \theta_{\mu,k}, \theta_{\mu,k} \rangle_{\mathrm{Pet}} + O_{\mu,k}\left(d_E^{-\delta_1}\right) + O_{\mu,k}\left(\frac{[\mathrm{CL}(E) : \mathrm{CL}(E)^2]}{\min\{d_E^{\delta_2}, \mathrm{wt}(E,\delta_3)^{\frac{1}{2}}\}}\right)$$

*as $d_E \to \infty$.*

**Organization**. The paper is organized as follows. In Section 2 we establish an explicit formula for the central value $L(\chi_\mu, k+1)$ which expresses it in terms of values of the fixed Hilbert modular theta function $\theta_{\mu,k}$ at CM points on the Hilbert modular variety $\Gamma_0(4\mathfrak{f}^2)\backslash\mathbb{H}^t$. In Section 3 we use the central value formula to obtain an exact formula for the average of $L(\chi_\mu\xi, k+1)$ as $\xi$ varies over $\mathrm{CL}^{\mathrm{ra}}(E)^\wedge$. In Section 4 we relate the CM points appearing in the average formula to an adelic toric orbit of CM points. In Sections 5, 6 and 7 we prove Theorems 1.8, 1.9 and 1.2, respectively. In Section 8 we use representation theory to establish necessary and sufficient conditions for the cuspidality of $\theta_{\mu,k}$. In Section 9 we summarize facts concerning algebraic Hecke characters and CM abelian varieties. Finally, in Sections 10, 11, and 12, we prove Theorem 1.3, Proposition 1.4, and Theorems 1.5 and 1.6, respectively.

**Acknowledgements**. We would like to thank Solomon Friedberg, Stephen Kudla, and Ken Ono for helpful discussions regarding this work. In addition, we thank the referee for a very careful reading of the manuscript leading to corrections and a much improved exposition.

## 2. A formula for the central value $L(\chi_\mu, k+1)$

The following notation and assumptions will remain fixed throughout this paper. Assume that $F$ is a totally real number field of degree $t$ with narrow class number 1. Let $\psi$ be the unramified additive character of $\mathbb{Q}\backslash\mathbb{Q}_{\mathbb{A}}$ such that

$$\psi_\infty(x) = e(x) = e^{2\pi i x}.$$

Let $\psi = \psi_F = \psi_{\mathbb{Q}} \circ \mathrm{tr}_{F/\mathbb{Q}}$ and $\psi_E = \psi_F \circ \mathrm{tr}_{E/F}$. Let $\mu = \prod \mu_v$ be a quadratic Hecke character of $F$ of conductor $\mathfrak{f} = f\mathcal{O}_F$ for some totally positive number $f \in \mathcal{O}_F$ prime to 2. It induces a quadratic character:

$$\mu' : (\mathcal{O}_F/\mathfrak{f})^* \to \{\pm 1\}, \quad \mu'(a) = \prod_{v|\mathfrak{f}} \mu_v(a). \tag{2.1}$$

We also denote $\mu_\infty = \prod_{v|\infty} \mu_v$.

Let $E$ be a CM number field with maximal totally real subfield $F$, and let $\Phi$ be a CM type for $E$. We aim to study the central values of $L$–functions associated to a fixed quadratic twist of canonical Hecke characters of type $(2k+1)\Phi$ as $(E, \Phi)$ varies and $F$ remains fixed (recall Definition 1.1). Note that if $\chi$ is a canonical Hecke character of type $(2k+1)\Phi$, so is $\chi\xi$ where $\xi$ an ideal class character of $E$. Clearly, if $\chi$ is a canonical Hecke character of CM type $\Phi$, $\chi^{2k+1}$ is a canonical Hecke character of CM type $(2k+1)\Phi$. When the class number of $\mathrm{CL}(E)$ is prime to $2k+1$, the characters of the form $\chi^{2k+1}$ give all of the canonical Hecke characters of type $(2k+1)\Phi$. In general, there are canonical Hecke characters of CM type $(2k+1)\Phi$ which are *not* of the form $\chi^{2k+1}$.

We will also view a canonical Hecke character $\chi$ as an idele class character, and let $\chi^{\mathrm{un}}$ be its associated unitary idele character. According to [R4, Proposition 1], the condition (1) in Definition 1.1 is equivalent to $\chi^{\mathrm{un}}|_{F_{\mathbb{A}}^*} = \epsilon_{E/F}$ where $\epsilon = \epsilon_{E/F}$ is the quadratic Hecke character of $F$ associated to the quadratic extension $E/F$.

Let $E = F(\sqrt{\Delta})$ be a totally imaginary quadratic extension of $F$ such that every prime factor of $2\mathfrak{f}$ is split in $E$ and $\mathcal{O}_E^* = \mathcal{O}_F^*$. Since $F$ has narrow class number one, we can and will choose $\Delta$ so that the relative discriminant $d_{E/F} = \Delta\mathcal{O}_F$. Let $\tilde{\mu} = \mu \circ N_{E/F}$ be the quadratic Hecke character of $E$ associated to $\mu$.

**Definition 2.1.** Let $X(\mu, k, E, \Phi)$ denote the set of Hecke characters $\chi$ of $E$ such that $\chi_\mu = \chi\tilde{\mu} \in X(k, E, \Phi)$.

If we fix a character $\chi$ in $X(\mu, k, E, \Phi)$, then

$$X(\mu, k, E, \Phi) = \{\chi\xi : \xi \in \mathrm{CL}(E)^\wedge\}.$$

Here $G^\wedge$ denotes the set of characters of $G$ for a finite abelian group $G$. Note that [RVY, Lemma 2.1] yields the following lemma.

**Lemma 2.2.** *Let $\chi \in X(\mu, k, E, \Phi)$, and fix $\delta = \sqrt{\Delta}$. Then the global root number of $\chi$ is $(-1)^{kt}\mu_\infty(-1)$. Locally, one has for every place $v$ of $F$,*

$$\prod_{w|v} \epsilon(\frac{1}{2}, \chi_w, \frac{1}{2}\psi_{E_w})\chi_w^{un}(\delta) = \begin{cases} 1 & \text{if } v \text{ is split in } E, \\ \mathrm{sgn}(\sigma(\delta)) & \text{if } v = \sigma \in \Phi, \\ (-1)^{n(\psi_v)} & \text{if } v \text{ is inert in } E, \\ \epsilon(\frac{1}{2}, \epsilon_v, \psi_v)\chi_v^{\mathrm{un}}(\delta) & \text{if } v \text{ is ramified in } E. \end{cases}$$

*Here $\epsilon(\frac{1}{2}, \chi_w, \frac{1}{2}\psi_{E_w})$ is Tate's local root number,*

$$n(\psi_v) = \max\{m : \psi_v(\pi_v^m \mathcal{O}_v) = 1\}$$

*is the conductor of $\psi_v$ where $\pi_v$ is a uniformizer of $F$, and $\mathrm{sgn}(ix) = \mathrm{sgn}(x)$ for $x \in \mathbb{R}$.*

When $\chi$ is the $(2k+1)$-th power of a canonical Hecke character of $E$ of type $\Phi$ with root number 1, Rodriguez-Villegas and the second author [RVY, Theorem 2.5] gave an explicit formula for the central value $L(\chi, k+1)$. We now show that the same formula holds for any $\chi \in X(\mu, k, E, \Phi)$. To state the formula, we need additional notation. Assume that $(-1)^{kt}\mu_\infty(-1) = 1$. Choose $\alpha \in F^*$ such that

$$\prod_{w|v} \epsilon(\frac{1}{2}, \chi_w^{un}, \frac{1}{2}\psi_{E_w})\chi_w^{un}(\delta) = \epsilon_v(\alpha) \tag{2.2}$$

for every place $v$ of $F$, and

$$n(\frac{\alpha}{4}\psi_v) \le 0$$

when $v$ is split in $E$. Define for $v \nmid \infty$ (recall that $\delta = \sqrt{\Delta}$)

$$n_v := \begin{cases} n(\alpha\psi_v) & \text{if } v \text{ is split}, \\ \frac{1}{2}n(\frac{\delta\alpha}{4}\psi_{E_v}) & \text{otherwise}. \end{cases}$$

We remark that $n_v$ is always an integer. Define

$$\mathfrak{l} = \mathfrak{f} \prod_{v \text{ ram}} \mathfrak{p}_v^{-[\frac{n_v}{2}]} \prod_{v \text{ unram}} \mathfrak{p}_v^{-n_v} = l\mathcal{O}_F$$

where $\mathfrak{p}_v$ is the ideal associated to $v$. Let $r$ be a fixed square-root of $\Delta \mod 4\mathfrak{f}^2 \prod_{v \text{ split}} \mathfrak{p}_v^{-n_v}$. For each ideal class $C \in \mathrm{CL}(E)$, choose a primitive ideal $\mathfrak{A} \in C^{-1}$ prime to $2\Delta f\alpha\mathcal{O}_E$, and write

$$\mathfrak{A}^2 = [a^2, \frac{b+\delta}{2}], \quad a, b \in \mathcal{O}_F$$

such that $a = N_{E/F}\mathfrak{A}$ is totally positive and

$$b^2 \equiv \Delta \mod a^2,$$
$$b \equiv r \mod 2f^2 \prod_{v \text{ split}} \mathfrak{p}_v^{-n_v},$$
$$b_v \equiv 0 \mod \mathfrak{p}_v^{n_v - 2[\frac{n_v}{2}]} \text{ if } v|\Delta.$$

This is possible since $F$ has narrow class number 1. In particular,

$$\tau_{\mathfrak{A}^2} = \frac{-\Delta\alpha(b+\delta)}{2(la)^2}$$

belongs to $\mathbb{H}^t$ via the CM type $\Phi$, i.e., $\Phi(\tau_{\mathfrak{A}^2}) \in \mathbb{H}^t$.

Finally, define the theta function

$$\theta_{\mu,k}(z) := \mathrm{Im}(2z)^{-\frac{k}{2}} \sum_{x \in \mathcal{O}_F, (x,f)=1} \mu'(x) H_{k,F}(x\sqrt{\mathrm{Im}(2z)}) e(x^2 z), \quad z \in \mathbb{H}^t, \tag{2.3}$$

where $\mu'$ is given in (2.1) and $H_k$ is the $k$-th normalized Hermite polynomial defined by

$$\frac{1}{2^k}\left(x - \frac{1}{2\pi}\frac{d}{dx}\right)^k e^{-\pi x^2} = H_k(x) e^{-\pi x^2}, \tag{2.4}$$

$$H_{k,F}(x) := \prod_{\sigma|\infty} H_k(\sigma(x)),$$

and

$$\mathrm{Im}(z) := \prod_{i=1}^{t} \mathrm{Im}(z_i)$$

for $z = (z_i) \in \mathbb{H}^t$. Note that the theta function $\theta_{\mu,k}$ is a real-analytic Hilbert modular form of weight $k + \frac{1}{2}$ for a certain subgroup of $\mathrm{SL}_2(F)$ (see Section 8).

**Theorem 2.3.** *Let $F$ be a totally real number field of narrow ideal class number 1 and degree $t$ over $\mathbb{Q}$. Let $\mu$ be a fixed quadratic Hecke character of $F$ of conductor $f\mathcal{O}_F$ such that $(2, f) = 1$. Let $E = F(\sqrt{\Delta})$ be a CM extension of $F$ with a CM type $\Phi$ such that every prime factor of $2f\mathcal{O}_F$ is split in $E$ and $\mathcal{O}_E^* = \mathcal{O}_F^*$. Let $k \geq 0$ be an integer with $(-1)^{kt}\mu_\infty(-1) = 1$, and let $\chi \in X(\mu, k, E, \Phi)$. Then the central value*

$$L(\chi, k+1) = \kappa \left| \sum_{C \in \mathrm{CL}(E)} \frac{\theta_{\mu,k}(\tau_{\mathfrak{A}^2})}{\chi(\bar{\mathfrak{A}})} \right|^2$$

*where*

$$\kappa = \frac{2^{\frac{1}{2}t}\pi^t |N_{F/\mathbb{Q}}(\Delta^3\alpha^2)|^{\frac{2k+1}{4}}}{|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}} N_{F/\mathbb{Q}}(l)^{2k+1}} \left(\frac{(4\pi)^k}{k!}\right)^t.$$

*Proof.* (sketch) Let $\eta$ be the character of $[E^1] = E^1 \backslash E^1_{\mathbb{A}}$ such that $\tilde{\mu} = \tilde{\eta}$, i.e., $\mu(z\bar{z}) = \eta(z/\bar{z})$ for every $z \in E^*_{\mathbb{A}}$. Let $\chi_0 = \chi^{un}\tilde{\mu}$. Then $(\chi_0, \eta, \alpha, \delta = \sqrt{\Delta}, \psi)$ satisfies [RVY, (0.6)]. Now, the proof is the same as that of [RVY, Theorem 2.5], except for the following small modification. Since we assume that $F$ has narrow class number 1, we can choose a totally positive generator

$a$ of $N_{E/F}\mathfrak{A}$, and $\mu'(c)\,\mathrm{sgn}(Nc)^k = 1$ for a totally positive unit $c$. So $\theta_{\mu,k}(\mathfrak{A})$ does not depend on the choice of the totally positive generator $a$, and the extra fudge factor in [RVY, Theorem 2.5] is not needed in this case. □

## 3. An exact formula for the average $L$-value

In this section we use Theorem 2.3 to derive an exact formula for the average of the central values $L(\chi\xi, k+1)$ as $\xi$ varies over $\mathrm{CL}^{\mathrm{ra}}(E)^\wedge$. Fix $\mu$ and $k \geq 0$ such that

$$(-1)^{kt}\mu_\infty(-1) = 1.$$

We also assume that $\Delta$ is squarefree and $d_{E/F} = \Delta\mathcal{O}_F$. This is not really a restriction since we have assumed that $F$ has (narrow) class number one and every prime factor of $2f\mathcal{O}_F$ is split in $E$.

Let

$$\mathrm{CL}^{\mathrm{ra}}(E) := E^*\backslash\hat{E}^*/(\hat{\mathcal{O}}_E^*\prod_{v|\Delta} E_v^*)$$

and

$$\mathrm{CL}_{\mathrm{ra}}(E) := E^*\backslash(E^*\hat{\mathcal{O}}_E^*\prod_{v|\Delta} E_v^*)/\hat{\mathcal{O}}_E^*.$$

Then $\mathrm{CL}_{\mathrm{ra}}(E)$ is a subgroup of $\mathrm{CL}(E) = E^*\backslash\hat{E}^*/\hat{\mathcal{O}}_E^*$, and we have the exact sequence

$$1 \to \mathrm{CL}_{\mathrm{ra}}(E) \to \mathrm{CL}(E) \to \mathrm{CL}^{\mathrm{ra}}(E) \to 1.$$

**Lemma 3.1.** *The map $z \mapsto \frac{z}{\bar{z}}$ from $\hat{E}^*$ to itself gives rise to a surjection from $\mathrm{CL}^{\mathrm{ra}}(E)$ to $\mathrm{CL}(E)^2$.*

*Proof.* Since $\frac{z}{\bar{z}} = \frac{z^2}{z\bar{z}}$, and $\hat{F}^* = F^*\hat{\mathcal{O}}_F^* \subset E^*\hat{\mathcal{O}}_E^*$, we have a surjective group homomorphism

$$\phi : \hat{E}^* \to \mathrm{CL}(E)^2 = E^*\backslash(E^*\hat{E}^{*,2}\hat{\mathcal{O}}_E^*)/\hat{\mathcal{O}}_E^*, \quad z \mapsto [\frac{z}{\bar{z}}] = [z^2].$$

Recall that for $v|\Delta$, $E_v^* = \omega_v^{\mathbb{Z}}\mathcal{O}_{E_v}^*$ for a uniformizer $\omega_v$ of $E_v$ such that $\bar{\omega}_v = -\omega_v$. Thus $\phi(\omega_v) = 1$ in $\mathrm{CL}(E)^2$, and thus $\ker\phi \supset \mathrm{CL}_{\mathrm{ra}}(E)$, so $\phi$ induces a surjection $\mathrm{CL}^{\mathrm{ra}}(E) \to \mathrm{CL}(E)^2$. □

**Definition 3.2.** For $\alpha \in F^*$, let $X(\mu, k, \alpha, E, \Phi)$ denote the set of Hecke characters $\chi \in X(\mu, k, E, \Phi)$ satisfying the condition (2.2).

Notice that if $\chi \in X(\mu, k, \alpha, E, \Phi)$, then $\chi \in X(\mu, k, \alpha\,\mathrm{N}_{E/F}\,z, E, \Phi)$ for any $z \in E^*$.

**Lemma 3.3.** *Let $\alpha \in \partial_F^{-1}$ with $-\mathrm{ord}_v\,\partial_F \leq \mathrm{ord}_v\,\alpha \leq -\mathrm{ord}_v\,\partial_F + 1$.*

(1) *The set $X(\mu, k, \alpha, E, \Phi)$ is not empty if and only if the following two conditions hold.*
  (a) *$\Phi(\alpha\delta) \in \mathbb{H}^t$, i.e., $\sigma(\alpha\delta) \in i\mathbb{R}_{>0}$ for every $\sigma \in \Phi$.*
  (b) *$\mathrm{ord}_v\,\alpha = -\mathrm{ord}_v\,\partial_F$ if $v$ is inert in $E$.*

(2) *If $\chi \in X(\mu, k, \alpha, E, \Phi)$, then*

$$X(\mu, k, \alpha, E, \Phi) = \{\chi\xi : \xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge\}.$$

*Proof.* Fix $\chi \in X(\mu, k, E, \Phi)$, and for every $\xi \in \mathrm{CL}(E)^\wedge$ and a place $v$ of $F$ define

$$\phi_v(\xi) := \frac{1}{\epsilon_v(\alpha)} \prod_{w|v} \epsilon(\frac{1}{2}, \chi_w^{\mathrm{un}}\xi_w, \frac{1}{2}\psi_{E_v})\chi_w^{\mathrm{un}}\xi_w(\delta).$$

Then Lemma 2.2 gives

$$\phi_v(\xi) = \begin{cases} 1 & \text{if } v \text{ split in } E, \\ \mathrm{sgn}(\sigma(\alpha\delta)) & \text{if } v = \sigma \in \Phi, \\ (-1)^{n(\psi_v) - \mathrm{ord}_v \alpha} & \text{if } v \text{ inert in } E, \\ \dfrac{\epsilon(\frac{1}{2}, \epsilon_v, \psi_v)\chi_w^{\mathrm{un}}(\delta)}{\epsilon_v(\alpha)}\xi_w(\delta) & \text{if } v|\Delta. \end{cases}$$

By definition, $\chi\xi \in X(\mu, k, \alpha, E, \Phi)$ if and only if $\phi_v(\xi) = 1$ for every place $v$ of $F$. Looking at infinite places and inert primes, we see that (a) and (b) are necessary for $X(\mu, k, \alpha, E, \Phi)$ to be non-empty. Assuming this, we have

$$\phi_v(\xi) = \begin{cases} 1 & \text{if } v \nmid \Delta, \\ \phi_v(1)\xi_w(\delta) & \text{if } v|\Delta. \end{cases}$$

Here $w$ is the place of $E$ above $v$. Define a group homomorphism

$$\mathrm{sgn} : \mathrm{CL}(E)^\wedge \to S := \{(d_v)_{v|\Delta} : d_v = \pm 1, \prod_v d_v = 1\}, \quad \xi \mapsto \mathrm{sgn}(\xi) = (\xi_w(\delta)_{v|\Delta}).$$

Since $\xi_w^2(\delta) = \xi_w(\delta^2) = 1$ ($\xi|_{\hat{F}^*} = 1$), $\xi_v(\delta) = \pm 1$, and sgn is well-defined. It is clearly a group homomorphism. For $v|\Delta$, it is easy to see that $\xi_v(\delta) = 1$ if and only if $\xi_v = 1$. So $\ker(\mathrm{sgn}) = \mathrm{CL}^{\mathrm{ra}}(E)^\wedge$, and sgn induces an injection, still denoted by sgn,

$$\mathrm{sgn} : \mathrm{CL}(E)^\wedge/(\mathrm{CL}(E)^{\mathrm{ra}})^\wedge \hookrightarrow S, \quad \xi \mapsto \mathrm{sgn}(\xi) = (\xi_w(\delta)_{v|\Delta}).$$

On the other hand,

$$\#\mathrm{CL}(E)^\wedge/(\mathrm{CL}(E)^{\mathrm{ra}})^\wedge = \#\mathrm{CL}_{\mathrm{ra}}(E).$$

We claim that

$$\#\mathrm{CL}_{\mathrm{ra}}(E) = \#S.$$

So sgn is actually an isomorphism. Therefore, there is a unique family of ideal class characters $\xi \in \mathrm{CL}(E)^\wedge/(\mathrm{CL}(E)^{\mathrm{ra}})^\wedge$ such that $\mathrm{sgn}(\xi) = (\phi_v(1))_{v|\Delta}$, i.e., $\phi_v(\xi) = 1$ for every $v$. When $\Delta$ is a unit, the above claim is clear—both are 1. In general, $s \geq 1$ be the number of prime factors of $\Delta$ so that $\#S = 2^{s-1}$. To see the above claim, one looks at the exact sequence (recall $\hat{F}^* = F^*\hat{\mathcal{O}}_F^* \subset E^*\hat{\mathcal{O}}_E^*$)

$$1 \to (\prod_{v|\Delta} F_v^*\mathcal{O}_{E_v}^*)\backslash E^*(\prod_{v|\Delta} F_v^*\mathcal{O}_{E_v}^*) \to \prod_{v|\Delta}(F_v^*\mathcal{O}_{E_v}^*\backslash E_v^*) = \{\pm 1\}^s \to \mathrm{CL}_{\mathrm{ra}}(E) \to 1,$$

and notice that the first quotient has order 2 and is generated by $\sqrt{\Delta} \in E^*$. So one has $\#\mathrm{CL}_{\mathrm{ra}}(E) = 2^{s-1} = \#S$ as claimed. This proves the lemma. ☐

We now choose a generator $\alpha_0$ for $\partial_F^{-1}$, i.e., $\partial_F^{-1} = \alpha_0\mathcal{O}_F$, and fix $\beta \in \mathcal{O}_F$ with $\mathrm{ord}_v \beta \leq 1$.

**Corollary 3.4.** *Write* $\partial_F^{-1} = \alpha_0\mathcal{O}_F$*, and fix* $\beta \in \mathcal{O}_F$ *with* $\operatorname{ord}_v \beta \leq 1$*. Then for* $\chi \in X(\mu, k, \alpha_0\beta, E, \Phi)$*, we have*

$$L(\chi, k+1) = \kappa \left| \sum_{C \in \mathrm{CL}(E)} \frac{\theta_{\mu,k}(\tau_{\mathfrak{A}^2,\beta})}{\chi(\overline{\mathfrak{A}})} \right|^2$$

*where*

$$\kappa := 2^{-\frac{1}{2}t}\pi^t N_{F/\mathbb{Q}}(\frac{\alpha_0}{\beta f^2})^{\frac{2k+1}{2}} |N_{F/\mathbb{Q}}(\Delta)|^{\frac{2k-1}{4}} \left(\frac{\pi^k}{k!}\right)^t$$

*and* $\tau_{\mathfrak{A}^2,\beta}$ *is given as follows. Fix a square root* $r$ *of* $\Delta \mod 16\beta f^2$*, and for a primitive integral ideal* $\mathfrak{A} \in C^{-1}$*, write*

$$\mathfrak{A}^2 = [a^2, \frac{b+\delta}{2}], \quad a, b \in \mathcal{O}_F$$

*with* $a$ *totally positive and* $b$ *satisfying*

$$b^2 \equiv \Delta \mod 16\beta f^2 a^2, \quad b \equiv r \mod 8\beta f^2.$$

*Then*

$$\tau_{\mathfrak{A}^2,\beta} = \frac{b\alpha_0 + \delta\alpha_0}{8\beta f^2 a^2}.$$

*Proof.* Let $\alpha = \frac{4\alpha_0\beta}{-\Delta}$. Then $\chi \in X(\mu, k, \alpha, E, \Phi)$, and $(\mu, k, \alpha, E, \Phi)$ satisfies all of the conditions of Theorem 2.3. The corollary now follows from Theorem 2.3. Indeed, a straightforward calculation gives

$$n_v = \begin{cases} -\operatorname{ord}_v 4\beta & \text{if } v \text{ is split },\\ -\operatorname{ord}_v \beta & \text{if } v \text{ is non-split.}\end{cases}$$

For example, when $v|\Delta$, $n(\frac{\delta\alpha}{4}\psi_{E_v}) = n(\frac{\alpha_0\beta}{-\delta}\psi_{E_v})$ is the smallest integer $m$ such that

$$\psi_{E_v}(\delta^m \frac{\alpha_0\beta}{\delta}\mathcal{O}_{E_0}) = 1,$$

i.e.,

$$\delta^m \in \delta(\alpha_0\beta)^{-1}\partial_{E_v}^{-1} = \delta(\alpha_0\beta)^{-1}\partial_{E_v/F_v}^{-1}\partial_{F_V}^{-1} = \beta^{-1}\mathcal{O}_{E_v},$$

and so $m \geq -2\operatorname{ord}_v \beta$. Thus $n_v = -\operatorname{ord}_v \beta = 0$ or 1. When $v$ is inert, a similar calculation gives $n_v = 0 = -\operatorname{ord}_v \beta$. So $l = 4\beta f$, and

$$\tau_{\mathfrak{A}^2} = \frac{-\Delta\alpha(b+\delta)}{2l^2a^2} = \frac{\alpha_0(b+\delta)}{8\beta f^2 a^2}$$

as expected. A straightforward calculation also gives $\kappa$ as stated in the corollary. □

For an integral ideal $N$ of $F$, define

$$\Gamma_0(N, \partial_F) := \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) : \, a, d \in \mathcal{O}_F, b \in \partial_F^{-1}, c \in N\partial_F\}.$$

The group $\Gamma_0(N, \partial_F)$ acts on $\mathbb{H}^t$ via the $t$ real embeddings of $F$, and $X_0(N, \partial_F) = \Gamma_0(N, \partial_F)\backslash\mathbb{H}^t$ is an open Hilbert modular variety. Fix a square root $r$ of $\Delta \mod 4N$, and for a primitive integral ideal $\mathfrak{A} \in C^{-1}$, write

$$\mathfrak{A} = [a, \frac{b+\delta}{2}], \quad a, b \in \mathcal{O}_F$$

with $a$ totally positive and $b$ satisfying

$$b^2 \equiv \Delta \mod 4Na, \quad b \equiv r \mod 2N.$$

Then

$$\tau_{\mathfrak{A},r} = \frac{b\alpha_0 + \delta\alpha_0}{2Na} \in X_0(N, \partial_F)$$

depends only on the ideal class $C$, and we will denote it from here forward by $\tau_C$. It is a CM point on $X_0(N, \partial_F)$.

**Theorem 3.5.** *For a Hecke character $\chi \in X(\mu, k, E, \Phi)$ we have*

$$\frac{1}{\#\mathrm{CL}^{\mathrm{ra}}(E)} \sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} \frac{L(\chi\xi, k+1)}{L(\epsilon_{E/F}, 1)} = c(k) \frac{1}{\#\mathrm{CL}(E)^2} \sum_{C \in \mathrm{CL}(E)^2} F_{\mu,k}(\tau_C)$$

*where*

$$F_{\mu,k}(z) := \mathrm{Im}(z)^{k+\frac{1}{2}} |\theta_{\mu,k}(z)|^2$$

*is a real-analytic function defined on $X_0(4f^2, \partial_F)$, $\tau_C$ is a CM point on $X_0(4f^2, \partial_F)$, and*

$$c(k) := \frac{2(8\pi)^{kt}}{(k!)^t} \sqrt{d_F}.$$

*Proof.* Since $\chi \in X(\mu, k, E, \Phi)$, there is an $\alpha \in F^*$ such that $\chi \in X(\mu, k, \alpha, E, \Phi)$. Furthermore, we can choose $\alpha = \frac{4\alpha_0\beta}{-\Delta}$ as in Corollary 3.4. The CM points $\tau_C$ in the theorem can be chosen to be $\tau_{\mathfrak{A}^2,\beta}$ for some $\mathfrak{A}^2 \in C^{-1}$. By Lemma 3.3, one has

$$\{\chi\xi : \xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge\} = X(\mu, k, \alpha, E, \Phi).$$

In particular, the constant $\kappa = \kappa(\chi\xi)$ in Theorem 2.3 (also Corollary 3.4) does not depend on $\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge$.

By definition of the canonical Hecke characters, one has $\overline{\chi(\mathfrak{A})} = \chi(\overline{\mathfrak{A}})$ and

$$\chi(\overline{\mathfrak{A}}\mathfrak{A}) = N_{E/\mathbb{Q}}(\mathfrak{A})^{2k+1}.$$

Using the formula for the central value $L(\chi\xi, k+1)$ in Theorem 2.3 (or equivalently, the formula in Corollary 3.4) and summing over $\mathrm{CL}^{\mathrm{ra}}(E)^\wedge$ yields

$$\begin{aligned}
\sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} L(\chi\xi, k+1) &= \kappa \sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} \sum_{C_1, C_2 \in \mathrm{CL}(E)} \frac{\theta_{\mu,k}(\tau_{\mathfrak{A}_1^2}) \overline{\theta_{\mu,k}(\tau_{\mathfrak{A}_2^2})}}{\chi(\overline{\mathfrak{A}_1}\mathfrak{A}_2)\xi(\overline{\mathfrak{A}_1}\mathfrak{A}_2)} \\
&= \kappa \sum_{C_1, C_2 \in \mathrm{CL}(E)} \frac{\theta_{\mu,k}(\tau_{\mathfrak{A}_1^2}) \overline{\theta_{\mu,k}(\tau_{\mathfrak{A}_2^2})}}{\chi(\overline{\mathfrak{A}_1}\mathfrak{A}_2)} \sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} \frac{1}{\xi(\overline{\mathfrak{A}_1}\mathfrak{A}_2)} \\
&= \kappa \#\mathrm{CL}^{\mathrm{ra}}(E) \sum_{C \in \mathrm{CL}(E)} \frac{|\theta_{\mu,k}(\tau_{\mathfrak{A}^2})|^2}{N_{E/\mathbb{Q}}(\mathfrak{A})^{2k+1}},
\end{aligned}$$

where we used

$$\sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} \frac{1}{\xi(\overline{\mathfrak{A}_1}\mathfrak{A}_2)} = \begin{cases} \#\mathrm{CL}^{\mathrm{ra}}(E) & \text{if} \quad C_1 C_2^{-1} \in \mathrm{CL}_{\mathrm{ra}}(E), \\ 0 & \text{if} \quad C_1 C_2^{-1} \notin \mathrm{CL}_{\mathrm{ra}}(E). \end{cases}$$

Since $\theta_{\mu,k}$ is a real-analytic Hilbert modular form of weight $k+\frac{1}{2}$ for $\Gamma_0(4f^2,\partial_F)$, a straightforward calculation shows that $F_{\mu,k}$ is a real-analytic function of weight zero for $\Gamma_0(4f^2,\partial_F)$. Hence $F_{\mu,k}$ is defined on $X_0(4f^2,\partial_F)$.

Recall that

$$\tau_{\mathfrak{A}^2} = -\frac{\Delta\alpha(b+\delta)}{2l^2a^2}$$

where

$$\delta \in E^\times \quad \text{and} \quad \Delta = \delta^2, \alpha, b, l, a \in F^\times.$$

Fix a CM type $\Phi = \{\sigma_1, \ldots, \sigma_t\}$ of $E$ (here $t = [F:\mathbb{Q}]$). Write

$$\tau_{\mathfrak{A}^2} = -\frac{\Delta\alpha b}{2l^2a^2} - \frac{\Delta\alpha\delta}{2l^2a^2}.$$

Since $\Delta\alpha b/2l^2a^2 \in F^\times$ and the restriction of $\sigma_i$ to $F$ is real, we have

$$\mathrm{Im}(\sigma_i(\tau_{\mathfrak{A}^2})) = \mathrm{Im}(\sigma_i(-\frac{\Delta\alpha\delta}{2l^2a^2})) = \frac{\mathrm{Im}(\sigma_i(-\Delta\alpha\delta))}{2\sigma_i(l)^2\sigma_i(a)^2},$$

and since $\alpha$ is totally positive we have

$$\mathrm{Im}(\sigma_i(-\Delta\alpha\delta)) = -\sigma_i(\delta^3\alpha) = |\sigma_i(\delta^3\alpha)| = |\sigma_i(\Delta^3\alpha^2)|^{1/2}.$$

It follows that

$$\mathrm{Im}(\tau_{\mathfrak{A}^2})^{k+\frac{1}{2}} = \left\{\prod_{i=1}^{t} \mathrm{Im}(\sigma_i(\tau_{\mathfrak{A}^2}))\right\}^{k+\frac{1}{2}} = \frac{|N_{F/\mathbb{Q}}(\Delta^3\alpha^2)|^{\frac{2k+1}{4}}}{2^{t(k+\frac{1}{2})}N_{F/\mathbb{Q}}(l)^{2k+1}N_{F/\mathbb{Q}}(a)^{2k+1}}.$$

Now, $a\mathcal{O}_F = N_{E/F}(\mathfrak{A})$, thus

$$N_{F/\mathbb{Q}}(a) = N_{F/\mathbb{Q}}N_{E/F}(\mathfrak{A}) = N_{E/\mathbb{Q}}(\mathfrak{A}).$$

By combining the preceding facts we obtain

$$\frac{|\theta_{\mu,k}(\tau_{\mathfrak{A}^2})|^2}{N_{E/\mathbb{Q}}(\mathfrak{A})^{2k+1}} = \frac{2^{t(k+\frac{1}{2})}N_{F/\mathbb{Q}}(l)^{2k+1}}{|N_{F/\mathbb{Q}}(\Delta^3\alpha^2)|^{\frac{2k+1}{4}}}F_{\mu,k}(\tau_{\mathfrak{A}^2}).$$

Recall that

$$\kappa = \frac{2^{t/2}\pi^t|N_{F/\mathbb{Q}}(\Delta^3\alpha^2)|^{\frac{2k+1}{4}}}{|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}}N_{F/\mathbb{Q}}(l)^{2k+1}}\left(\frac{(4\pi)^k}{k!}\right)^t.$$

Then substituting in the average formula and simplifying yields

$$\frac{1}{\#\mathrm{CL}^{\mathrm{ra}}(E)}\sum_{\xi\in\mathrm{CL}^{\mathrm{ra}}(E)^\wedge} L(\chi\xi, k+1) = \frac{2^{t/2}2^{t(k+\frac{1}{2})}\pi^t\left(\frac{(4\pi)^k}{k!}\right)^t}{|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}}}\sum_{C\in\mathrm{CL}(E)} F_{\mu,k}(\tau_{\mathfrak{A}^2}).$$

By Lemma 3.6 and our assumption that $F$ has narrow class number one (thus class number one)

$$L(\epsilon_{E/F}, 1) = \frac{2^{t-1}\pi^t\#\mathrm{CL}(E)}{|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}}\sqrt{d_F}}.$$

Hence we get

$$\frac{1}{\#\mathrm{CL}^{\mathrm{ra}}(E)} \sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} \frac{L(\chi\xi, k+1)}{L(\epsilon_{E/F}, 1)} = c(k) \frac{1}{\#\mathrm{CL}(E)} \sum_{C \in \mathrm{CL}(E)} F_{\mu,k}(\tau_{\mathfrak{A}^2})$$

where

$$c(k) = \frac{2^{t/2} 2^{t(k+\frac{1}{2})} \pi^t \left(\frac{(4\pi)^k}{k!}\right)^t}{2^{t-1}\pi^t} \sqrt{d_F} = \frac{2(8\pi)^{kt}}{(k!)^t} \sqrt{d_F}.$$

Finally, using the isomorphism $\mathrm{CL}(E)/\mathrm{CL}_2(E) \cong \mathrm{CL}(E)^2$ we obtain

$$\frac{1}{\#\mathrm{CL}(E)} \sum_{C \in \mathrm{CL}(E)} F_{\mu,k}(\tau_{\mathfrak{A}^2}) = \frac{1}{\#\mathrm{CL}(E)^2} \sum_{C \in \mathrm{CL}(E)^2} F_{\mu,k}(\tau_C).$$

□

**Lemma 3.6.** *One has*

$$L(\epsilon_{E/F}, 1) = \frac{2^{t-1}\pi^t \#\mathrm{CL}(E)}{|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}} \#\mathrm{CL}(F)\sqrt{d_F}}.$$

*Proof.* For a number field $L$, let $\zeta_L(s)$ denote its Dedekind zeta function, and let $\kappa_L$ denote the residue of $\zeta_L(s)$ at $s = 1$. It is well-known that

$$\zeta_E(s) = L(\epsilon_{E/F}, s)\zeta_F(s).$$

Thus $L(\epsilon_{E/F}, 1) = \kappa_E/\kappa_F$. By Dirichlet's analytic class number formula,

$$\kappa_E = \frac{(2\pi)^t \#\mathrm{CL}(E) R_E}{w_E \sqrt{d_E}}$$

where $R_E$ is the regulator, $w_E$ is the number of roots of unity, and $d_E$ is the absolute discriminant. Similarly,

$$\kappa_F = \frac{2^t \#\mathrm{CL}(F) R_F}{w_F \sqrt{d_F}}.$$

Since we chose $\Delta$ so that $d_{E/F} = \Delta\mathcal{O}_F$, we have

$$d_E = d_F^2 N_{F/\mathbb{Q}} d_{E/F} = d_F^2 |N_{F/\mathbb{Q}}(\Delta)|$$

and by [Was, Proposition 4.16],

$$\frac{R_E}{R_F} = \frac{2^{t-1}}{[\mathcal{O}_E^\times : U_E \mathcal{O}_F^\times]} = 2^{t-1}$$

since $\mathcal{O}_E^* = \mathcal{O}_F^*$ by assumption. Hence

$$\frac{\kappa_E}{\kappa_F} = \frac{2^{t-1}\pi^t \#\mathrm{CL}(E)}{|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}} \#\mathrm{CL}(F)\sqrt{d_F}}.$$

□

*Remark* 3.7. It is interesting to note that $F_{\mu,k}$ does not depend on $\chi$, $\Delta$, or the CM type $\Phi$ of $E = F(\sqrt{\Delta})$. Moreover, the average formula in Theorem 3.5 does not depend on the choice of Hecke character $\chi \in X(\mu, k, E, \Phi)$.

## 4. Toric orbits of CM points

In this section we give the explicit relation between the classical and adelic representations of the CM points which appear in the average formula in Theorem 3.5. We then use this to express the average formula in terms of an adelic toric integral.

Recall that for an integral ideal $N$ of $F$,

$$\Gamma_0(N,\partial_F)=\{\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(F):\ a,d\in\mathcal{O}_F, b\in\partial_F^{-1}, c\in N\partial_F\},$$

and define

$$K_0(N,\partial_F):=\{g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(\hat{F}^*):\ a,d\in\hat{\mathcal{O}}_F, b\in\hat{\partial}_F^{-1}=\partial_F^{-1}\otimes\hat{\mathcal{O}}_F, c\in\widehat{N\partial_F}\}.$$

Then

$$K_0(N,\partial_F)\cap \mathrm{GL}^+(F)=\mathcal{O}_F^*\Gamma_0(N,\partial_F).$$

Since $F$ has narrow class number one,

$$\mathrm{GL}_2(\hat{F})=\hat{F}^*\,\mathrm{GL}_2^+(F)K_0(N,\partial_F).$$

Therefore the Hilbert modular variety associated to $K_0(N,\partial_F)$ is

$$X_0(N,\partial_F)=\mathrm{PGL}_2(F)\backslash(\mathbb{H}^{\pm})^t\times\mathrm{PGL}_2(\hat{F})/\hat{F}^*K_0(N,\partial_F)\cong\Gamma_0(N,\partial_F)\backslash\mathbb{H}^t,\quad [z,g_rka]\mapsto[g_r^{-1}z]$$

for $z\in\mathbb{H}^t$, $g_r\in\mathrm{GL}_2^+(F)$, $k\in K_0(N,\partial_F)$, and $a\in\hat{F}^*$.

Now let $N=4\beta f^2\mathcal{O}_F$ be as above, and recall that $\alpha_0\mathcal{O}_F=\partial_F^{-1}$. Let $M\in\mathcal{O}_F$ be totally positive such that every prime factor of $M$ is split in $E$, and choose $b\equiv r \mod 8\beta f^2$ as before, and

$$b^2\equiv\Delta \mod 4NM.$$

Write

$$\tau_0=\frac{\alpha_0(b+\delta)}{2N}\in X_0(N,\partial_F)$$

via the CM type $\Phi$. Then $E=F\tau_0+F$ and $\mathcal{O}_E=\mathcal{O}_F+\mathcal{O}_F\tau_0\frac{N}{\alpha_0}$. Using the basis $\{\tau_0,1\}$, one obtains an injection

$$j:E^*\hookrightarrow\mathrm{GL}_2(F),\quad j(x+y\frac{b+\delta}{2})=\begin{pmatrix} x+by & \frac{(\Delta-b^2)\alpha_0}{4N}\\ y\frac{N}{\alpha_0} & x\end{pmatrix}.$$

Notice that $j(z)\in K_0(N,\partial_F)$ if $z\in\hat{\mathcal{O}}_E^*$. It is straightforward to check that $\tau_0$ is a fixed point of the torus $j(E^*)$. So the associated toric orbit is

$$\{[\tau_0,j(z)]:\ z\in E^*\backslash\hat{E}^*/\hat{\mathcal{O}}_E^*=\mathrm{CL}(E)\}.$$

For each ideal class $C\in\mathrm{CL}(E)$, choose a split prime ideal $\mathfrak{p}$ of $E$ representing $C^{-1}$, i.e., $[\mathfrak{p}]=C^{-1}$, so $\mathfrak{p}\bar{\mathfrak{p}}=p\mathcal{O}_F$ for a totally positive (prime) element $p$. Let $M$ be the square of the product of $h_E$ such primes $p$ (one for each ideal class). Then classically the CM points which appear in Theorem 3.5 are given by

$$\tau_{\mathfrak{p}}=\frac{(b+\delta)\alpha_0}{2Np}=\frac{1}{p}\tau_0,\quad \tau_{\mathfrak{p}^2}=\frac{(b+\delta)\alpha_0}{2Np^2}=\frac{1}{p^2}\tau_0.$$

For each prime $p|M$, $p$ is split in $E$, and so $\Delta\in F_p$ has two square roots in $F_p$. If we denote one of them by $\delta$, then the other one is $-\delta$. We fix an isomorphism $E_{\mathfrak{p}}\cong F_p$ so that $\delta\mapsto\delta$. So in the isomorphism $E_{\bar{\mathfrak{p}}}\cong F_p$, $\delta\mapsto-\delta$.

**Proposition 4.1.** *Let* $\pi_p = (1, p) \in E_p^* = E_{\mathfrak{p}}^* \oplus E_{\bar{\mathfrak{p}}}^*$. *Then*

$$[\tau_{\mathfrak{p}}] = [\tau_0, j(\pi_p)], \quad [\tau_{\mathfrak{p}^2}] = [\tau_0, j(\pi_p^2)]$$

*in* $X_0(4\beta f^2, \partial_F)$, *and thus in* $X_0(4f^2, \partial_F)$. *In particular,* $\{\tau_{\mathfrak{a}} : [\mathfrak{a}] \in \mathrm{CL}(E)\}$ *is a toric orbit of* $E^*$ *in* $X_0(4f^2, \partial_F)$, *and* $\{\tau_{\mathfrak{a}^2} : [\mathfrak{a}^2] \in \mathrm{CL}(E)^2\}$ *is the toric suborbit associated to* $\mathrm{CL}(E)^2 = E^* \backslash E^* \hat{E}^{*,2} \hat{\mathcal{O}}_E^* / \hat{\mathcal{O}}_E^*$.

*Proof.* Write

$$\pi_p = (x + y\frac{b+\delta}{2}, x - y\frac{b-\delta}{2}), \quad x, y \in F_p.$$

Then

$$y = \frac{1-p}{\delta} \in \mathcal{O}_p, \quad x = \frac{p(b+\delta)}{2\delta} + \frac{\delta - b}{2\delta} \in \mathcal{O}_p,$$

and so

$$x + by = \frac{p(b+\delta)}{2\delta} - \frac{pb}{\delta} - \frac{b+\delta}{2\delta} \in p\mathcal{O}_p.$$

Here we used that

$$\frac{\delta + b}{2} \in \mathfrak{p} = [p, \frac{b+\delta}{2}] \subset \mathfrak{p}\mathcal{O}_{\mathfrak{p}} = p\mathcal{O}_p.$$

This implies that

$$k_p = \begin{pmatrix} \frac{x+by}{p} & \frac{(\Delta - b^2)\alpha_0}{4Np} y \\ y\frac{N}{\alpha_0} & x \end{pmatrix} \in K_0(N, \partial_F),$$

and $j(\pi_p) = g_r k$ with $g_r = \mathrm{diag}(p, 1) \in \mathrm{GL}_2(F)$, and $k = g_r^{-1}(g_r)_p k_p \in K_0(N, \partial_F)$. Here $(g_r)_p$ is the $p$-component in $\mathrm{GL}_2(F_p)$ of $g_r$. So

$$[\tau_0, j(\pi_p)] = [\tau_0, g_r k] = [g_r^{-1}\tau_0, 1] = [\tau_{\mathfrak{p}}].$$

The argument gives $[\tau_0, j(\pi_p)^2] = [\tau_{\mathfrak{p}^2}]$. □

As a consequence of Proposition 4.1 we obtain the following corollary of Theorem 3.5.

**Corollary 4.2.** *Let* $\mathcal{T}' = E^* \backslash E^* \hat{E}^{*,2} \hat{\mathcal{O}}_E^* / \hat{\mathcal{O}}_E^*$ *be the suborbit of* $\mathcal{T} = \hat{F}^* E^* \backslash \hat{E}^* / \hat{\mathcal{O}}_E^*$ *associated to* $\mathrm{CL}(E)^2$ *in the Hilbert modular variety* $X_0(4f^2, \partial_F)$. *Then for a Hecke character* $\chi \in X(\mu, k, E, \Phi)$ *we have*

$$\frac{1}{\#\mathrm{CL}^{\mathrm{ra}}(E)} \sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} \frac{L(\chi\xi, k+1)}{L(\epsilon_{E/F}, 1)} = c(k) \frac{1}{\mathrm{vol}(\mathcal{T}')} \int_{\mathcal{T}'} F_{\mu,k}([\tau_0, j(t)])\, dt.$$

## 5. Proof of Theorem 1.8

First, recall that we have chosen $\Delta$ so that the relative discriminant $d_{E/F} = \Delta\mathcal{O}_F$ and thus

$$d_E = d_F^2 N_{F/\mathbb{Q}} d_{E/F} = d_F^2 |N_{F/\mathbb{Q}}(\Delta)|.$$

In particular, $d_E$ and $|N_{F/\mathbb{Q}}(\Delta)|$ are of the "same size" as $d_E \to \infty$.

Clozel and Ullmo [CU], Cohen [C], and Zhang [Zh] independently proved that toric orbits of CM points on Hilbert modular varieties become equidistributed as $|N_{F/\mathbb{Q}}(\Delta)| \to \infty$. Their proofs were conditional on a subconvexity bound which was eventually established by Venkatesh in [V]. As described in [MV1, Section 5.2], the subconvexity bound in [MV1, Theorem 6, (3.4)] implies the equidistribution of *big enough* toric suborbits of the toric orbit. This subconvexity bound was established by Michel and Venkatesh in [MV2], and the resulting equidistribution of toric suborbits is stated in [MV1, Theorem 9]. In our setting, the theorem implies that there exists an absolute constant $0 < \eta < 1$ such that if $\#\mathcal{T}' \geq (\#\mathcal{T})^{\eta}$, then $\mathcal{T}'$ is equidistributed on $X_0(4f^2, \partial_F)$ as $|N_{F/\mathbb{Q}}(\Delta)| \to \infty$.

Since $\#\mathcal{T} = \#\mathrm{CL}(E)$, and $\#\mathcal{T}' = \#\mathrm{CL}(E)/\#\mathrm{CL}_2(E)$ via the isomorphism $\mathrm{CL}(E)^2 \cong \mathrm{CL}(E)/\mathrm{CL}_2(E)$, to establish equidistribution of the toric suborbit $\mathcal{T}'$ it suffices to show that

$$(\#\mathrm{CL}(E))^{1-\eta} \geq \#\mathrm{CL}_2(E). \tag{5.1}$$

By Lemma 5.1, for all $\epsilon > 0$ we have

$$(\#\mathrm{CL}(E))^{1-\eta} \geq c_1(\epsilon)|N_{F/\mathbb{Q}}(\Delta)|^{(1-\eta)(\frac{1}{2}-\epsilon)}$$

where the constant $c_1(\epsilon) > 0$ is ineffective. Furthermore, one has the following estimate (see [Zh, Corollary 6.4])

$$\#\mathrm{CL}_2(E) \leq c_2(\epsilon)|N_{F/\mathbb{Q}}(\Delta)|^{\epsilon}$$

for some constant $c_2(\epsilon) > 0$. Thus inequality (5.1) is implied by

$$|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}(1-\eta)-(2-\eta)\epsilon} \geq c_2(\epsilon)/c_1(\epsilon). \tag{5.2}$$

Clearly, for any $\epsilon < (1-\eta)/2(2-\eta)$, inequality (5.2) holds for all sufficiently large $|N_{F/\mathbb{Q}}(\Delta)|$.

Now, by Proposition 8.1 the conditions $k \geq 0$ is odd or $\mu_v(-1) = -1$ for some prime $v|\mathfrak{f}$ are necessary and sufficient for $\theta_{\mu,k}$ to be cuspidal. Therefore, $F_{\mu,k}$ is smooth and all of its derivatives have exponential decay in the cusps of $X_0(4f^2, \partial_F)$. Then as a consequence of Corollary 4.2 and the equidistribution of $\mathcal{T}'$ we conclude that

$$\frac{1}{\mathrm{vol}(\mathcal{T}')}\int_{\mathcal{T}'} F_{\mu,k}([\tau_0, j(t)])\,dt = \int_{X_0(4f^2,\partial_F)} F_{\mu,k}(z)d\mu_X + o(1)$$

as $|N_{F/\mathbb{Q}}(\Delta)| \to \infty$. □

**Lemma 5.1.** *For all $\epsilon > 0$, one has*

$$\#\mathrm{CL}(E) \geq c(\epsilon)\frac{d_F^{1-2\epsilon}}{(2\pi)^t 2^{t-1} R_F}|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}-\epsilon}$$

*where the constant $c(\epsilon) > 0$ is ineffective.*

*Proof.* By Siegel's lower bound for the residue of $\zeta_E(s)$ at $s = 1$ (see e.g. [St]), for all $\epsilon > 0$,

$$\kappa_E \geq c(\epsilon)d_E^{-\epsilon}$$

where the constant $c(\epsilon) > 0$ is ineffective. Using the formulas for $\kappa_E$, $d_E$ and $R_E/R_F$ in the proof of Lemma 3.6, we obtain

$$\#\mathrm{CL}(E) \geq c(\epsilon)\frac{w_E}{(2\pi)^t R_E}d_E^{\frac{1}{2}-\epsilon} \geq c(\epsilon)\frac{d_F^{1-2\epsilon}}{(2\pi)^t 2^{t-1} R_F}|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}-\epsilon}.$$

□

## 6. Proof of Theorem 1.9

Recall again that we have chosen $\Delta$ so that $d_{E/F} = \Delta\mathcal{O}_F$ and thus $d_E = d_F^2|N_{F/\mathbb{Q}}(\Delta)|$. Let $\phi \in C^\infty(X_0(4f^2, \partial_F), d\mu_X)$ be a smooth function whose derivatives have exponential decay in the cusps of $X_0(4f^2, \partial_F)$. Define the measure

$$\mu_E^{\mathcal{T}'}(\phi) := \frac{1}{\mathrm{vol}(\mathcal{T}')}\int_{\mathcal{T}'}\phi([\tau_0, j(t)])dt$$

where $dt$ is the Haar measure on $\mathcal{T}$. We will show that

$$\mu_E^{\mathcal{T}'} \to \mu_X$$

in the weak-$*$ topology as $|N_{F/\mathbb{Q}}(\Delta)| \to \infty$ with a quantitative bound on the rate of convergence.

Let $\chi$ be a character of $\mathcal{T}$ which is trivial on $\mathcal{T}'$. Define the measure

$$\mu_E^{\mathcal{T},\chi}(\phi) := \frac{1}{\mathrm{vol}(\mathcal{T})}\int_{\mathcal{T}}\chi(t)\phi([\tau_0, j(t)])dt.$$

By Fourier analysis one has the decomposition

$$\mu_E^{\mathcal{T}'}(\phi) = \sum_{\substack{\chi\in\widehat{\mathcal{T}} \\ \chi_{|\mathcal{T}'}\equiv 1}} \mu_E^{\mathcal{T},\chi}(\phi). \tag{6.1}$$

Let $\mathbf{1}$ denote the trivial character on $\mathcal{T}$. By [V, Theorem 7.1], there exists a $\delta > 0$ and constants $d, \beta$ such that

$$\mu_E^{\mathcal{T},\mathbf{1}}(\phi) = \int_{X_0(4f^2,\partial_F)}\phi(z)d\mu_X + O(|N_{F/\mathbb{Q}}(\Delta)|^{-\delta}S^*_{\infty,d,\beta}(\phi))$$

as $|N_{F/\mathbb{Q}}(\Delta)| \to \infty$. Here $S^*_{\infty,d,\beta}(\phi)$ is the Sobolev norm defined in [V, Section 2.10].

Let $\delta_1 > 0$ be arbitrary. By [V, (7.6)], for each nontrivial character $\chi \in \widehat{\mathcal{T}}$ which is trivial on $\mathcal{T}'$ one has the estimate

$$\mu_E^{\mathcal{T},\chi}(\phi) \ll (|N_{F/\mathbb{Q}}(\Delta)|^{3\delta_1\beta-\delta/2} + |N_{F/\mathbb{Q}}(\Delta)|^{\delta_1(\alpha-1/2)} + \mathrm{wt}(E,\delta_1)^{-1/2})S^*_{\infty,d,\beta}(\phi)$$

where $\alpha > 0$ is the constant in the bound towards the Ramanujan conjecture given in [V, eq. (7.5)]. It follows that

$$\sum_{\substack{\chi\in\widehat{\mathcal{T}}\setminus\{\mathbf{1}\} \\ \chi_{|\mathcal{T}'}\equiv 1}} \mu_E^{\mathcal{T},\chi}(\phi) \ll m_E(|N_{F/\mathbb{Q}}(\Delta)|^{3\delta_1\beta-\delta/2} + |N_{F/\mathbb{Q}}(\Delta)|^{\delta_1(\alpha-1/2)} + \mathrm{wt}(E,\delta_1)^{-1/2})S^*_{\infty,d,\beta}(\phi)$$

where

$$m_E := [\mathcal{T} : \mathcal{T}'] = [\mathrm{CL}(E) : \mathrm{CL}(E)^2].$$

From the decomposition (6.1), we conclude that

$$\begin{aligned}\mu_E^{\mathcal{T}'}(\phi) = &\int_{X_0(4f^2,\partial_F)}\phi(z)d\mu_X + O(|N_{F/\mathbb{Q}}(\Delta)|^{-\delta}S^*_{\infty,d,\beta}(\phi)) \\ &+ O(m_E(|N_{F/\mathbb{Q}}(\Delta)|^{3\delta_1\beta-\delta/2} + |N_{F/\mathbb{Q}}(\Delta)|^{\delta_1(\alpha-1/2)} + \mathrm{wt}(E,\delta_1)^{-1/2})S^*_{\infty,d,\beta}(\phi))\end{aligned}$$

as $|N_{F/\mathbb{Q}}(\Delta)| \to \infty$.

By Corollary 4.2 we have

$$\frac{1}{\#\mathrm{CL}^{\mathrm{ra}}(E)}\sum_{\xi\in\mathrm{CL}^{\mathrm{ra}}(E)^\wedge}\frac{L(\chi\xi,k+1)}{L(\epsilon_{E/F},1)}=c(k)\mu_E^{\mathcal{T}'}(F_{\mu,k}).$$

Because $F_{\mu,k}$ is smooth and its derivatives have exponential decay in the cusps of $X_0(4f^2,\partial_F)$, it follows from the preceding analysis with $\phi=F_{\mu,k}$ that

$$\begin{aligned}\frac{1}{\#\mathrm{CL}^{\mathrm{ra}}(E)}\sum_{\xi\in\mathrm{CL}^{\mathrm{ra}}(E)^\wedge}\frac{L(\chi\xi,k+1)}{L(\epsilon_{E/F},1)}&=c(k)\int_{X_0(4f^2,\partial_F)}F_{\mu,k}(z)d\mu_X\\&+O_k(|N_{F/\mathbb{Q}}(\Delta)|^{-\delta}S^*_{\infty,d,\beta}(F_{\mu,k}))\\&+O_k(m_E(|N_{F/\mathbb{Q}}(\Delta)|^{3\delta_1\beta-\delta/2}+|N_{F/\mathbb{Q}}(\Delta)|^{\delta_1(\alpha-1/2)}+\mathrm{wt}(E,\delta_1)^{-1/2})S^*_{\infty,d,\beta}(F_{\mu,k}))\end{aligned}$$

as $|N_{F/\mathbb{Q}}(\Delta)|\to\infty$. By choosing $\delta_1$ to be sufficiently small, we conclude that the error term has the desired form.

□

## 7. Proof of Theorem 1.2

Fix a Hecke character $\chi\in X(\mu,k,E,\Phi)$. Then by Corollary 4.2 and Lemma 3.6 we have

$$\sum_{\xi\in\mathrm{CL}^{\mathrm{ra}}(E)^\wedge}L(\chi\xi,k+1)=\tilde{c}(k)\#\mathrm{CL}^{\mathrm{ra}}(E)\frac{\#\mathrm{CL}(E)}{|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}}}\frac{1}{\mathrm{vol}(\mathcal{T}')}\int_{\mathcal{T}'}F_{\mu,k}([\tau_0,j(t)])dt,$$

where

$$\tilde{c}(k):=c(k)\frac{2^{t-1}\pi^t}{\sqrt{d_F}}.$$

Let $\phi:X_0(4f^2,\partial_F)\to[0,1]$ be a smooth, compactly supported function such that

$$\mu_X(\mathrm{supp}(F_{\mu,k})\cap\mathrm{supp}(\phi))>0.$$

Define $G_{\mu,k}:=F_{\mu,k}\phi$. Then $G_{\mu,k}\in C_c^\infty(X_0(4f^2,\partial_F),d\mu_X)$, and since $F_{\mu,k}\geq 0$,

$$F_{\mu,k}\geq G_{\mu,k}\geq 0.$$

By Lemma 3.1 there is a surjection

$$\mathrm{CL}^{\mathrm{ra}}(E)\twoheadrightarrow\mathrm{CL}(E)^2\cong\mathrm{CL}(E)/\mathrm{CL}_2(E).$$

Thus

$$\#\mathrm{CL}^{\mathrm{ra}}(E)\geq\#\mathrm{CL}(E)/\#\mathrm{CL}_2(E).$$

Combining these estimates yields

$$\sum_{\xi\in\mathrm{CL}^{\mathrm{ra}}(E)^\wedge}L(\chi\xi,k+1)\geq\tilde{c}(k)\frac{(\#\mathrm{CL}(E))^2}{\#\mathrm{CL}_2(E)|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}}}\frac{1}{\mathrm{vol}(\mathcal{T}')}\int_{\mathcal{T}'}G_{\mu,k}([\tau_0,j(t)])\,dt.$$

Because the toric suborbit $\mathcal{T}'$ is equidistributed (see Section 5) we have

$$\frac{1}{\mathrm{vol}(\mathcal{T}')}\int_{\mathcal{T}'}G_{\mu,k}([\tau_0,j(t)])\,dt=\int_{X_0(4f^2,\partial_F)}G_{\mu,k}(z)d\mu_X+o(1)$$

as $|N_{F/\mathbb{Q}}(\Delta)| \to \infty$. Thus

$$\sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} L(\chi\xi, k+1) \geq \tilde{c}(k) \frac{(\#\mathrm{CL}(E))^2}{\#\mathrm{CL}_2(E)|N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}}} \left( \int_{X_0(4f^2, \partial_F)} G_{\mu,k}(z) d\mu_X + o(1) \right)$$

as $|N_{F/\mathbb{Q}}(\Delta)| \to \infty$. By combining the estimates

$$\#\mathrm{CL}(E) \gg_\epsilon |N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}-\epsilon}$$

and

$$\#\mathrm{CL}_2(E) \ll_\epsilon |N_{F/\mathbb{Q}}(\Delta)|^{\epsilon},$$

we find that

$$\sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} L(\chi\xi, k+1) \gg_\epsilon |N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}-\epsilon}$$

as $|N_{F/\mathbb{Q}}(\Delta)| \to \infty$. In particular, this proves that for $|N_{F/\mathbb{Q}}(\Delta)|$ sufficiently large, there exists at least one $\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge$ such that $L(\chi\xi, k+1) \neq 0$.

Recall that $\chi^{\mathrm{un}}$ is the unitary idele character associated to $\chi$. Then

$$L(\chi\xi, k+1) = L(\chi^{\mathrm{un}}\xi, \frac{1}{2}).$$

The conductor of $\chi^{\mathrm{un}}$ is $\mathfrak{f}\sqrt{\Delta}\mathcal{O}_E$. By Hypothesis A.2 in [ELMV], one has the subconvexity bound

$$L(\chi^{\mathrm{un}}\xi, \frac{1}{2}) \ll_t C_\infty(\chi^{\mathrm{un}}\xi, \frac{1}{2})^N (d_E N_{E/\mathbb{Q}}(\mathfrak{f}\sqrt{\Delta}\mathcal{O}_E))^{\frac{1}{4}-\delta} \tag{7.1}$$

for some absolute constants $N, \delta > 0$. Here $C_\infty(\chi^{\mathrm{un}}\xi, \frac{1}{2})$ is the archimedean part of the analytic conductor of $\chi^{\mathrm{un}}\xi$ (see e.g. [ELMV, p. 54, eq. (83)]), which depends on $k$ since $\chi^{\mathrm{un}}\xi$ does. As remarked in the second case following Hypothesis A.2 at the bottom of [ELMV, p. 55], that the bound (7.1) holds for the (varying quadratic) CM extensions $E/F$ follows (by quadratic base change) from the works [BHM], [DFI], and [MV2].

Now, we have $d_E = d_F^2|N_{F/\mathbb{Q}}(\Delta)|$, and

$$\begin{aligned} N_{E/\mathbb{Q}}(\mathfrak{f}\sqrt{\Delta}\mathcal{O}_E) &= N_{F/\mathbb{Q}} N_{E/F}(\mathfrak{f}\sqrt{\Delta}\mathcal{O}_E) \\ &= N_{F/\mathbb{Q}}(\mathfrak{f}^2\Delta\mathcal{O}_F) \\ &= N_{F/\mathbb{Q}}(\mathfrak{f})^2|N_{F/\mathbb{Q}}(\Delta)|. \end{aligned}$$

Substituting these calculations in the subconvexity bound yields

$$L(\chi\xi, k+1) \ll_k |N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}-2\delta},$$

and thus

$$\sum_{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge} L(\chi\xi, k+1) \ll_k \#\{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge : \ L(\chi\xi, k+1) \neq 0\} |N_{F/\mathbb{Q}}(\Delta)|^{\frac{1}{2}-2\delta}.$$

Finally, by combining the upper and lower bounds for the sum of the central values, we see that there exists an absolute constant $\delta_1 > 0$ such that

$$\#\{\xi \in \mathrm{CL}^{\mathrm{ra}}(E)^\wedge : \ L(\chi\xi, k+1) \neq 0\} \gg_{\delta_1, k} |N_{F/\mathbb{Q}}(\Delta)|^{\delta_1}$$

as $|N_{F/\mathbb{Q}}(\Delta)| \to \infty$. $\square$

## 8. The theta function $\theta_{\mu,k}$

In this section we study the Hilbert modular theta function $\theta_{\mu,k}$ using representation theory. In particular, we establish necessary and sufficient conditions for the cuspidality of theta function $\theta_{\mu,k}$ defined in (2.3). For an algebraic group $H$ over $F$, we denote $[H] = H(F)\backslash H(\mathbb{A})$ where $\mathbb{A} = F_{\mathbb{A}}$ is the adeles of $F$. Let $\psi_{\mathbb{Q}} = \prod_p \psi_{\mathbb{Q}_p}$ be the 'canonical' unramified additive character of $\mathbb{Q}\backslash\mathbb{Q}_{\mathbb{A}}$ such that $\psi_{\mathbb{Q}_\infty}(x) = e(x)$, and let $\psi = \psi_{\mathbb{Q}} \circ \mathrm{tr}_{F/\mathbb{Q}}$ be the additive character of $[F]$.

**Proposition 8.1.** *Let $F$ be a totally real number field of degree $t$. Let $k \geq 0$ be an integer and let $\mu$ be a fixed quadratic Hecke character of $F$ of conductor $\mathfrak{f}$ such that $(-1)^{kt}\mu_\infty(-1) = 1$. Then the theta function*

$$\theta_{\mu,k}(z) = \mathrm{Im}(2z)^{-\frac{k}{2}} \sum_{n\in\mathcal{O}_F,(n,\mathfrak{f})=1} \mu'(n)H_{k,F}(n\sqrt{\mathrm{Im}(2z)})e(n^2z)$$

*is a real-analytic Hilbert modular form for $\Gamma_0(4f^2, \partial_F)$ of weight $k+\frac{1}{2}$ (with respect to some multiplier system). It is cuspidal if and only if $k$ is odd or $\mu_v(-1) = -1$ for some prime $v|\mathfrak{f}$.*

*Proof.* Let $V = F$ be the quadratic space with quadratic form $(x, y) = 2xy$ (so $Q(x) = x^2$), and let $G(V) = O(1)$ be its isometry group, as an algebraic group over $F$ (notice that $G(V)(F) = \{\pm1\}$). Let $W = F^2$ with the standard symplectic form with isometry group $\mathrm{Sp}(W) = \mathrm{SL}_2$. Then we have a dual reductive pair $(O(1), \mathrm{SL}_2)$ in $\mathrm{Sp}(V \otimes W) = \mathrm{SL}_2$. This introduces a Weil representation $\omega_\psi = \omega_{V,\psi}$ of $O(1)(\mathbb{A}) \times \mathrm{Mp}_{1,\mathbb{A}}$ on the space $S(\mathbb{A})$ of Schwartz functions on $\mathbb{A}$ (see for example [KRY, Chapter 8] and [Ge]), where $\mathrm{Mp}_{1,\mathbb{A}}$ is the metaplectic $\mathbb{C}^1$-cover of $\mathrm{SL}_2(\mathbb{A})$, where $\mathbb{C}^1$ is the unit circle in the complex plane. Explicitly, we use the normalized cocycles $g' = [g, t]$ as in [KRY, Section 8.5] to identify $\mathrm{Mp}_{1,\mathbb{A}}$ with $\mathrm{SL}_2(\mathbb{A}) \times \mathbb{C}^1$. One has for $\phi = \prod \phi_v \in S(\mathbb{A})$

$$\begin{aligned}
\omega_\psi(h)\phi(x) &= \phi(h^{-1}x), \quad h \in O(1)(\mathbb{A}),\\
\omega_\psi([n(b), 1])\phi(x) &= \psi(bx^2)\phi(x), \quad b \in \mathbb{A}, n(b) = \left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right),\\
\omega_\psi([m(a), 1])\phi(x) &= (2, a)_{\mathbb{A}}|a|_{\mathbb{A}}^{\frac{1}{2}}\phi(ax), \quad a \in \mathbb{A}^\times, m(a) = \mathrm{diag}(a, a^{-1}),\\
\omega_\psi([w_v, 1])\phi(x) &= \gamma_v \prod_{v'\neq v} \phi_{v'}(x) \int_{F_v} \phi_v(y)\psi_v(-2xy)dy.
\end{aligned} \tag{8.1}$$

Here $w_v$ is $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ in the $v$-component and the identity in all other components, and $\gamma_v$ is a root of unity (which is not important to us) such that

$$\prod_v \gamma_v = 1.$$

For every Schwartz function $\phi \in S(\mathbb{A})$, the theta kernel function

$$\theta_\phi(g', h) = \sum_{x\in F} \omega_\psi(g')\phi(h^{-1}x)$$

is an automorphic form of two variables on $(\mathrm{SL}_2(F)\backslash\mathrm{Mp}_{1,\mathbb{A}}) \times [O(1)]$. For a character $\xi$ of $[O(1)]$, one produces an automorphic form on $\mathrm{SL}_2(F)\backslash\mathrm{Mp}_{1,\mathbb{A}}$ (the global theta lifting of $\xi$) by integration against the theta kernel:

$$\theta_\phi(\xi)(g') = \int_{[O(1)]} \theta_\phi(g', h)\xi(h)dh.$$

Here we assign on $O(1)(F_v) = \{\pm 1\}$ and $O(1)(F) = \{\pm 1\}$ the probability measure as its Haar measure, and assign on $O(1)(\mathbb{A})$ the product measure, and on $[O(1)]$ the quotient measure, with total mass 1.

It is natural to ask whether this lifting is nonvanishing (not identically zero) and cuspidal. In general these questions are very interesting and challenging. Our case is well-known, yet it is difficult to find a precise reference so we provide the details. Choose $\phi = \otimes \phi_v \in S(\mathbb{A})$ as follows:

$$\phi_v(x) = \begin{cases} H_k(\sqrt{2}x)e^{-2\pi x^2} & \text{if } v|\infty, \\ \mu_v(x)\mathrm{char}(\mathcal{O}_v^*) & \text{if } v|f, \\ \mathrm{char}(\mathcal{O}_v) & \text{if } v \nmid f\infty. \end{cases}$$

Here $H_k$ is the Hermite function defined in (2.4). Let $\xi = \prod_v \xi_v$ be a character of $O(1)(\mathbb{A})$ given by

$$\xi_v(-1) = \begin{cases} (-1)^k & \text{if } v|\infty, \\ \mu_v(-1) & \text{if } v|\mathfrak{f}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\xi$ defines a character of $[O(1)]$ if and only if

$$\xi(-1) = \prod_v \xi_v(-1) = (-1)^{kt} \prod_{v|\mathfrak{f}} \mu_v(-1) = (-1)^{kt}\mu_\infty(-1)$$

is equal to 1, which is precisely our assumption in this proposition. By the strong approximation theorem, we only need to compute for $g' = g'_z = [n(u_j)m(\sqrt{v}_j), 1] \in \mathrm{Mp}_{1,F_\infty} = \prod_{j=1}^t \mathrm{Mp}_{1,\mathbb{R}}$ where $z = (z_1, \cdots, z_t) \in \mathbb{H}^t$ and $z_j = u_j + iv_j$. Notice that $n(u_j)m(\sqrt{v}_j)(i) = u_j + iv_j = z_j$. A straightforward calculation gives

$$\theta_\phi(\xi)(g'_z) = \sum_{n \in F} \omega_\psi(g'_z)\phi(n) = \mathrm{Im}(z)^{\frac{1}{4}} \sum_{n \in \mathcal{O}_F, (n,\mathfrak{f})=1} \mu'(n) H_{k,F}(n\sqrt{\mathrm{Im}(2z)})e(n^2 z).$$

So

$$\theta_{\mu,k}(z) = 2^{\frac{1}{4}} \,\mathrm{Im}(2z)^{-\frac{2k+1}{4}} \theta_\phi(\xi)(g'_z) \neq 0.$$

By the well-known relationship between classical modular forms and automorphic forms in the adelic language (see [KRY, Proposition 8.5.20]), $\theta_{\mu,k}$ is a Hilbert modular form of weight $k + \frac{1}{2}$ if and only if $\phi_v$ is an eigenfunction of $\tilde{K}_v$ with eigencharacter '$e^{\pi i(k+\frac{1}{2})x}$' for every $v|\infty$ , where $\tilde{K}_v$ is the preimage of $K_v = \mathrm{SO}_2(\mathbb{R}) \subset \mathrm{SL}_2(F_v) = \mathrm{SL}_2(\mathbb{R})$ in $\mathrm{Mp}_{1,F_v}$. This is true by [Wal, Section 5.6]. Here is the rough idea for the convenience of the reader. Write

$$\tilde{K}_v := \{[k_\theta, t] : \, k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, -\pi < \theta \le \pi, t \in \mathbb{C}^1\}.$$

Then

$$\chi_{k+\frac{1}{2}} : \tilde{K}_v \to \mathbb{C}^1, \quad [k_\theta, t] \mapsto te^{i(k+\frac{1}{2})\theta}$$

is a character of $\tilde{K}_v$ of weight $k + \frac{1}{2}$. First, one checks by direct computation that $\phi_v^0 = e^{-2\pi x^2}$ is an eigenfunction of $\tilde{K}_v$ with character $\chi_{\frac{1}{2}}$ (the shift from $e^{-\frac{1}{2}x^2}$ in [Wal] to our $e^{-2\pi x^2}$ is due to the different choice of $\psi_v$ in the Weil representation and the quadratic space we used). Next, using the weight raising operator $A^+$ in [Wal, Section 5.6] (with minor modification), one shows that

$$\phi_v = \phi_v^{(k)} = H_k(\sqrt{2}x)e^{-2\pi x^2} = (A^+)^k \phi_v^0$$

(up to a constant multiple) is an eigenfunction of $\tilde{K}_v$ of weight $k+\frac{1}{2}$ (our $H_k$ is basically $h_k$ in [Wal, Section 5.6]).

Next, we claim that $\theta_{\mu,k}$ is cuspidal if and only if $\xi$ is non-trivial, i.e., either $k$ is odd or $\mu_v(-1)=-1$ for some finite prime $v|\mathfrak{f}$. This is a very special case of the first occurrence theorem of Rallis [Ra] (although he only proved this when the dimension of the quadratic space is even, the proof goes through for odd dimension without trouble). Since our case is simple, we give a direct proof rather than appealing to [Ra]. Whether $\theta_{\mu,k}$ is cuspidal is the same as whether the following integral (the constant term of $\theta_\phi(\xi)$)

$$\int_{[F]} \theta_\phi(\xi)([n(b),1]g')db$$

is zero (different choices of $g'$ correspond to different cusps). A simple calculation gives

$$\begin{aligned}\int_{[F]} \theta_\phi(\xi)([n(b),1]g')db &= \int_{[O(1)]} \xi(h)dh \sum_{x\in F} \omega_\psi(g')\phi(h^{-1}x) \int_{[F]} \psi(bx^2)db \\ &= \omega_\psi(g')\phi(0) \int_{[O(1)]} \xi(h)dh \\ &= \begin{cases} \omega_\psi(g')\phi(0) & \text{if } \xi = 1, \\ 0 & \text{if } \xi \neq 1. \end{cases}\end{aligned}$$

Furthermore, when $\xi=1$, $k$ is even, $H_k(0)\neq 0$, and so

$$\omega_\psi(1)\phi(0) = \phi(0) = H_k(0)^{[F:\mathbb{Q}]} \neq 0.$$

This proves the claim.

Finally, to determine the level, it is enough to check that $\phi$ is $K_0(4f^2,\partial_F)$-invariant up to some multiplier system, which comes from the splitting of $K_0(4,\partial_F)$ to $\mathrm{Mp}_{1,\hat{F}}$. Here

$$K_0(4f^2,\partial_F) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\hat{F}) : \, a,d \in \hat{\mathcal{O}}_F, c \in 4f^2\partial_F\hat{\mathcal{O}}_F, b \in \partial_F^{-1}\hat{\mathcal{O}}_F\}.$$

Let

$$K_0(4,\partial_F) \to \mathrm{Mp}_{1,\hat{F}}, \quad g \mapsto [g,\lambda(g)], \quad \lambda(g) = \prod_{v\nmid\infty} \lambda_v(g)$$

be the splitting as in [KRY, Section 8.5]. For almost all $v$, $\lambda_v(g)=1$. Since we are not concerned with the multiplier system, we can ignore $\lambda_v(g)$.

We now check that under the splitting, $\hat{\phi} = \prod_{v\nmid\infty}\phi_v$ is an eigenfunction of $K_0(4f^2,\partial_F)$.

First, we assume $v\nmid 2f$. For $m(a)=\mathrm{diag}(a,a^{-1})$ with $a\in\mathcal{O}_v^*$,

$$\omega_{V,\psi_v}(m(a))\phi_v(x) = (2,a)_v\phi_v(ax) = \phi_v(x).$$

For $n(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ with $b\in(\partial_F)_v^{-1} = \pi_v^{n(\psi_v)}$, where $\pi_v$ is a uniformizer of $F_v$, we have

$$\omega_{V,\psi_v}(n(b))\phi_v(x) = \psi_v(bQ(x))\phi_v(x) = \phi_v(x).$$

Finally for $n_-(c) = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ with $c\in 4(\partial_F)_v$, we have

$$n_-(c) = w^{-1}n(-c)w, \quad w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

Then

$$\begin{aligned}\omega_{V,\psi_v}(w)\phi_v(x) &= \gamma_v \int_{F_v} \phi_v(y)\psi_v(-(x,y))dy \\ &= \gamma_v \int_{\mathcal{O}_v} \psi_v(-2xy)dy \\ &= \gamma_v \mathrm{char}(\frac{1}{2}\pi_v^{n(\psi_v)}\mathcal{O}_v)(x)\mathrm{vol}(\frac{1}{2}\pi_v^{n(\psi_v)}\mathcal{O}_v).\end{aligned}$$

So

$$\begin{aligned}\omega_{V,\psi_v}(n(-c)w)\phi_v(x) &= \gamma_v \mathrm{vol}(\frac{1}{2}\pi_v^{n(\psi_v)}\mathcal{O}_v)\psi_v(-cx^2)\mathrm{char}(\frac{1}{2}\pi_v^{n(\psi_v)}\mathcal{O}_v)(x) \\ &= \omega_{V,\psi_v}(w)\phi_v(x)\end{aligned}$$

since $\psi_v(-cx^2) = 1$ under our assumption $c \in 4\pi_v^{-n(\psi_v)}\mathcal{O}_v$. So

$$\omega_{V,\psi_v}(n_-(c))\phi_v(x) = \phi_v(x).$$

(Here we ignore the interference of $\lambda_v$ and cocycles, so the above calculation is only true up to a function of $g \in K_0(4f^2, \partial_F)$, which does not affect our conclusion below). Since $K_0(4f^2, \partial_F)$ is generated by $m(a)$, $n(b)$, and $n_-(c)$, we have verified that $\phi_v$ is an eigenfunction of $K_0(4f^2, \partial_F)_v$.

For $v|f$, the proof is similar for $m(a)$ and $n(b)$, and

$$\omega_{V,\psi_v}(m(a))\phi_v(x) = \mu_v(a)\phi_v(x).$$

For $n_-(c) = w^{-1}n(-c)w$, we have

$$\omega_{V,\psi_v}(w)\phi_v(x) = \gamma_v \int_{\mathcal{O}_v^*} \mu_v(y)\psi_v(-2xy)dy = 0$$

unless $n(\psi_v) - \mathrm{ord}_v(2x) = \mathrm{ord}_v f$, i.e., $\mathrm{ord}_v x = n(\psi_v) - \mathrm{ord}_v(2f)$. So

$$\omega_{V,\psi_v}(n(-c)w)\phi_v(x) = \psi_v(-cx^2)\omega_{V,\psi_v}(w)\phi_v(x) = \omega_{V,\psi_v}(w)\phi_v(x)$$

since $\mathrm{ord}_v cx^2 \ge n(\psi_v)$ under the above condition on $x$. So we have again

$$\omega_{V,\psi_v}(n_-(c))\phi_v(x) = \phi_v(x)$$

and $\phi_v$ is an eigenfunction of $K_0(4f^2, \partial_F)_v$. The case $v|2$ is similar and is left to the reader. This completes the proof.

□

## 9. Algebraic Hecke characters and CM abelian varieties

In this section we summarize some facts which we will need concerning CM types, algebraic Hecke characters and their associated CM abelian varieties, and reflex fields. We refer the reader to [Sch] for basic facts about algebraic Hecke characters and to [Sh1] for CM abelian varieties. Let $E$ be a totally imaginary number field. Then a *type* of $E$ is a formal sum

$$\Phi = \sum_{\sigma: E \hookrightarrow \mathbb{C}} n_\sigma \sigma$$

with $n_\sigma \in \mathbb{Z}$. If $L$ is a finite extension of $E$, one can extend $\Phi$ to a type $\Phi_L$ of $L$ by

$$\Phi_L = \sum_{\sigma: L \hookrightarrow \mathbb{C}} n_\sigma \sigma, \quad n_\sigma = n_{\sigma|_E}.$$

A type $\Phi$ is called *simple* if it cannot be extended from a proper subfield. When $E$ is Galois over $\mathbb{Q}$, the embeddings of $E$ into $\mathbb{C}$ are elements of the Galois group $\mathrm{Gal}(E/\mathbb{Q})$. In this case, one defines its *reflex type* by

$$\Phi' = \sum_{\sigma : E \hookrightarrow \mathbb{C}} n_\sigma \sigma^{-1}.$$

In general, given a type $(E, \Phi)$, its reflex type $(E', \Phi')$ is defined as follows. First, the *reflex field* $E'$ of $E$ is the subfield of $\mathbb{C}$ generated by all

$$\mathrm{N}_\Phi(\alpha) = \prod \sigma(\alpha)^{n_\sigma}, \quad \alpha \in E^*.$$

Second, let $L$ be a finite Galois extension of $\mathbb{Q}$ containing both $E$ and $E'$, and extend $\Phi$ to $\Phi_L$. Then it is a standard fact [Sh1, Chapter II, Proposition 28] that there is a unique type $\Phi'$ of $E'$ such that its extension to $L$ is

$$(\Phi')_L = (\Phi_L)'.$$

This $\Phi'$ is independent of the choice of $L$ and is called the reflex type of $\Phi$. Note that the *type norm* $\mathrm{N}_\Phi$ induces a group homomorphism

$$\mathrm{N}_\Phi : \mathrm{CL}(E) \to \mathrm{CL}(E'), \quad \mathfrak{a} \mapsto \prod_{\sigma \in \Phi} \sigma(\mathfrak{a})^{n_\sigma} \mathcal{O}_L \cap E'.$$

An *algebraic Hecke character* of $E$ of infinity type $\Phi$ and modulus $\mathfrak{f}$ (an integral ideal of $E$) is a group homomorphism

$$\chi : I(\mathfrak{f}) \longrightarrow \mathbb{C}^*$$

such that

$$\chi(\alpha \mathcal{O}_E) = \mathrm{N}_\Phi(\alpha) \quad \text{for} \quad \alpha \equiv 1 \mod \mathfrak{f}.$$

Here $I(\mathfrak{f})$ denotes the group of fractional ideals of $E$ prime to $\mathfrak{f}$. The Dirichlet unit theorem implies that there is an algebraic Hecke character of $E$ of type $\Phi$ if and only if $w(\Phi) = n_\sigma + n_{\sigma\rho}$ is independent of the choice of $\sigma$. Here $\rho$ is complex conjugation of $\mathbb{C}$. In such a case, $\Phi$ is called a *Serre type* of weight $w(\Phi)$. Notice that the subfield $\mathbb{Q}(\chi)$ of $\mathbb{C}$ generated by $\chi(\mathfrak{a})$ for $\mathfrak{a} \in I(\mathfrak{f})$ is a number field containing the reflex field $E'$ of $(E, \Phi)$. We say that $\chi$ has *values in* $T$ if $\mathbb{Q}(\chi) \subset T$.

When $E$ is a CM number field, i.e., a totally imaginary quadratic extension of a totally real number field, a type $\Phi$ of $E$ is a *CM type* if $n_\sigma \geq 0$ and $n_\sigma + n_{\sigma\rho} = 1$ for every complex embedding $\sigma$ of $E$. In this case, $\Phi$ is often identified with the set of embeddings $\{\sigma : n_\sigma = 1\}$. In general, any extension of a CM type is also called a CM type.

An abelian variety $A$ defined over a subfield $L$ of $\mathbb{C}$ is said to be a *CM abelian variety* over $L$ if there is a number field $T$ of degree $[T : \mathbb{Q}] = 2 \dim A = 2d$ together with an embedding

$$i : T \hookrightarrow \mathrm{End}^0_L A = \mathrm{End}_L A \otimes \mathbb{Q}.$$

In this case, $T$ acts on the differentials $\Omega_{A/\mathbb{C}}$ diagonally via the $d$ embeddings $\Phi = \{\phi_1, \cdots, \phi_d\}$. That is, there is a basis $\omega_j$ for $\Omega_{A/\mathbb{C}}$ such that for every $\alpha \in i^{-1}(\mathrm{End}_L A)$,

$$i(\alpha)^*(\omega_j) = \phi_j(\alpha)\omega_j.$$

We will identify $\Phi$ with the formal sum $\sum_{\phi \in \Phi} \phi$. We usually say $(A, i)$ is of CM type $(T, \Phi)$. It is a fact that the two seemingly different definitions of CM types are the same (see [Sh1, Theorem 1, p. 40]).

The following theorem ([Ya3, Theorem 1.1]) summarizes the basic relationship between algebraic Hecke characters and CM abelian varieties. Parts (1) and (2) are due to Shimura and Taniyama, and Casselman, respectively ([Sh1, Theorems 19.8 and 19.11], [Sh1, Theorem 21.4]).

**Theorem 9.1.** *Let $E \subset \mathbb{C}$ be a number field. Let $(T, \Phi)$ be a CM type, and let $(T', \Phi')$ be its reflex type. Assume that $T' \subset E$.*

(1) *If $(A, i)$ is a CM abelian variety over $E$ of CM type $(T, \Phi)$, then there is a unique algebraic Hecke character $\chi$ of $E$ of infinity type $\Phi'_E$ with values in $T$ such that $i(\chi(\mathfrak{p}))$ reduces to the Frobenius endomorphism of $A$ modulo $\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ where $A$ has good reduction. In particular,*

$$L(A/E, s) = \prod_{\sigma : T \hookrightarrow \mathbb{C}} L(\chi^\sigma, s).$$

*Here $\chi^\sigma = \sigma \circ \chi$. We call $\chi$ the (algebraic) Hecke character associated to $(A, i)$.*

(2) *Conversely, if $\chi$ is an (algebraic) Hecke character of $E$ of infinity type $\Phi'_E$ with values in $T$, then there is a CM abelian variety $(A, i)$ over $E$ of CM type $(T, \Phi)$, unique up to isogeny, such that the algebraic Hecke character associated to $(A, i)$ is $\chi$. We call $(A, i)$ the abelian variety associated to $\chi$, and denote it by $A(\chi)$.*

(3) *Let $(A, i)$ be a CM abelian variety over $E$ of CM type $(T, \Phi)$, and let $\chi$ be the Hecke character of $E$ associated to $(A, i)$. Then the following are equivalent:*
   *(a) The CM abelian variety $(A, i)$ is simple over $E$.*
   *(b) $T = \mathbb{Q}(\chi)$.*

## 10. Proof of Theorem 1.3

By Theorem 1.2, for all sufficiently large discriminants $d_E$ there exists a $\chi \in X(0, E, \Phi)$ such that $L(\chi_\mu, 1) \neq 0$. Now, the $L$–function of $A(\chi)^\mu$ factors as

$$L(A(\chi)^\mu, 1) = \prod_{\sigma : \mathbb{Q}(\chi_\mu) \hookrightarrow \mathbb{C}} L(\chi_\mu^\sigma, 1).$$

A well-known theorem of Shimura [Sh2] implies that

$$L(\chi_\mu, 1) \neq 0 \quad \text{if and only if} \quad L(\chi_\mu^\sigma, 1) \neq 0.$$

Thus $L(A(\chi)^\mu, 1) \neq 0$, and it follows from Tian and Zhang [TZ] that $\mathrm{Rank} A(\chi)^\mu(E) = 0$. □

## 11. Proof of Proposition 1.4

Let $\chi$ be a Hecke character of $E$ of CM type $\Phi$. By Theorem 9.1, there is a (simple) CM abelian variety $A(\chi)$ over $E$ associated to $\chi$ of CM type $(T, \Phi'_T)$ with $T = \mathbb{Q}(\chi)$.

Let $A$ be an algebraic variety over a number field $H$, and let $E$ be a subfield of $H$. Then the scalar restriction $\mathrm{Res}_{H/E} A$ is an algebraic variety over $E$ such that for any $E$-algebra $R$, one has

$$\mathrm{Res}_{H/E} A(R) = A(R \otimes_E H).$$

It is clear that $L(A, s) = L(\mathrm{Res}_{H/E} A, s)$ when $A$ is an abelian variety over $H$.

**Lemma 11.1.** *Let $E$ be a totally imaginary number field with a CM type $\Phi$, let $\chi$ be an algebraic Hecke character of $E$ of type $\Phi$, and let $A(\chi)$ be the CM abelian variety over $E$ associated to $\chi$. Let $H$ be a finite abelian extension of $E$ with Galois group $G$, and let $A(\chi_H)$ be the CM abelian variety over $H$ associated to $\chi_H = \chi \circ N_{H/E}$. Then the following are equivalent:*

(1) $\mathrm{Res}_{H/E} A(\chi_H) = A(\chi)$ *(isogenous over $E$).*

(2) *One has*

$$\{\chi^\sigma : \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\chi_H))\} = \{\chi\phi : \phi \in G^\wedge\}.$$

(3) $[\mathbb{Q}(\chi) : \mathbb{Q}(\chi_H)] = [H : E]$.

(4) $[H : E] \dim A(\chi_H) = \dim A(\chi)$.

*Proof.* Recall that two abelian varieties over a number field are isogenous if and only if they have the same $L$-function. By class field theory, we can identify characters of $G = \mathrm{Gal}(H/E)$ with Hecke characters of $E$ of finite order. With this identification, we have

$$L(\chi_H, s) = \prod_{\phi \in G^\wedge} L(\chi\phi, s),$$

and thus

$$L(A(\chi_H), s) = \prod_{\sigma:\mathbb{Q}(\chi_H)\hookrightarrow\mathbb{C}} \prod_{\phi\in G^\wedge} L(\chi^\tau\phi, s)$$

where $\tau$ is a complex embedding of $\mathbb{Q}(\chi)$ which equals $\sigma$ when restricted to $\mathbb{Q}(\chi_H)$. So $L(A(\chi_H), s) = L(A(\chi), s)$ if and only if

$$\prod_{\phi\in G^\wedge} L(\chi\phi, s) = \prod_{\tau:\mathbb{Q}(\chi)\hookrightarrow\mathbb{C}, \tau|_{\mathbb{Q}(\chi_H)}=1} L(\chi^\tau, s).$$

When $\tau = 1$ on $\mathbb{Q}(\chi_H)$, $\phi = \chi^{\tau-1}$ has the property

$$\phi(N_{H/E}(\mathfrak{a})) = 1$$

for every ideal of $H$ prime to the conductor of $\chi_H$. Thus $\phi \in G^\wedge$ by class field theory, and we have proved that

$$\{\chi^\tau : \tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\chi_H))\} \subset \{\chi\phi : \phi \in G^\wedge\}.$$

So $L(A(\chi_H), s) = L(A(\chi), s)$ if and only if this inclusion is actually an equality. Since

$$\#\{\chi^\tau : \tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\chi_H))\} = [\mathbb{Q}(\chi) : \mathbb{Q}(\chi_H)],$$

we obtain the lemma. □

*Remark* 11.2. It is easy to see from the proof that in general, $\dim \mathrm{Res}_{H/E} A(\chi_H) \geq \dim A(\chi)$.

**Corollary 11.3.** *Let notation be as in Lemma 11.1. Let $H = H(\Phi)$ be the CM class field of $E$ associated to* $\ker \mathrm{N}_\Phi$*, and assume that* $\chi(\mathfrak{a}) \in E'$ *when* $[\mathfrak{a}] \in \ker \mathrm{N}_\Phi$*. Then* $\mathrm{Res}_{H/E} A(\chi_H) = A(\chi)$ *and*

$$\dim A(\chi_H) = \frac{1}{2}[E' : \mathbb{Q}], \quad \dim A(\chi) = \frac{1}{2}[E' : \mathbb{Q}][\mathrm{CL}(E) : \ker \mathrm{N}_\Phi].$$

*Proof.* This follows from Lemma 11.1 and [R3, Theorem 1]. □

*Remark* 11.4. When a Hecke character of $E$ satisfying the conditions of Corollary 11.3 exists, $A(\chi)$ should have the smallest dimension $\frac{1}{2}[E' : \mathbb{Q}][\mathrm{CL}(E) : \ker \mathrm{N}_\Phi]$ among all CM abelian varieties defined over $E$ whose associated Hecke characters have CM type $\Phi$. A very interesting question is the following: does such a Hecke character always exists, and if not, what is the minimal dimension among all CM abelian varieties defined over $E$ whose associated Hecke characters have CM type $\Phi$?

**Proof of Proposition 1.4**: (a) Under our assumptions on $E$ and $F$, it follows from [BY, Lemma 5.3] that the type norm $\mathrm{N}_\Phi$ induces an isomorphism between $\mathrm{CL}(E)$ and $\mathrm{CL}(E')$. If $\chi$ is any fixed canonical Hecke character of $E$ of CM type $\Phi$, we see that $\chi(\alpha\mathcal{O}_E) \in E'$ by definition. Therefore by [R3, Theorem 1] we have

$$\{\chi^\sigma : \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/E')\} = \{\chi\xi : \xi \in \mathrm{CL}(E)^\wedge\}. \tag{11.1}$$

By Corollary 11.3, $\dim A_E = 2h_E$ since $\ker \mathrm{N}_\Phi = \{1\}$.

(b) Since $\ker \mathrm{N}_\Phi = \{1\}$, $H = H(\Phi)$ is the Hilbert class field of $E$. The equality (11.1) implies that $\chi_H = \chi \circ N_{H/E}$ is independent of the choice of $\chi$. Therefore the CM abelian variety $A_H = A(\chi_H)$ associated to $\chi_H$ is unique up to $H$-isogeny. Finally, Corollary 11.3 implies that $\dim A_H = 2$ and $A_E = \mathrm{Res}_{H/E} A_H$. □

## 12. Proof of Theorems 1.5 and 1.6

Let $\chi$ be a fixed canonical Hecke character of $E$ of CM type $(2k+1)\Phi$. Arguing as in the proof of Proposition 1.4, where we now use the condition $(2k+1, h_E) = 1$ in [R3, Theorem 1], we find that

$$\{\chi^\sigma : \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/E')\} = \{\chi\xi : \xi \in \mathrm{CL}(E)^\wedge\}.$$

By Theorem 1.2, for all sufficiently large discriminants $d_E$ there exists a $\chi \in X(k, E, \Phi)$ such that $L(\chi_\mu, k+1) \neq 0$. Then using the results of Shimura [Sh2] and Tian and Zhang [TZ] as in the proof of Theorem 1.3, we find that $L(\chi_\mu, k+1) \neq 0$ for all $\chi \in X(k, E, \Phi)$ and $\mathrm{Rank} A_H^\mu(H) = \mathrm{Rank} A_E^\mu(E) = 0$. □

*Remark* 12.1. We remark that when $E$ is a CM number field with maximal totally real number field $F$, the type norm $\mathrm{N}_\Phi$ becomes the norm $N_{F/\mathbb{Q}}$ when restricting to $F$, so we have

$$\mathrm{CL}(F) \subset \ker \mathrm{N}_\Phi$$

where $\mathrm{CL}(F)$ stands for the natural image of the class group $\mathrm{CL}(F)$ in $\mathrm{CL}(E)$.

## References


[BHM] V. Blomer, G. Harcos, and Ph. Michel, *Bounds for modular L–functions in the level aspect*. Ann. Sci. Ecole Norm. Sup. **40** (2007), 697–740.

[BY] J. H. Bruinier and T. H. Yang, *CM values of Hilbert modular functions.* Invent. Math. **163** (2006), 229–288.

[CU] L. Clozel and E. Ullmo, *Équidistribution de mesures algébriques.* Compos. Math. **141** (2005), 1255–1309.

[C] P. Cohen, *Hyperbolic equidistribution problems on Siegel 3-folds and Hilbert modular varieties.* Duke Math. J. **129** (2005), 87–127.

[DFI] W. Duke, J. B. Friedlander, and H. Iwaniec, *The subconvexity problem for Artin L-functions*. Invent. Math. **149** (2002), 489–577.

[ELMV] M. Einsiedler, E. Lindenstrauss, P. Michel, and A. Venkatesh, *Distribution of periodic torus orbits and Duke's theorem for cubic fields*. Ann. of Math., to appear.

[FM1] A. Folsom and R. Masri, *Equidistribution of Heegner points and the partition function.* Math. Ann. **348** (2010), 289–317.

[FM2] A. Folsom and R. Masri, *The asymptotic distribution of traces of Maass-Poincaré series.* Adv. Math. **226** (2011), 3724–3759.

[Ge] S. Gelbart, *Weil's representation and the spectrum of the metaplectic group.* Lecture Notes in Mathematics, Vol. 530. Springer-Verlag, Berlin-New York, 1976.

[Gr] B. Gross, *Arithmetic on elliptic curves with complex multiplication. With an appendix by B. Mazur.* Lecture Notes in Mathematics, 776. Springer, Berlin, 1980. iii+95 pp.

[KMY] B. D. Kim, R. Masri and T. H. Yang, *Nonvanishing of Hecke $L$–functions and the Bloch-Kato conjecture.* Math. Ann. **349** (2011), 301–343.

[KL] V. A. Kolyvagin and D. Yu. Logachev, *Finiteness of the Shafarevich-Tate group and the group of rational points for some modular abelian varieties.* (Russian) Algebra i Analiz **1** (1989), no. 5, 171–196; translation in Leningrad Math. J. **1** (1990), 1229–1253.

[KRY] S. Kudla, M. Rapoport, and T. H. Yang, *Modular forms and special cycles on Shimura curves.* Annals of Math. Studies series, vol 161, Princeton Univ. Publ., 2006.

[La] S. Lang, *Complex multiplication.* Springer-Verlag, New York, 1983.

[Ma1] R. Masri, *Asymptotics for sums of central values of canonical Hecke L-series.* Int. Math. Res. Not. IMRN 2007, no. 19, Art. ID rnm065, 27 pp.

[Ma2] R. Masri, *Quantitative nonvanishing of L-series associated to canonical Hecke characters.* Int. Math. Res. Not. IMRN 2007, no. 19, Art. ID rnm070, 16 pp.

[MV1] P. Michel and A. Venkatesh, *Equidistribution, L-functions and ergodic theory: on some problems of Yu. Linnik.* International Congress of Mathematicians. Vol. II, 421–457, Eur. Math. Soc., Zrich, 2006.

[MV2] P. Michel and A. Venkatesh, *The subconvexity problem for $GL_2$.* Publ. Math. Inst. Hautes tudes Sci. No. 111 (2010), 171–271

[MY] S. D. Miller and T. H. Yang, *Nonvanishing of the central derivative of canonical Hecke $L$–functions.* Math. Res. Lett. **7** (2000), 263–277.

[MR] H. Montgomery and D. Rohrlich, *On the L-functions of canonical Hecke characters of imaginary quadratic fields. II.* Duke Math. J. **49** (1982), 937–942.

[Ra] S. Rallis, *On the Howe duality conjecture. Compositio Math*, **51** (1984), 333–399.

[R1] D. Rohrlich, *The nonvanishing of certain Hecke L-functions at the center of the critical strip.* Duke Math. J. **47** (1980), 223–232.

[R2] D. Rohrlich, *On the L-functions of canonical Hecke characters of imaginary quadratic fields.* Duke Math. J. **47** (1980), 547–557.

[R3] D. E. Rohrlich, *Galois conjugacy of unramified twists of Hecke characters.* Duke Math. J. **47** (1980), 695–703.

[R4] D. E. Rohrlich, *Root numbers of Hecke L-functions of CM fields.* Amer. J. Math. **104** (1982), 517–543.

[RVY] F. Rodriguez-Villegas and T. H. Yang, *Central values of Hecke L-functions of CM number fields.* Duke Math. J. **98** (1999), 541–564.

[Ru] K. Rubin, *Elliptic curves with complex multiplication and the conjecture of Birch and Swinnerton-Dyer.* Invent. Math. **64** (1981), 455–470.

[Sch] M. Schappacher, *Periods of Hecke characters.* Lecture Notes in Mathematics **1301**, 1988, xvi+160, Springer-Verlag, Berlin.

[Sh1] G. Shimura, *Abelian varieties with complex multiplication and modular functions.* Princeton Univ. Press, 1998.

[Sh2] G. Shimura, *The special values of the zeta functions associated with cusp forms.* Comm. Pure Appl. Math. **29** (1976), 783–804.

[St] H. Stark, *Some effective cases of the Brauer-Siegel theorem.* Invent. Math. **23** (1974), 135–152.

[T] N. Templier, *On asymptotic values of canonical quadratic $L$–functions*. Int. J. Number Theory. **6** (2010), 1717–1730.

[TZ] Y. Tian and S. W. Zhang, *Kolyvagin systems of CM points on Shimura curves.* preprint, 2008.

[V] A. Venkatesh, *Sparse equidistribution problems, period bounds and subconvexity.* Ann. of Math. **172** (2010), 989–1094.

[Wal] N. R. Wallach *Symplectic geometry and Fourier analysis*, Lie groups: history frontiers and applications, Volume V, Math Sci. Press, 1977.

[Was] L. Washington, *Introduction to cyclotomic fields.* Second edition. Graduate Texts in Mathematics, 83. Springer-Verlag, New York, 1997. xiv+487 pp.

[Ya1] T. H. Yang, *Theta liftings and Hecke L-functions.* J. Reine Angew. Math. **485** (1997), 25–53.

[Ya2] T. H. Yang, *Nonvanishing of central Hecke L-values and rank of certain elliptic curves.* Compositio Math. **117** (1999), 337–359.

[Ya3] T. H. Yang, *On CM abelian varieties over imaginary quadratic fields.* Math. Ann. **329** (2004), 87–117.

[Zh] S. W. Zhang, *Equidistribution of CM-points on quaternion Shimura varieties.* Int. Math. Res. Not. 2005, 3657–3689.

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA
*E-mail address*: masri@math.tamu.edu

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA
*E-mail address*: thyang@math.wisc.edu