FOURIER COEFFICIENTS OF HARMONIC WEAK MAASS FORMS
AND THE PARTITION FUNCTION

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ABSTRACT. In a recent preprint, Bruinier and Ono [BO2] proved that certain harmonic weak Maass forms have the property that the Fourier coefficients of their holomorphic parts are algebraic traces of weak Maass forms evaluated on Heegner points. As a special case, they obtained a remarkable finite algebraic formula for the Hardy-Ramanujan partition function \( p(n) \), which counts the number of partitions of a positive integer \( n \). We establish an asymptotic formula with a power savings in the error term for the Fourier coefficients in the Bruinier-Ono formula. As a consequence, we obtain a new asymptotic formula for \( p(n) \). One interesting feature of this formula is that the main term contains essentially \( 3 \cdot h(-24n+1) \) fewer terms than the truncated main term in Rademacher’s exact formula for \( p(n) \), where \( h(-24n+1) \) is the class number of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-24n+1}) \).

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1. INTRODUCTION AND STATEMENT OF RESULTS

During the last ten years there have been remarkable advances in the study of \( q \)-series, modular forms, and \( L \)-functions through their connection to harmonic weak Maass forms (see for example the excellent survey articles of Ono [O] and Zagier [Za]). Loosely speaking, a harmonic weak Maass form of weight \( k \in \frac{1}{2}\mathbb{Z} \) is a real-analytic function on the complex upper half-plane \( \mathbb{H} \) which transforms like a modular form with respect to some congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \), vanishes under the weight \( k \) hyperbolic Laplacian, and has at most linear exponential growth in the cusps of the group. A detailed discussion of harmonic weak Maass forms can be found in the fundamental paper of Bruinier and Funke [BF] (see also section 3).

A harmonic weak Maass form has a Fourier expansion into a non-holomorphic and holomorphic part. There is a great amount of interest in understanding the arithmetic meaning of these Fourier coefficients. For example, Bruinier and Ono [BO] related the Fourier coefficients of the non-holomorphic and holomorphic parts of weight \( 1/2 \) harmonic weak Maass forms to central values and central derivatives of modular \( L \)-functions, respectively. Bruinier [B2] has since related the Fourier coefficients of the holomorphic parts of weight \( 1/2 \) harmonic weak Maass forms to periods of algebraic differentials of the third kind on modular and elliptic curves.

In a recent preprint, Bruinier and Ono [BO2] constructed a theta lift from the space of harmonic weak Maass forms of weight \(-2\) to the space of vector-valued harmonic weak Maass forms of weight \(-1/2\). They used this lift to prove that certain vector-valued harmonic weak Maass forms of weight \(-1/2\) have the property that the Fourier coefficients of their holomorphic parts are algebraic traces of fixed, weight 0 weak Maass forms evaluated on Heegner points. As a special case they established a finite algebraic formula for the Hardy-Ramanujan partition function \( p(n) \), which counts the number of partitions of a positive
integer $n$. In this paper we will establish an asymptotic formula with a power savings in the error term for the Fourier coefficients in the Bruinier-Ono formula (see Theorem 1.2). As a consequence, we will obtain a new asymptotic formula for $p(n)$ (see Theorem 1.7). One interesting feature of this formula is that the main term contains essentially $3 \cdot h(-24n + 1)$ fewer terms than the truncated main term in Rademacher’s exact formula for $p(n)$, where $h(-24n + 1)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-24n + 1})$ (see the discussion following Theorem 1.7).

In order to state our results we briefly review the Bruinier-Ono “period formula”. Bruinier and Ono [BO2, section 3.1] defined a theta lift $\Lambda : W_{-2}(N) \to W_{-\frac{1}{2},\rho}$ from the space $W_{-2}(N)$ weight -2 weak Maass forms of level $N$ to the space of weight -1/2 vector-valued weak Maass forms $W_{-\frac{1}{2},\rho}$ with Weil representation $\rho$ and level $N$. When restricting to the spaces of harmonic weak Maass forms and weakly holomorphic modular forms, respectively, one has

$$
\Lambda : H_{-2}(N) \to H_{-\frac{1}{2},\rho},
\Lambda : M_{-2}(N) \to M_{-\frac{1}{2},\rho}.
$$

The image of $f \in H_{-2}(N)$ under the theta lift $\Lambda$ has a Fourier expansion of the form

$$
\Lambda(f, w) = \Lambda^-(f, w) + \Lambda^+(f, w),
$$

with non-holomorphic part

$$
\Lambda^-(f, w) := \sum_{h \mod 2N} \sum_{D \in \mathbb{Z}} c_{\Lambda_f}(D, h) \Gamma(1 - k, \pi |D| v / N) q^{D/4N} \mathbf{e}_h,
$$

and holomorphic part

$$
\Lambda^+(f, w) := \sum_{h \mod 2N} \sum_{D=1}^N c_{\Lambda_f^+}(D, h) q^{-D/4N} \mathbf{e}_h + \sum_{h \mod 2N} \sum_{D=0}^\infty c_{\Lambda_f^+}^+(D, h) q^{D/4N} \mathbf{e}_h,
$$

where $N_\infty \geq 0$ is an integer, $\Gamma(a, t)$ is the incomplete Gamma function, and $\{\mathbf{e}_h\}$ is the standard basis for the group algebra $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$. Note that $c_{\Lambda_f^+}(D, h) = 0$ unless $D \equiv h^2$ mod $4N$.

Assume now that $N$ is squarefree and $-D < -4$ is an odd fundamental discriminant coprime to $N$ such that every prime divisor of $N$ splits in $K = \mathbb{Q}(\sqrt{-D})$. Fix a solution $h \in \mathbb{Z}/2N\mathbb{Z}$ of $h^2 \equiv -D$ mod $4N$, and for a primitive integral ideal $\mathfrak{A}$ of $K$ write

$$
\mathfrak{A} = Za + \mathbb{Z} \frac{b + \sqrt{-D}}{2},
$$

with $b \equiv h \mod 2N$ and $b^2 \equiv -D \mod 4Na$. Then

$$
\tau^{(h)}_\mathfrak{A} = \frac{b + \sqrt{-D}}{2Na}
$$

is a Heegner point on the modular curve $X_0(N)$. It is known that $\tau^{(h)}_\mathfrak{A}$ depends only on the ideal class of $\mathfrak{A}$ and on $h \mod 2N$, so we denote it by $\tau^{(h)}_{[\mathfrak{A}]}$. For details concerning these facts, see [GZ, part II, section 1].
Let $\text{CL}(K)$ be the ideal class group of $K$ and $h(-D) = \#\text{CL}(K)$ be the class number. By Minkowski’s theorem we may choose a primitive integral ideal $\mathfrak{A}$ in each ideal class $[\mathfrak{A}] \in \text{CL}(K)$ such that

$$N_{K/Q}(\mathfrak{A}) \leq \frac{2}{\pi} \sqrt{D}.$$ Having fixed such a choice $\mathfrak{A}$ for each ideal class, we define

$$\mathcal{O}_{D,N,h} := \left\{ \tau^{(h)} : [\mathfrak{A}] \in \text{CL}(K) \right\}.$$ Let $\Gamma_\tau$ be the image of the stabilizer of $\tau = \tau^{(h)}_{[\mathfrak{A}]}$ in $\text{PSL}_2(\mathbb{Z})$.

Recall that the Maass level-raising operator $R_{-2} := 2i \frac{\partial}{\partial z} - 2 \text{Im}(z)^{-1}$, $z \in \mathbb{H}$ defines a map

$$R_{-2} : W_{-2}(N) \rightarrow W_0(N)$$ from weak Maass forms of weight -2 to weak Maass forms of weight 0. Define the operator

$$\partial := \frac{1}{4\pi} R_{-2}.$$

Bruinier and Ono [BO2, Theorem 3.6] established a period formula for the Fourier coefficients of the holomorphic parts of weight $-1/2$ vector-valued harmonic weak Maass forms in $\Lambda(H_{-2}(N))$ which in our setup is given as follows.

**Theorem 1.1** (Bruinier-Ono). For each $f \in H_{-2}(N)$, the $(D,h)$-th Fourier coefficient of the holomorphic part of $\Lambda(f,w) \in H_{-\frac{1}{2},\rho}$ is given by

$$c_{\Lambda_f}^+(D,h) = -\frac{4N}{D} \sum_{\tau \in \mathcal{O}_{D,N,h}} \frac{\partial f(\tau)}{\# \Gamma_\tau}.$$ (1.1)

We will establish the following asymptotic formula for the Fourier coefficients $c_{\Lambda_f}^+(D,h)$ in the case of weakly holomorphic forms.

**Theorem 1.2.** Suppose $f \in M_{-2}(N)$ has the Fourier expansion

$$f(z) = \sum_{n=1}^{N_{\infty}} c_f(-n)q^{-n} + \sum_{n=0}^{\infty} c_f(n)q^n.$$ Then the $(D,h)$-th Fourier coefficient of the holomorphic part of $\Lambda(f,w) \in M_{-\frac{1}{2},\rho}$ satisfies

$$c_{\Lambda_f}^+(D,h) = \frac{1}{D} \sum_{\tau \in \mathcal{O}_{D,N,h}, \text{Im}(\tau) > \frac{4N}{\pi} + D^{-\frac{1}{16}}} M_{f,N}(\tau) + C_{f,N} \frac{h(-D)}{D} + O_{c,N}(D^{-\frac{\log 9}{176} + \epsilon}),$$ where

$$M_{f,N}(z) := -4N \sum_{n=1}^{N_{\infty}} c_f(-n)n \left(1 - \frac{1}{2\pi n \text{Im}(z)} \right) e(-nz)$$
and
\[ C_{f,N} := -\frac{N}{\pi} \int_{\text{reg}} R_{-2} f(z) d\mu(z) \]
is a Borcherds-type regularized integral.

**Remark 1.3.** Using a suitable decomposition of the space \( H_{-2}(N) \) as in \([BO2, \text{Corollary 3.4}]\), Theorem 1.2 can be generalized to give an asymptotic formula for the Fourier coefficients of the holomorphic parts of every form in \( \Lambda(H_{-2}(N)) \).

**Remark 1.4.** Let \( \ell \mid N \) with \( (\ell, N/\ell) = 1 \), and let \( f \mapsto f|_{-2} W_{\ell}^{N} \) be the Atkin-Lehner involution of \( M_{1,2}^{!}(N) \) defined in \([BO2, \text{section 4.3}]\). Assume that \( f|_{-2} W_{\ell}^{N} \) has coefficients in \( \mathbb{Z} \) for every such \( \ell \). Bruinier and Ono \([BO2, \text{Theorem 4.5}]\) proved that \( 6D \cdot \partial f(\tau) \) is an algebraic integer in the ring class field for the order \( \mathcal{O}_D \subset \mathbb{Q}(\sqrt{-D}) \).

**Remark 1.5.** The Fourier coefficients \( c_{\Lambda(f)}^{+}(D,h) \) are likely related to periods of differentials of the third kind in a manner similar to \([B2]\).

We now discuss our application to the partition function \( p(n) \). Define the weakly holomorphic modular form \( f_p \in M_{1,2}^{!}(6) \) by
\[
 f_p(z) := \frac{1}{2} \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta(z)^2 \eta(2z)^2 \eta(3z)^2 \eta(6z)^2} = q^{-1} - 10 - 29q - \cdots,
\]
where
\[
 \eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)
\]
is the Dedekind eta function and
\[
 E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \left( \sum_{d|n} d \right) q^n
\]
is the usual weight 2 Eisenstein series. In \([BO2, \text{section 3.2}]\), Bruinier and Ono proved that
\[
 p(n) = -\frac{1}{24} c_{\Lambda(f_p)}^{+}(24n - 1, 1),
\]
which combined with (1.1) yields the finite algebraic formula (see\([BO2, \text{Theorem 1.1}]\))
\[
 p(n) = \frac{1}{24n - 1} \sum_{\tau \in \mathbb{Q}_{24n-1,6,1}} \partial f_p(\tau).
\]

**Remark 1.6.** This formula is algebraic in the following sense. Bruinier and Ono \([BO2, \text{Theorem 4.2}]\) showed that \( f_p(z) \) satisfies the assumption on the Atkin-Lehner involutions in Remark 1.4, and hence that \( 6(24n - 1) \partial f_p(\tau) \) is an algebraic integer in the ring class field of discriminant \(-24n + 1\). Note that Larson and Rolen \([LR]\) later showed that this result holds without the 6, answering a question of Bruinier and Ono \([BO2, \text{section 5, (1)}]\).

By combining Theorem 1.2 with (1.3), we will obtain the following new asymptotic formula for \( p(n) \).
Theorem 1.7. Let $n$ be a positive integer with $24n - 1$ squarefree. Then
\[
p(n) = \frac{1}{24n - 1} \sum_{\tau \in \mathcal{O}_{24n - 1, 6, 1} \atop \text{Im}(\tau) > \frac{2}{3} + (24n - 1)^{-1/4}} \left(1 - \frac{1}{2\pi \text{Im}(\tau)}\right) e(-\tau) + C_{f_p,6}^* \frac{h(-24n + 1)}{24n - 1} + O(n^{-\frac{89}{176} + \epsilon}),
\]
where
\[
C_{f_p,6}^* := \frac{1}{4\pi} \int_{\text{reg}} R_{-2 f_p} (z) d\mu(z).
\]

The asymptotic distribution of $p(n)$ has been studied extensively since the early part of the 20th century. Hardy and Ramanujan [HR] invented the “circle method” and used it to establish the well-known asymptotic
\[
p(n) \sim \frac{1}{4\sqrt{3n}} \exp \left(2\pi \sqrt{n / 6}\right).
\]
Rademacher [R] later used a refinement of the circle method to establish the exact formula
\[
p(n) = \frac{1}{2\pi \sqrt{2}} \sum_{c=1}^{\infty} \sqrt{c} A_c(n) \frac{d}{dn} \exp \left(\frac{\pi \lambda_n \sqrt{2/3}}{c}\right),
\]
where $\lambda_n := \sqrt{n - 1/24}$ and $A_c(n)$ is the exponential sum
\[
A_c(n) := \sum_{\substack{b \mod c \mod 1 \atop (b, c) = 1}} \omega_{b,c} e \left(\frac{-nb}{c}\right),
\]
where $\omega_{b,c}$ is a certain root of unity. By truncating the series at $N := \lceil \sqrt{n/6} \rceil$ and estimating the remainder, Rademacher [R2] obtained the asymptotic formula
\[
p(n) = \text{MT}(n) + O(n^{-\frac{3}{8}}),
\]
where
\[
\text{MT}(n) := \frac{1}{2\pi \sqrt{2}} \sum_{c=1}^{N} \sqrt{c} A_c(n) \frac{d}{dn} \exp \left(\frac{\pi \lambda_n \sqrt{2/3}}{c}\right).
\]
Lehmer [L] improved the error term in (1.4) to $O(n^{-\frac{1}{4} + \epsilon})$. Using an arithmetic reformulation of Rademacher’s exact formula due to Bringmann and Ono [BrO], the author and Folsom [FM] established an asymptotic formula for $p(n)$ with an error term which is $O(n^{-\delta})$ for some absolute $\delta > 1/2$.

We wish to compare the main term in Theorem 1.7 with Rademacher’s truncated main term $\text{MT}(n)$, which reveals some interesting new features in the asymptotic distribution of $p(n)$. Suppose that $n \neq 6\ell^2$ for any $\ell \in \mathbb{Z}$. Then using an analysis with exponential sums (see [FM, Proposition 6.1]), one can show that
\[
\text{MT}(n) = \frac{1}{24n - 1} \sum_{\tau \in \mathcal{O}_{24n - 1, 6} \atop \text{Im}(\tau) > 1} \chi_{12}(\tau) \left(1 - \frac{1}{2\pi \text{Im}(\tau)}\right) e(-\tau),
\]
where
\[ \mathcal{O}_{24n-1,6} := \bigcup_{h \mod 12 \atop h^2 \equiv -24n+1 \mod 24} \mathcal{O}_{24n-1,6,h} \] (1.6)
is the set of Heegner points of discriminant \(-24n+1\) on \(X_0(6)\) and \(\chi_{12}(\tau) = (12b)^{-1}\) where \(-b\) is the middle coefficient of the quadratic form \(Q(X,Y) = 6aX^2 - bXY + cY^2\) corresponding to the Heegner point \(\tau\). The main term in Theorem 1.7 contains no character twist and is summed over one Galois orbit of Heegner points on \(X_0(6)\) and \(\chi_{12}(\tau) = (12b)^{-1}\) where \(-b\) is the middle coefficient of the quadratic form \(Q(X,Y) = 6aX^2 - bXY + cY^2\) corresponding to the Heegner point \(\tau\). The size of each Galois orbit is \(h(-24n+1)\), and there are 4 orbits corresponding to the solutions \(h = 1, 5, 7,\) and 11 of the congruence in (1.6), so the main term in Theorem 1.7 contains essentially 3 \(\cdot h(-24n+1)\) fewer terms than Rademacher’s main term \(MT(n)\). It would be very interesting to know whether the shorter main term (or the form of the main term itself) in Theorem 1.7 offers any computational advantages.

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2. Outline of the proof of Theorem 1.2

In this section we give a brief outline of the proof of Theorem 1.2. We want to establish an asymptotic formula as \(D \to \infty\) for the trace
\[ \sum_{\tau \in \mathcal{O}_{D,N,h}} \frac{\partial f(\tau)}{\#\mathcal{H}_\tau} \]
appearing in the Bruinier-Ono formula (1.1). The image of a weakly holomorphic modular form \(f\) in \(M^!_{2}(\mathbb{Q})\) under the Maass level-raising operator \(\partial = \frac{1}{4\pi}R_{-2}\) is in the space \(W_0(\mathbb{Q})\) of weight 0 weak Maass forms of level \(\mathbb{Q}\). We will show that the function \(\partial f\) can be expressed as a finite linear combination of certain Maass-Poincaré series \(\{F_{N,n}\}_{n=1}^{\infty}\) in \(W_0(\mathbb{Q})\), thus reducing us to the study of the trace of \(F_{N,n}\). We will compute the Fourier expansion of \(F_{N,n}\) in the cusps of \(\Gamma_0(\mathbb{Q})\) and find that it has a part with linear exponential growth in the cusps and a part with polynomial growth in the cusps. The Galois orbit of Heegner points \(\mathcal{O}_{D,N,h}\) is becoming quantitatively equidistributed with respect to the invariant hyperbolic measure on the open modular curve \(Y_0(\mathbb{Q}) = \Gamma_0(\mathbb{Q}) \setminus \mathbb{H}\) as \(D \to \infty\). However, we cannot directly use this fact to obtain an asymptotic formula for the trace because the “test function” \(F_{N,n}\) grows very rapidly in the cusps and hence is not admissible. We overcome this using two different regularizations. We first prove an equidistribution theorem for Galois orbits of Heegner points in which the test functions are allowed to grow polynomially in the cusps (see Theorem 6.3). We then construct for each \(\eta > 0\) a certain smooth Poincaré series \(\mathcal{P}_{n,\eta}\) which regularizes the linear exponential growth of \(F_{N,n}\) in the cusps (this is influenced by a construction of Duke [D] to regularize the pole of the \(j\)-function). Upon substituting the regularized function \(F_{N,n} - \mathcal{P}_{n,\eta}\) into the equidistribution theorem, we obtain a “smooth” asymptotic formula for the trace of \(F_{N,n}\). Finally, using a Borcherds-type integration along with a delicate analysis to unsmooth the main term and bound (as a function of \(\eta\)) the Sobolev norm of \(F_{N,n} - \mathcal{P}_{n,\eta}\) appearing in the error term, we obtain the desired asymptotic
formula for the trace of $F_{N,n}$ (see Theorem 9.1). Our proof has some elements in common with [FM2], though considerable new difficulties arise because of the presence of level in all of our arguments (which is crucial for our application to the partition function).

3. harmonic weak Maass forms

In this section we review some facts concerning harmonic weak Maass forms. For more details, see the fundamental paper of Bruinier and Funke [BF]. Let $z = x + iy \in \mathbb{H}$. Let $k \in \frac{1}{2}\mathbb{Z}$ and $N$ be a positive integer. The weight $k$ hyperbolic Laplacian is defined by

$$\Delta_k := -y^2 (\partial_x^2 + \partial_y^2) + iky(\partial_x + i\partial_y).$$

A weak Maass form of weight $k$ on $\Gamma_0(N)$ is a smooth function $f: \mathbb{H} \to \mathbb{C}$ satisfying:

1. $f|_k M = f$ for all $M \in \Gamma_0(N)$;
2. There is a complex number $\lambda$ such that $\Delta_k f = \lambda f$;
3. There is a constant $C > 0$ such that $f(z) = O(e^{Cy})$ as $y \to \infty$. An analogous condition is required at all cusps.

Note. The slash operator $|_k$ is defined as in Shimura’s theory of half-integral weight forms. A weak Maass form is harmonic if $\Delta_k f = 0$. Every harmonic weak Maass form has a Fourier expansion of the form

$$f(z) = f^-(z) + f^+(z)$$

with non-holomorphic part

$$f^-(z) := \sum_{n < 0} c_f^-(n) \Gamma(1 - k, 4\pi |n| y)q^n$$

and holomorphic part

$$f^+(z) := \sum_{n=1}^{N_\infty} c_f^+(n) q^{-n} + \sum_{n=0}^{\infty} c_f^+(n) q^n,$$

where $N_\infty \geq 0$ is an integer and $\Gamma(a,t)$ is the incomplete Gamma function. A harmonic weak Maass form with trivial non-holomorphic part is called a weakly holomorphic modular form.

Let

$$M_k^0(N) \subset H_k(N) \subset W_k(N)$$

denote the spaces of weakly holomorphic modular forms, harmonic weak Maass forms, and weak Maass forms, respectively.

We also require the notion of a vector-valued weak Maass form. Let $w = u + iv \in \mathbb{H}$, and let $Mp_2(\mathbb{R})$ be the metaplectic two-fold cover of $SL_2(\mathbb{R})$ realized as the group of pairs $(M, \phi(w))$ where $M \in SL_2(\mathbb{R})$ and $\phi: \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi(w)^2 = cw + d$. The multiplication is defined by

$$(M, \phi(w))(M', \phi'(w)) = (MM', \phi(M'w)\phi'(w)).$$

Let $\tilde{\Gamma} = Mp_2(\mathbb{Z})$ be the inverse image of $SL_2(\mathbb{Z})$ under the covering map. The group $\tilde{\Gamma}$ is generated by $T = ((1 1), 1)$ and $S = ((0 -1), \sqrt{7})$. For $h \in \mathbb{Z}/2N\mathbb{Z}$, let $e_h$ be the corresponding standard basis vector of the group algebra $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$. The Weil representation...
\( \rho \) is the unitary representation of \( \tilde{\Gamma} \) on \( \mathbb{C}[\mathbb{Z}/2NZ] \) defined in terms of the generators \( T \) and \( S \) of \( \tilde{\Gamma} \) by

\[
\rho(T)(\epsilon_h) = e\left(\frac{h^2}{4N}\right) \epsilon_h,
\]
\[
\rho(S)(\epsilon_h) = \frac{1}{\sqrt{2iN}} \sum_{h' \mod 2N} e\left(-\frac{hh'}{2N}\right) \epsilon_{h'}.
\]

A vector-valued weak Maass form of weight \( k \) (with respect to \( \tilde{\Gamma} \) and \( \rho \)) is a smooth function \( g : \mathbb{H} \to \mathbb{C}[\mathbb{Z}/2NZ] \) satisfying:

1. \( g(Mw) = \phi(w)^{2k} \rho(M, \phi)g(w) \) for all \( (M, \phi) \in \tilde{\Gamma} \).
2. There is a complex number \( \lambda \) such that \( \Delta_k g = \lambda g \).
3. There is a constant \( C > 0 \) such that \( g(w) = O(e^{Cv}) \) as \( v \to \infty \). An analogous condition is required at all cusps.

Every vector-valued harmonic weak Maass form has a Fourier expansion of the form \( g = g^- + g^+ \) with non-holomorphic part

\[
g^-(w) := \sum_{h \mod 2N} \sum_{D < 0} c_g^- (D, h) \Gamma(1 - k, \pi|D|v/N) q^{D/4N} \epsilon_h
\]
and holomorphic part

\[
g^+(w) := \sum_{D \equiv h^2 \mod 4N} \sum_{h \mod 2N} c_g^+ (D, h) q^{-D/4N} \epsilon_h + \sum_{h \mod 2N} \sum_{D = 0} c_g^+ (D, h) q^{D/4N} \epsilon_h.
\]

Note that \( c_g^\pm (D, h) = 0 \) unless \( D \equiv h^2 \mod 4N \).

Let \( M^i_{k, \rho} \subset H_{k, \rho} \subset W_{k, \rho} \) denote the vector spaces of vector-valued weakly holomorphic modular forms, vector-valued harmonic weak Maass forms, and vector-valued weak Maass forms, respectively.

### 4. The Bruinier-Ono theta lift

Let \( z \in \mathbb{H} \) and \( k \in \frac{1}{2} \mathbb{Z} \). The Maass level raising and lowering differential operators are defined by

\[
R_{k, z} := 2i \frac{\partial}{\partial z} + k \text{Im}(z)^{-1},
\]
\[
L_{k, z} := -2i \text{Im}(z)^2 \frac{\partial}{\partial z}.
\]

Bruinier and Ono [BO2, sections 2.3 and 3.1] defined a theta lift

\[
\Lambda : W_{-2}(N) \to W_{-\frac{3}{2}, \rho}
\]
by

\[
\Lambda(f, w) := L_{3/2, w} \int_{T_0(N) \backslash \mathbb{H}} (R_{-2, z} f(z)) \Theta(w, z, \phi_{KM}),
\]

where \( \Theta(w, z, \phi_{KM}) \) is a \( \mathbb{C}[\mathbb{Z}/2NZ] \)-valued theta function constructed from the Kudla-Millson Schwartz function \( \phi_{KM} \) (see [KM] and [BO2, section 2.3]). As a function of \( z \), the theta kernel \( \Theta(w, z, \phi_{KM}) \) is a \( \Gamma_0(N) \)-invariant harmonic \( (1, 1) \)-form on \( \mathbb{H} \). When restricting \( \Lambda \) to
harmonic weak Maass forms and weakly holomorphic modular forms, respectively, one has (see [BO2, Corollary 3.4 and Theorem 3.5])

\[
\Lambda : H_{-2}(N) \to \mathbf{H}_{-\frac{1}{2}, \rho}, \\
\Lambda : M^!_{-2}(N) \to M^!_{-\frac{1}{2}, \rho}.
\]

5. Differential operators and weak Maass forms

Let \( m \in \mathbb{Z}^+ \) and define the Maass-Poincaré series (see [B, section 1.3])

\[
F_{N,m}(z) := \pi m \sum_{\gamma \in \Gamma \setminus \Gamma_0(N)} \text{Im} (m \gamma z)^{1/2} I_{3/2}(2\pi m \text{Im}(\gamma z)) e(-m \text{Re}(\gamma z)), \quad z = x + iy \in \mathbb{H}
\]

where \( I_{3/2} \) is the usual \( I \)-Bessel function of order \( 3/2 \). It is known that \( F_{N,m} \) is a weak Maass form of weight 0 with eigenvalue \(-2\) with respect to the weight 0 hyperbolic Laplacian \( \Delta := -y^2(\partial_x^2 + \partial_y^2) \).

Define the differential operator

\[
\partial_{-2} := \frac{1}{2\pi i} \frac{\partial}{\partial z} + \frac{1}{2\pi y}.
\]

Then \( \partial_{-2} \) defines a map

\[
\partial_{-2} : M^!_{-2}(N) \to W_0(N).
\]

Suppose that \( f \in M^!_{-2}(N) \) has the Fourier expansion

\[
f(z) = \sum_{n=1}^{N_{\infty}} c_f(-n) q^{-n} + \sum_{n=0}^{\infty} c_f(n) e(nz).
\]

Then by [MP, Theorem 1.2],

\[
\partial_{-2}(f) = -2 \sum_{n=1}^{N_{\infty}} c_f(-n) F_{N,n}.
\]

In particular, the identity

\[
\frac{1}{4\pi} R_{-2} = -\partial_{-2}
\]

implies that

\[
\frac{1}{4\pi} R_{-2}(f) = 2 \sum_{n=1}^{N_{\infty}} c_f(-n) F_{N,n}.
\]

6. Quantitative equidistribution

For an integer \( A \geq 0 \), let \( \Delta^A \) be the composition of the weight 0 hyperbolic Laplacian with itself \( A \)-times, and for cusp \( \mathfrak{a} \) of \( \Gamma_0(N) \), let \( \sigma_{\mathfrak{a}} \in \text{SL}_2(\mathbb{R}) \) be a scaling matrix such that \( \sigma_{\mathfrak{a}}(\infty) = \mathfrak{a} \) (see [I, p. 47]).
Proposition 6.1. Let \( g : \mathbb{H} \to \mathbb{C} \) be a \( C^\infty \), \( \Gamma_0(N) \)-invariant function, and suppose that for each cusp \( \alpha \) we have

\[
\Delta^A g(\sigma_\alpha z) = O(e^{-Cy}), \quad A = 0, 1, 2, \ldots
\]

for some constant \( C > 0 \) (depending on \( \alpha \) and \( A \)). Then

\[
\sum_{\tau \in \mathcal{O}_{D,N,h}} g(\tau) = h(-D) \int_{Y_0(N)} g(z) d\mu(z) + O_{\epsilon,N}(||\Delta^2 g||_2 2^{1-\frac{1}{16}+\epsilon}).
\]

Proof. Let \( \{u_m(z)\}_{m=1}^\infty \) be an orthonormal basis of Maass cusp forms for \( \Gamma_0(N) \) satisfying \( \Delta u_m = \lambda_m u_m \) where the eigenvalues \( \lambda_m = t_m^2 + \frac{1}{4} \) for \( m \in \mathbb{Z}^+ \) are ordered so that \( \lambda_1 \leq \lambda_2 \leq \cdots \). For each cusp \( \alpha \) define the real-analytic Eisenstein series

\[
E_\alpha(z, s) := \sum_{\gamma \in \mathcal{O}_\alpha \backslash \Gamma_0(N)} \text{Im}(\sigma_\alpha^{-1} \gamma z)^s, \quad z \in \mathbb{H}, \quad \text{Re}(s) > 1.
\]

By the bound (6.1) the spectral decomposition of \( L^2(Y_0(N)) \) with respect to \( \Delta \) yields the expansion

\[
g(z) = \langle g, 1 \rangle + \sum_{m=1}^\infty \langle g, u_m \rangle u_m(z) + \sum_\alpha \frac{1}{4\pi} \int_\mathbb{R} \langle g, E_\alpha(\cdot, \frac{1}{2} + it) \rangle E_\alpha(z, \frac{1}{2} + it) dt,
\]

which converges pointwise absolutely and uniformly on compact subsets of \( Y_0(N) \). Summing over the spectral expansion yields

\[
\sum_{\tau \in \mathcal{O}_{D,N,h}} g(\tau) = h(-D) \langle g, 1 \rangle + \sum_{m=1}^\infty \langle g, u_m \rangle W_m(D) + \sum_\alpha \frac{1}{4\pi} \int_\mathbb{R} \langle g, E_\alpha(\cdot, \frac{1}{2} + it) \rangle W_\alpha(D, t) dt,
\]

(6.2)

where the hyperbolic Weyl sums are defined by

\[
W_m(D) := \sum_{\tau \in \mathcal{O}_{D,N,h}} u_m(\tau) \quad \text{and} \quad W_\alpha(D, t) := \sum_{\tau \in \mathcal{O}_{D,N,h}} E_\alpha(\tau, \frac{1}{2} + it).
\]

To estimate the contribution of the discrete spectrum, it suffices to consider only \( L^2 \)-normalized Maass newforms for \( \Gamma_0(N) \) (see e.g. the proof of [HM, Theorem 6]). By Zhang [Z], for such a newform \( u_m \) one has the period formula

\[
\left| \sum_{\tau \in \mathcal{O}_{D,N,h}} u_m(\tau) \right|^2 = c_D \sqrt{D} |a_m(1)|^2 \Lambda(u_m, \frac{1}{2}) \Lambda(u_m \otimes (-D), \frac{1}{2}),
\]

(6.3)

where \( c_D \) is a positive constant which takes only finitely many different values, \( a_m(1) \) is the first Fourier coefficient of \( u_m \), and

\[
\Lambda(\Pi, s) := L_\infty(\Pi, s) L(\Pi, s)
\]

is the completed \( L \)-function. Note that the term \( |a_m(1)|^2 \) appears on the right hand side because in Zhang’s formula the Maass newform in the Rankin-Selberg \( L \)-function is arithmetically normalized.

By a subconvexity bound of Jutila and Motohashi [JM] we have

\[
L(u_m, \frac{1}{2}) \ll_{\epsilon,N} \lambda_m^{\frac{1}{2}+\epsilon},
\]
and by a hybrid subconvexity bound of Blomer and Harcos [BH, Theorem 2] we have
\[ L(u_m \otimes (-D), \frac{1}{2}) \ll_{\epsilon, N} \lambda_m^{\frac{7}{8} + \epsilon} D^{\frac{1}{2} - \frac{1}{6} + \epsilon}. \]

Then using the following estimate of Hoffstein and Lockhart [HL]
\[ |a_m(1)|^2 \ll \lambda_m \epsilon^{|t|}, \]
and the fact that the contribution from the infinite parts of the \( L \)-functions in (6.3) is \( \ll e^{-\pi |t_m|} \) by Stirling’s formula, we obtain the estimate
\[ W_m(D) \ll_{\epsilon, N} \lambda_m^{\frac{23}{24} + \epsilon} D^{\frac{1}{2} - \frac{1}{16} + \epsilon}. \]

(6.4)

Following the argument in [HM, section 6.4], one can reduce the estimate of \( W_a(D, t) \) to an analogous estimate for
\[ \sum_{\tau \in O_{D,1,1}} E(\tau, \frac{1}{2} + it), \]
where \( E(z, s) \) is the real-analytic Eisenstein series for \( \text{SL}_2(\mathbb{Z}) \). One has the identity (see e.g. [GZ, p. 248])
\[ \sum_{\tau \in O_{D,1,1}} E(\tau, s) = 2^{-s}D^{s/2} \zeta(2s)^{-1} \zeta(s) L((\frac{-D}{\cdot}), s). \]

Then using a standard lower bound for \( \zeta(2s) \), the subconvexity bound
\[ \zeta(\frac{1}{2} + it) \ll (\frac{1}{4} + t^2)^{\frac{1}{2} + \epsilon}, \]
and the hybrid subconvexity bound of Heath-Brown [HB]
\[ L((\frac{-D}{\cdot}), \frac{1}{2} + it) \ll \epsilon (\frac{1}{4} + t^2)^{\frac{1}{2} + \epsilon} D^{\frac{1}{2} - \frac{1}{16} + \epsilon}, \]
we obtain
\[ W_a(D, t) \ll \epsilon (\frac{1}{4} + t^2)^{\frac{1}{2} + \epsilon} D^{\frac{1}{2} - \frac{1}{16} + \epsilon}. \]

(6.5)

By the bound (6.1), a repeated application of Stokes’ theorem (see e.g. [I, Lemma 1.18]) yields the following identities,
\[ \langle g, u_n \rangle = \lambda_n^{-A} \langle \Delta^A g, u_n \rangle \]
and
\[ \langle g, E_a(\cdot, \frac{1}{2} + it) \rangle = (\frac{1}{4} + t^2)^{-A} \langle \Delta^A g, E_a(\cdot, \frac{1}{2} + it) \rangle. \]
By Parseval’s formula (see [IK, (15.17)]),
\[ \sum_{n=1}^{\infty} |\langle \Delta^A g, u_n \rangle|^2 + \sum_a \frac{1}{4\pi} \int_{\mathbb{R}} |\langle \Delta^A g, E_a(\cdot, \frac{1}{2} + it) \rangle|^2 dt = ||\Delta^A g||^2_2. \]
Then by the Cauchy-Schwarz inequality we have
\[ \sum_{n=1}^{\infty} |\langle g, u_n \rangle_2| \lambda_n^{\frac{23}{24} + \epsilon} \ll ||\Delta^2 g||_2. \]

(6.6)
and
\[ \int_{\mathbb{R}} |\langle g, E_a(\cdot, \frac{1}{2} + it) \rangle| \left(\frac{1}{4} + t^2\right)^{\frac{3}{2} + \epsilon} dt \ll \|\Delta^2 g\|_2. \] (6.7)

Finally, the proposition follows by combining (6.2) with the estimates (6.4)–(6.5) and (6.6)–(6.7). \( \square \)

**Definition 6.2.** Let \( g : \mathbb{H} \to \mathbb{C} \) be a \( C^\infty \), \( \Gamma_0(N) \)-invariant function and \( \alpha \) be a real number. We say that \( g \) has **cuspidal growth of power** \( \alpha \) if for each cusp \( a \) of \( \Gamma_0(N) \) there exists a constant \( c_a \in \mathbb{C} \) such that
\[ \Delta^A (g(a^z) - c_a y^\alpha) = O(\epsilon^{-C}) \]
for some constant \( C > 0 \) (depending on \( a \) and \( A \)).

We now use Proposition 6.1 and a suitable regularization to establish the following

**Theorem 6.3.** Suppose that \( F \) has cuspidal growth of power \( \alpha < 5/8 \). Then
\[ \sum_{\tau \in O_{D,N,h}} F(\tau) = h(-D) \int_{Y_0(N)} F(z) d\mu(z) \]
\[ + O_{\epsilon,N}(\|\Delta^2 F_{T_0}\|_2 D^{\frac{3}{2} - \frac{3}{16} + \epsilon}) + O_{\epsilon,N}(D^{\frac{1}{2} - (1 - \alpha) + \epsilon}) + O_{\epsilon,N}(D^{\frac{1}{2} - c(\alpha) + \epsilon}) \]
where \( F_{T_0} \) is a regularized version of \( F \) for some constant \( T_0 > 1 \) (see (6.9)) and
\[ c(\alpha) := \begin{cases} \frac{1}{16}, & \alpha \leq \frac{1}{2} \\ \frac{1}{16} - \left(\frac{\alpha}{2} - \frac{1}{4}\right), & \frac{1}{2} < \alpha < \frac{5}{8}. \end{cases} \]

**Proof.** For \( T > 1 \), define
\[ \psi_T(t) := t^\alpha \chi(t/T), \]
where \( \chi : \mathbb{R}^+ \to [0, 1] \) is a \( C^\infty \) function such that
\[ \chi(t) = \begin{cases} 0, & t < 1 \\ 1, & t > 2. \end{cases} \]

Let
\[ E_b(\psi_T|z) := \sum_{\gamma \in \Gamma_b \setminus \Gamma_0(N)} \psi_T(\text{Im}(\sigma_b^{-1} \gamma z)), \]
and define
\[ \eta_T(z) := \sum_b c_b E_b(\psi_T|z). \]

Then by [I, (3.17)] and (8.2) we have
\[ \eta_T(a^z) = \begin{cases} 0, & 1 < y < T \\ c_a y^\alpha \chi(y/T), & T \leq y \leq 2T \\ c_a y^\alpha, & y > 2T. \end{cases} \]

It follows that for \( y > 2T \) the regularized function
\[ F_T(z) := F(z) - \eta_T(z) \]
satisfies the bound (6.1).

Now, let

\[ T = T_D := 1 + \max\left\{ \sqrt{D}, \frac{4N}{\pi} \right\}. \]

Then by Lemma 6.4 we have \( \eta_T(\tau) = 0 \) for all \( \tau \in \mathcal{O}_{D,N,h} \) and thus

\[ \sum_{\tau \in \mathcal{O}_{D,N,h}} F(\tau) = \sum_{\tau \in \mathcal{O}_{D,N,h}} F_T(\tau). \tag{6.8} \]

Let \( T_0 \) be a constant (independent of \( D \)) with \( T > T_0 > 1 \) and decompose the regularized function as

\[ F_T(z) = F_{T_0}(z) + \tilde{\eta}_T(z) \tag{6.9} \]

where

\[ \tilde{\eta}_T(z) := \eta_{T_0}(z) - \eta_T(z). \]

Since \( F_{T_0} \) satisfies the bound (6.1), it follows from Proposition 6.1 that

\[ \sum_{\tau \in \mathcal{O}_{D,N,h}} F_{T_0}(\tau) = h(-D) \int_{Y_0(N)} F_{T_0}(z) d\mu(z) + O_{\epsilon,N}(\| \Delta^2 F_{T_0} \|_2 D^{\frac{1}{2} - \frac{1}{16} + \epsilon}). \]

Then by (6.8)–(6.9), to complete the proof it suffices to show that

\[ \sum_{\tau \in \mathcal{O}_{D,N,h}} \tilde{\eta}_T(\tau) = h(-D) \int_{Y_0(N)} \eta_{T_0}(z) d\mu(z) + O_{\epsilon,N}(D^{\frac{1}{2} - \frac{1}{2} + \epsilon}) + O_{\epsilon}(1). \]

By [I, Theorem 11.3 and (7.12)–(7.13)] we have

\[ \tilde{\eta}_T(z) = \langle \tilde{\eta}_T, 1 \rangle + \frac{1}{2\pi i} \sum_b c_b \int_{\mathbb{R}} \left( \hat{\psi}_{T_0}(\frac{1}{2} + it) - \hat{\psi}_T(\frac{1}{2} + it) \right) E_b(z, \frac{1}{2} + it) dt, \]

where

\[ \hat{\psi}(t) := \int_0^\infty \psi(t) t^{-(s+1)} dt. \]

It follows that

\[ \sum_{\tau \in \mathcal{O}_{D,N,h}} \tilde{\eta}_T(\tau) = h(-D) \int_{Y_0(N)} \eta_{T_0}(z) d\mu(z) - h(-D) \int_{Y_0(N)} \eta_T(z) d\mu(z) + E(D,T), \]

where

\[ E(D,T) := \frac{1}{2\pi i} \sum_b c_b \int_{\mathbb{R}} \left( \hat{\psi}_{T_0}(\frac{1}{2} + it) - \hat{\psi}_T(\frac{1}{2} + it) \right) W_b(D,t) dt. \]

By [KMY, Lemma 5.6], for all \( B > 0 \) we have

\[ \int_{\mathbb{R}} |\hat{\psi}_{T_0}(\frac{1}{2} + it) - \hat{\psi}_T(\frac{1}{2} + it)| (1 + |t|)^B dt \ll C(\alpha,T), \tag{6.10} \]

where

\[ C(\alpha,T) := \begin{cases} \log(T), & \alpha \leq \frac{1}{2} \\ T^{\alpha - \frac{1}{2}}, & \alpha > \frac{1}{2}. \end{cases} \]
Since \( T < (1 + \frac{4N}{\pi})\sqrt{D} \), we combine (6.5) and (6.10) to obtain
\[
E(D,T) = \begin{cases} 
O_{\epsilon,N}(D^{\frac{1}{2} - \frac{1}{16} + \epsilon}), & \alpha \leq \frac{1}{2} \\
O_{\epsilon,N}(D^{\frac{1}{2} - (\frac{1}{16} - (\frac{1}{2} - \frac{1}{4})) + \epsilon}), & \frac{1}{2} < \alpha < \frac{5}{8}.
\end{cases}
\]
Finally, a straightforward estimate yields
\[
h(-D) \int_{Y_0(N)} \eta_T(z) d\mu(z) = O(h(-D)T^{\alpha-1}) = O_{\epsilon,N}(D^{\frac{1}{2} - (\frac{1}{2} - \frac{1}{4}) + \epsilon}),
\]
where we used
\[
h(-D) \ll \epsilon D^{\frac{1}{2} + \epsilon}.
\]
\[\square\]

**Lemma 6.4.** For \( \gamma \in \Gamma_0(N) \) and \( \tau \in \mathcal{O}_{D,N,h} \) we have
\[
\text{Im}(\sigma_a^{-1}\gamma \tau) \leq \max\{\text{Im}(\tau), \frac{4N}{\pi}\} \leq \max\{\frac{\sqrt{D}}{2N}, \frac{4N}{\pi}\}.
\]
**Proof.** Recall that we chose \( \mathcal{O}_{D,N,h} \) so that a Heegner point \( \tau \in \mathcal{O}_{D,N,h} \) has the form
\[
\tau = \tau^{(h)} = \frac{b + \sqrt{-D}}{2Na}
\]
with
\[
a \leq \frac{2}{\pi} \sqrt{D}.
\]
Write \( \sigma_a^{-1}\gamma = \left( \begin{array}{cc} a' & b' \\
c' & d' \end{array} \right) \in \text{SL}_2(\mathbb{R}) \). Then we have
\[
\text{Im}(\sigma_a^{-1}\gamma \tau) = \frac{\text{Im}(\tau)}{|c' \tau + d'|^2} = \frac{1}{|c' \tau + d'|^2} \frac{\sqrt{D}}{2Na}.
\]
If \( c' = 0 \), then \( d' = 1 \) (see [I, (2.15)-(2.17)]) so that
\[
\text{Im}(\sigma_a^{-1}\gamma \tau) \leq \frac{\sqrt{D}}{2N}
\]
(recall that \( a = N_{K/Q}(\mathfrak{a}) \geq 1 \)). On the other hand, if \( c' \neq 0 \), then by (8.2) we have \( (c')^2 \geq 1 \) so that
\[
|c' \tau + d'|^2 = \left( \frac{c'b}{2Na} + d' \right)^2 + \left( \frac{c'\sqrt{D}}{2Na} \right)^2 \geq \frac{D}{4N^2a^2}.
\]
Then using (6.11) we obtain
\[
\text{Im}(\sigma_a^{-1}\gamma \tau) \leq \frac{4N^2a^2}{D} \frac{\sqrt{D}}{2Na} = 2Na \frac{\sqrt{D}}{\sqrt{D}} = \frac{4N}{\pi}.
\]
\[\square\]
7. Fourier expansion of $F_{N,m}(z)$

Write

$$F_{N,m}(z) = \sum_{\gamma \in \Gamma \setminus \Gamma_0(N)} p_m(\gamma z),$$

where

$$p_m(z) := \psi_m(\text{Im}(z))e(-mz)$$

and

$$\psi_m(t) := \pi m \sqrt{mt} I_{3/2}(2\pi mt)e(mt), \quad t \in \mathbb{R}^+. \tag{1}$$

Then by [I, p. 60] the Fourier expansion of $F_{N,m}$ in the cusp $b$ is given by

$$F_{N,m}(\sigma_b z) = \delta_{\infty,b} \psi_m(y)e(-mz) + \sum_{n \in \mathbb{Z}} e(nx) \sum_{c \in \mathbb{R}^+} S_{\infty,b}(-m, n; c) A_m(n, c, y),$$

where $S_{\infty,b}(-m, n; c)$ is the Kloosterman sum

$$S_{\infty,b}(-m, n; c) := \sum_{\left(\begin{array}{cc}a & * \\ c & d \end{array}\right) \in B \setminus \sigma_b^{-1} \Gamma_0(N) \sigma_b / B} e\left(-\frac{md + na}{c}\right)$$

with group of integral translations

$$B := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\},$$

and

$$A_m(n, c, y) := \int \psi_m \left( \frac{y}{c^2(t^2 + y^2)} \right) e \left( \frac{m}{c^2(i t + iy)} - nt \right) dt. \tag{2}$$

In Lemma 7.1 we evaluate the integral $A_m(n, c, y)$. Inserting this into the Fourier expansion yields

$$F_{N,m}(\sigma_b z) = \delta_{\infty,b} \frac{\pi m^{3/2}}{\sqrt{y}} I_{3/2}(2\pi my)e(-mx) + C_b(m)y^{-1} + \sum_{\substack{n \in \mathbb{Z} \setminus \{0\}}} C_b(m, n) \sqrt{y} K_{3/2}(2\pi |n| y)e(nx),$$

where

$$C_b(m) := \frac{2\pi^3 m^3}{3} \sum_{c \in \mathbb{R}^+} S_{\infty,b}(-m, 0; c)$$

and

$$C_b(m, n) := \begin{cases} 2\pi m^{3/2} \sum_{c \in \mathbb{R}^+} \frac{S_{\infty,b}(-m, n; c)}{c} J_3 \left( \frac{4\pi \sqrt{m |n|}}{c} \right), & n < 0 \\ 2\pi m^{3/2} \sum_{c \in \mathbb{R}^+} \frac{S_{\infty,b}(-m, n; c)}{c} I_3 \left( \frac{4\pi \sqrt{mn}}{c} \right), & n > 0. \end{cases}$$
Finally, using the identities
\[
\sqrt{t} I_{3/2}(t) = \frac{1}{\sqrt{2\pi}} \left( e^{t} \left( 1 - \frac{1}{t} \right) + e^{-t} \left( 1 + \frac{1}{t} \right) \right)
\]
and
\[
\sqrt{t} K_{3/2}(t) = \frac{\sqrt{\pi}}{2} e^{-t} \left( 1 + \frac{1}{t} \right),
\]
we obtain
\[
F_{N,m}(\sigma_{b}z) = C_{b}(m) y^{-1} + \delta_{\infty,b} \frac{m}{2} \left( 1 - \frac{1}{2\pi my} \right) e(-mz) + E_{b}(m, x, y), \tag{7.1}
\]
where
\[
E_{b}(m, x, y) := \delta_{\infty,b} \frac{m}{2} e^{-2\pi my} \left( 1 + \frac{1}{2\pi my} \right) e(-mx) + \frac{1}{2} \sum_{n \in \mathbb{Z}, n \neq 0} C_{b}(m, n) \left| n \right|^{-1/2} e^{-2\pi \left| n \right| y} \left( 1 + \frac{1}{2\pi \left| n \right| y} \right) e(nx).
\]

**Lemma 7.1.** We have
\[
A_{m}(n, c, y) = \begin{cases} 
\frac{2\pi m^{3/2}}{c} \sqrt{y} K_{3/2}(2\pi |n| y) J_{3} \left( \frac{4\pi \sqrt{m |n|}}{c} \right), & n < 0 \\
\frac{2\pi^{3} m^{3}}{3 \pi y^{1}}, & n = 0 \\
\frac{2\pi m^{3/2}}{c} \sqrt{y} K_{3/2}(2\pi ny) I_{3} \left( \frac{4\pi \sqrt{mn}}{c} \right), & n > 0.
\end{cases}
\]

**Proof.** Using the identity
\[
M_{0,3/2}(2u) = 2^{7/2} \Gamma(5/2) \sqrt{u} I_{3/2}(u)
\]
where $M_{0,3/2}$ is the usual $M$-Whittaker function of order $(0, 3/2)$, we find that
\[
\psi_{m}(t) = C_{1} m M_{0,3/2}(4\pi mt) e(mt),
\]
where
\[
C_{1} := \frac{\pi}{\sqrt{2\pi} 2^{7/2} \Gamma(5/2)} = \frac{1}{12}.
\]
Therefore
\[
A_{m}(n, c, y) = \frac{m}{12} I_{m}(n, c, y),
\]
where
\[
I_{m}(n, c, y) := \int_{\mathbb{R}} M_{0,3/2} \left( \frac{4\pi my}{c^{2}(t^{2} + y^{2})} \right) e \left( \frac{mt}{c^{2}(t^{2} + y^{2})} - nt \right) dt.
\]
By a simple change of variables, one can show that $I_m(n, c, y)$ equals the integral $I$ in [B, p. 33] with the choices $k = 0$ and $s = 2$. Then using the evaluation of the integral $I$ given there, we have

$$
I_m(n, c, y) = \begin{cases} 
C_2 \frac{\sqrt{m/|n|}}{c} W_{0,3/2}(4\pi |n| y) J_3 \left( \frac{4\pi \sqrt{m} |n|}{c} \right), & n < 0 \\
C_3 \frac{m^2}{c^4} y^{-1}, & n = 0 \\
C_2 \frac{\sqrt{m/n}}{c} W_{0,3/2}(4\pi ny) I_3 \left( \frac{4\pi \sqrt{mn}}{c} \right), & n > 0,
\end{cases}
$$

where

$$
C_2 := \frac{2\pi \Gamma(4)}{\Gamma(2)} = 12\pi, \quad C_3 := \frac{4\pi^3 \Gamma(4)}{3\Gamma(2)^2} = 8\pi^3,
$$

$K_3$ and $J_3$ are the usual $K$ and $J$-Bessel functions of order 3, respectively, and $W_{0,3/2}$ is the usual $W$-Whittaker function of order $(0, 3/2)$. Using the identity

$$
W_{0,3/2}(2u) = \sqrt{2u/\pi} K_{3/2}(u),
$$

where $K_{3/2}$ is the usual $K$-Bessel function of order $3/2$, we have

$$
W_{0,3/2}(4\pi |n| y) = 2\sqrt{|n|} y K_{3/2}(2\pi |n| y).
$$

The result now follows after simplification. \hfill \Box

8. POINCARÉ SERIES

Let $\lambda : \mathbb{R} \to [0, 1]$ be a $C^\infty$ function such that

$$
\lambda(t) = \begin{cases} 
0, & t \leq 0 \\
1, & t \geq 1.
\end{cases}
$$

Let $\eta > 0$, and define

$$
\mathcal{P}_{m,\eta}(z) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} \psi_{m,\eta}(\text{Im}(\gamma z)) e(-m\gamma z),
$$

where

$$
\psi_{m,\eta}(t) := \lambda \left( \frac{t - \frac{4N}{\pi}}{\eta} \right) m \left( 1 - \frac{1}{2\pi mt} \right).
$$

Then define the regularized function

$$
F_{N,m,\eta}(z) := F_{N,m}(z) - \mathcal{P}_{m,\eta}(z).
$$

Proposition 8.1. For $y > \frac{4N}{\pi} + \eta$ we have

$$
F_{N,m,\eta}(\sigma_b z) = C_b(m) y^{-1} + E_b(m, x, y).
$$

In particular, the regularized function $F_{N,m,\eta}$ has cuspidal growth of power $\alpha = -1$. 
Proof. We have the Fourier expansion
\[
\mathcal{P}_{m,\eta}(\sigma_b z) = \delta_{\infty,b} \psi_{m,\eta}(y) e(-mz) + \sum_{n \in \mathbb{Z}} e(nx) \sum_{c \in \mathbb{R}^+} S_{\infty,b}(-m, n; c) A_{m,\eta}(n, c, y),
\]
where
\[
A_{m,\eta}(n, c, y) := \int_{\mathbb{R}} \psi_{m,\eta}(y) \left( \frac{y}{c^2(t^2 + y^2)} \right) e \left( \frac{m}{c^2(t + iy)} - nt \right) dt.
\]
The function \(\psi_{m,\eta} : \mathbb{R} \rightarrow [0, m]\) is \(C^\infty\) and satisfies
\[
\psi_{m,\eta}(t) = \begin{cases} 
0, & t \leq \frac{4N\pi}{\eta}, \\
\frac{m}{2} \left( 1 - \frac{1}{2\pi mt} \right), & t \geq \frac{4N\pi}{\eta} + \eta.
\end{cases}
\]
Moreover, since
\[
\min\{c \in \mathbb{R}^+: (\ast \ast c \ast \ast) \in \sigma_a^{-1} \Gamma_0(N) \sigma_b \} \geq 1
\]
for all cusps \(a, b\) of \(\Gamma_0(N)\) (see [I, eqs. (2.28)–(2.31)]), we have
\[
\frac{y}{c^2(t^2 + y^2)} \leq \frac{4N}{\pi}
\]
for \(y \geq \frac{4N}{\pi}\). It follows from (8.1) that
\[
\mathcal{P}_{m,\eta}(\sigma_b z) = \begin{cases} 
\delta_{\infty,b} \psi_{m,\eta}(y) e(-mz), & y \geq \frac{4N}{\pi}, \\
\delta_{\infty,b} \frac{m}{2} \left( 1 - \frac{1}{2\pi my} \right) e(-mz), & y \geq \frac{4N}{\pi} + \eta.
\end{cases}
\]
The proposition now follows from the Fourier expansion (7.1).

9. TRACES OF WEAK MAASS FORMS

Define the trace
\[
\text{Tr}_D(F_{N,m}) := \sum_{\tau \in \mathcal{O}_{D,N,h}} F_{N,m}(\tau).
\]

**Theorem 9.1.** We have
\[
\text{Tr}_D(F_{N,m}) = \frac{m}{2} \sum_{\tau \in \mathcal{O}_{D,N,h}} \left( 1 - \frac{1}{2\pi m \text{Im}(\tau)} \right) e(-m\tau) + h(-D)c_{N,m} + O_{\epsilon,N}(D^{\frac{1}{2} - \frac{1}{16\pi} + \epsilon}),
\]
where
\[
c_{N,m} := \int_{\text{reg}} F_{N,m}(z) d\mu(z)
\]
is a Borcherds-type regularized integral.
Proof. Let $0 < \eta < 1/4$. By Proposition 8.1, the regularized function $F_{N,m,\eta}$ has cuspidal growth of power $\alpha = -1$. Moreover, this growth is uniform in $\eta$ for $y > \frac{4N}{\pi} + \frac{1}{4}$, hence the same choice of constant $T_0 = \frac{4N}{\pi} + \frac{1}{2}$ and corresponding function $\eta_{T_0}$ given by

$$
\eta_{T_0}(\sigma_b z) = \begin{cases} 
0, & 1 < y < \frac{4N}{\pi} + \frac{1}{2} \\
C_b(m)y^{-1}\chi(y/2), & \frac{4N}{\pi} + \frac{1}{2} \leq y \leq 2\left(\frac{4N}{\pi} + \frac{1}{2}\right) \\
C_b(m)y^{-1}, & y > 2\left(\frac{4N}{\pi} + \frac{1}{2}\right)
\end{cases}
(9.1)
$$
can be used to regularize each function $F_{N,m,\eta}$ as in the proof of Theorem 6.3. Upon substituting $F_{N,m,\eta}$ into Theorem 6.3, we obtain the asymptotic formula

$$
\text{Tr}_D(F_{N,m}) = \text{Tr}_D(\mathcal{P}_{m,\eta}) + h(-D) \int_{\mathcal{Y}_0(N)} F_{N,m,\eta}(z)d\mu(z) \\
+ O_{\epsilon,N}(||\Delta^2 F_{N,m,\eta,T_0}||_2)D^{\frac{1}{2} - \frac{1}{16}\epsilon}) + O_{\epsilon,N}(D^{\frac{1}{2} - \frac{1}{16}\epsilon}) + O_N(D^{-\frac{1}{2} + \epsilon}),
(9.2)
$$
where

$$
F_{N,m,\eta,T_0}(z) := F_{N,m,\eta}(z) - \eta_{T_0}(z).
$$

By Lemma 9.2 we have

$$
\text{Tr}_D(\mathcal{P}_{m,\eta}) = \sum_{\text{Im}(\tau) > \frac{4N}{\pi}} \psi_{m,\eta}(\text{Im}(\tau))e(-m\tau).
$$

Split the sum on the right hand side into the ranges $\text{Im}(\tau) \leq \frac{4N}{\pi} + \eta$ and $\text{Im}(\tau) > \frac{4N}{\pi} + \eta$, and define

$$
R_{N,m,\eta}(D) := \text{Tr}_D(F_{N,m}) - \frac{m}{2} \sum_{\text{Im}(\tau) > \frac{4N}{\pi} + \eta} \left(1 - \frac{1}{2\pi m \text{Im}(\tau)}\right) e(-m\tau).
$$

Then (9.2) can be written as

$$
R_{N,m,\eta}(D) = h(-D) \int_{\mathcal{Y}_0(N)} F_{N,m,\eta}(z)d\mu(z) \\
+ O_{\epsilon,N}(||\Delta^2 F_{N,m,\eta,T_0}||_2)D^{\frac{1}{2} - \frac{1}{16}\epsilon}) + O_{\epsilon,N}(D^{\frac{1}{2} - \frac{1}{16}\epsilon}) + O_N(D^{-\frac{1}{2} + \epsilon}) \\
+ \sum_{\frac{4N}{\pi} < \text{Im}(\tau) \leq \frac{4N}{\pi} + \eta} \psi_{m,\eta}(\text{Im}(\tau))e(-m\tau).
$$

In Lemma 13.2 we will show that

$$
\int_{\mathcal{Y}_0(N)} F_{N,m,\eta}(z)d\mu(z) = \int_{\text{reg}} F_{N,m}(z)d\mu(z) =: c_{N,m}
$$
where the right hand side is a Borcherds-type regularized integral (note that the right hand side is independent of $\eta$).

A straightforward estimate yields

$$
\sum_{\frac{4N}{\pi} < \text{Im}(\tau) \leq \frac{4N}{\pi} + \eta} \psi_{m,\eta}(\text{Im}(\tau))e(-m\tau) \ll m \cdot e^{2\pi m \left(\frac{4N}{\pi} + \frac{1}{4}\right)} \#\Lambda_{N,h,\eta}(D),
$$
where

$$
\Lambda_{N,h,\eta}(D) := \{\tau \in \mathcal{O}_{D,N,h} : \frac{4N}{\pi} < \text{Im}(\tau) \leq \frac{4N}{\pi} + \eta\}.
$$
By Lemma 11.1 we have
\[
\# \Lambda_{N,h,\eta}(D) = O_N(\eta h(-D)) + O_{\epsilon,N}(\eta^{-\frac{1}{2}} \frac{1}{16} + \epsilon).
\]

By Lemma 12.1 we have the estimate
\[
||\Delta^2 F_{N,m,\eta,T_0}||_2 = O_{N,m}(\eta^{-10}).
\]

Combining the preceding estimates yields
\[
R_{N,m,\eta}(D) = h(-D)c_{N,m} + O_{\epsilon,N}(\eta^{-10} D^{\frac{1}{2} - \frac{1}{16} + \epsilon}) + O_N(\eta h(-D)).
\]

If we let \( \eta = D^{-b} \) for any \( 0 < b < 1/160 \), then
\[
R_{N,m,\eta}(D) = h(-D)c_{N,m} + O_{\epsilon,N}(D^{\frac{1}{2} - \frac{1}{16} + \epsilon}),
\]
where we used
\[
h(-D) \ll D^{\frac{1}{2} + \epsilon}.
\]

The exponent is optimized when
\[
\frac{1}{16} - 10b = b, \text{ or } b = 1/176.
\]
Thus
\[
R_{N,m,\eta}(D) = h(-D)c_{N,m} + O_{\epsilon,N}(D^{\frac{1}{2} - \frac{1}{176} + \epsilon}).
\]

**Lemma 9.2.** We have
\[
\text{Tr}_D(P_{m,\eta}) = \sum_{\text{Im}(\tau) > \frac{4N}{\pi}} \psi_{m,\eta}(\text{Im}(\tau)) e(-m\tau).
\]

**Proof.** Write
\[
\text{Tr}_D(P_{m,\eta}) = \sum_{\text{Im}(\tau) \leq \frac{4N}{\pi}} P_{m,\eta}(\tau) + \sum_{\text{Im}(\tau) > \frac{4N}{\pi}} P_{m,\eta}(\tau) =: \text{I} + \text{II}.
\]

Let \( \gamma \in \Gamma_\infty \setminus \Gamma_0(N) \) and \( \tau \in \mathcal{O}_{D,N,h} \) with \( \text{Im}(\tau) \leq \frac{4N}{\pi} \). Then by Lemma 6.4, \( \text{Im}(\gamma \tau) \leq \frac{4N}{\pi} \). Since \( \psi_{m,\eta}(t) = 0 \) for \( t \leq \frac{4N}{\pi} \), it follows that
\[
P_{m,\eta}(\tau) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} \psi_{m,\eta}(\text{Im}(\gamma \tau)) e(-m\gamma \tau) = 0,
\]
and thus \( \text{I} = 0 \). On the other hand, if \( \tau \in \mathcal{O}_{D,N,h} \) with \( \text{Im}(\tau) > \frac{4N}{\pi} \), by (8.3) we have
\[
P_{m,\eta}(\tau) = \psi_{m,\eta}(\text{Im}(\tau)) e(-m\tau),
\]
and thus
\[
\text{II} = \sum_{\text{Im}(\tau) > \frac{4N}{\pi}} \psi_{m,\eta}(\text{Im}(\tau)) e(-m\tau).
\]
10. PROOF OF THEOREMS 1.2 AND 1.7

**Proof of Theorem 1.2.** By combining the Bruinier-Ono period formula (1.1), the identity (5.1), and Theorem 9.1 we obtain

\[
\Lambda_f^+(D, h) = -\frac{4N}{D} \text{Tr}_D \left( \frac{1}{4\pi} R_{-2f} \right)
\]

\[
= -\frac{8N}{D} \sum_{n=1}^{N_\infty} c_f(-n) \text{Tr}_D (F_{N,n})
\]

\[
= \frac{1}{D} \sum_{\text{Im}(\tau) > \frac{4N}{\pi} + D^{-\frac{1}{16}}} M_{f,N}(\tau) + C_{f,N} \frac{h(-D)}{D} + O_{\epsilon,N}(D^{-\frac{9}{176} + \epsilon}),
\]

where

\[
M_{f,N}(z) = -4N \sum_{n=1}^{N_\infty} c_f(-n) n \left( 1 - \frac{1}{2\pi n \text{Im}(z)} \right) e(-nz)
\]

and

\[
C_{f,N} = -8N \sum_{n=1}^{N_\infty} c_f(-n)c_{N,n} = -8N \int_{\text{reg}} \sum_{n=1}^{N_\infty} c_f(-n) F_{N,n}(z) d\mu(z) = -\frac{N}{\pi} \int_{\text{reg}} R_{-2f}(z) d\mu(z),
\]

where for the last equality we again used (5.1). □

**Proof of Theorem 1.7.** Recall the Bruinier-Ono formula for the partition function,

\[
p(n) = -\frac{1}{24} c_{f_p}^+(24n - 1, 1),
\]

where for the weakly holomorphic modular form \( f_p \in M_{-2}(6) \) defined by (1.2) we have \( N_\infty = 1 \) and \( c_{f_p}(-1) = 1 \). Then by Theorem 1.2 we obtain

\[
c_{f_p}^+(24n - 1, 1) = \frac{1}{24n - 1} \sum_{\tau \in \mathbb{C}_{24n-1,6,1}} M_{f_p,6}(\tau) + C_{f_p,6} \frac{h(-24n + 1)}{24n - 1} + O_{\epsilon}(n^{-\frac{9}{176} + \epsilon}),
\]

where

\[
M_{f_p,6}(z) = -24 \left( 1 - \frac{1}{2\pi \text{Im}(z)} \right) e(-z)
\]

and

\[
C_{f_p,6} = -\frac{6}{\pi} \int_{\text{reg}} R_{-2f_p}(z) d\mu(z).
\]

The theorem now follows after multiplying by \(-1/24\). □

11. PROOF OF LEMMA 11.1

In this section we establish the following estimate (see also [D, p. 248-249]).

**Lemma 11.1.** For \( 0 < \eta < 1/4 \) we have

\[
\# \Lambda_{N,h,\eta}(D) = O_N(\eta h(-D)) + O_{\epsilon}(\eta^{-1} D^{\frac{1}{2} - \frac{1}{4} + \epsilon}).
\]
Proof. Let $\phi_{\eta} : \mathbb{R} \to [0,1]$ be a $C^\infty$ function which is supported on $(4N/\pi - \eta, 4N/\pi + 2\eta)$, which equals 1 on $[4N/\pi, 4N/\pi + \eta]$, and which satisfies the bound
\[ \phi^{(A)}_{\eta} \ll \eta^{-A}, \quad A = 0, 1, 2. \] (11.1)
Define
\[ g_{\eta}(z) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} \phi_{\eta}(\text{Im}(\gamma z)). \]
Then we have
\[ \sum_{\tau \in \mathcal{O}_{D,N,h}} g_{\eta}(\tau) = \sum_{\tau \in \mathcal{O}_{D,N,h}} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} \phi_{\eta}(\text{Im}(\gamma \tau)) + \sum_{\tau \in \mathcal{O}_{D,N,h}} \phi_{\eta}(\text{Im}(\tau)) \geq \sum_{\tau \in \mathcal{O}_{D,N,h}} \phi_{\eta}(\text{Im}(\tau)) \]
\[ = \sum_{\tau \in \mathcal{O}_{D,N,h}} 1 + \sum_{\tau \in \mathcal{O}_{D,N,h}} \phi_{\eta}(\text{Im}(\tau)) \]
\[ \geq \# \Lambda_{N,h,\eta}(D). \]
The real-analytic Eisenstein series $E_\infty(z,s)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ with residue $1/\text{vol}(Y_0(N))$ (see [I, Theorem 11.3 and Proposition 6.13]). Then by [I, eq. (7.12)] we have
\[ g_{\eta}(z) = \frac{1}{\text{vol}(Y_0(N))} \hat{\phi}_{\eta}(1) + \frac{1}{2\pi i} \int_{\mathbb{R}} \hat{\phi}_{\eta}(\frac{1}{2} + it) E_\infty(z, \frac{1}{2} + it) dt, \]
where
\[ \hat{\phi}_{\eta}(s) := \int_{0}^{\infty} \phi_{\eta}(u) u^{-s} \, du. \]
Thus
\[ \sum_{\tau \in \mathcal{O}_{D,N,h}} g_{\eta}(\tau) = \frac{1}{\text{vol}(Y_0(N))} \hat{\phi}_{\eta}(1) h(-D) + \frac{1}{2\pi i} \int_{\mathbb{R}} \hat{\phi}_{\eta}(\frac{1}{2} + it) W_\infty(D,t) dt. \]
Now, a straightforward estimate yields (here we use $\eta < 1/4$)
\[ \hat{\phi}_{\eta}(1) \leq 3\eta. \]
Moreover, by (6.5) we have
\[ W_\infty(D,t) \ll \epsilon \left( \frac{1}{4} + t^2 \right)^{17/56 + \epsilon} D^{\frac{1}{2} - \frac{1}{16} + \epsilon}. \]
It follows that
\[ \sum_{\tau \in \mathcal{O}_{D,N,h}} g_{\eta}(\tau) = O_N(\eta h(-D)) + O_\epsilon(C_\eta D^{\frac{1}{2} - \frac{1}{16} + \epsilon}), \]
where
\[ C_\eta := \int_{\mathbb{R}} |\hat{\phi}_{\eta}(\frac{1}{2} + it)| (\frac{1}{4} + t^2)^{17/56 + \epsilon} dt. \]
We integrate by parts 2 times and use the bound (11.1) to obtain
\[ \hat{\phi}_\eta(\frac{1}{2} + it) \ll \eta^{-1} \prod_{j=1}^{2} |\frac{3}{2} + it - j|^{-1}, \]
which yields \( C_\eta \ll \eta^{-1} \).

\[ \square \]

12. PROOF OF LEMMA 12.1

**Lemma 12.1.** For \( \eta < 1/4 \) we have
\[ ||\Delta^A F_{N,m,\eta,T_0}||_2 \ll_{N,m} \eta^{-2(4A+2)}, \quad A = 1, 2, \ldots. \]

**Proof.** Fix \( Y > 0 \), and define
\[ P(Y) := \{ z = x + iy : 0 < x < 1, \ y \geq Y \} \]
and
\[ L(Y) := \{ z = x + iY : 0 < x < 1 \}. \]
One can choose a fundamental domain \( F \) for \( Y_0(N) \) such that
\[ F = F(Y) \cup \bigcup_a F_a(Y), \]
where \( F_a(Y) := \sigma_a P(Y) \) and
\[ F(Y) := F \setminus \bigcup_a F_a(Y) \]
has compact closure and is adjacent to each cuspidal zone \( F_a(Y) \) along the horocycle \( \sigma_a L(Y) \) (see [I, section 2.2]). Using the \( SL_2(\mathbb{R}) \)-invariance of the normalized hyperbolic measure
\[ d\mu(z) := \frac{1}{\text{vol}(Y_0(N))} \frac{dx dy}{y^2}, \]
we find that
\[ ||\Delta^A F_{N,m,\eta,T_0}||_2^2 := \int_F |\Delta^A F_{N,m,\eta,T_0}(z)|^2 d\mu(z) \leq I + II, \]
where
\[ I := \int_{F(\frac{4N}{\pi})} |\Delta^A F_{N,m,\eta,T_0}(z)|^2 d\mu(z) \]
and
\[ II := \sum_b \int_{P(\frac{4N}{\pi})} |\Delta^A F_{N,m,\eta,T_0}(\sigma_b z)|^2 d\mu(z). \]

First we estimate \( II \). By (7.1), (8.3) and (9.1), we have
\[ F_{N,m,\eta,T_0}(\sigma_b z) = \begin{cases} F_{N,m}(\sigma_b z) - \delta_{\infty,b} \frac{m}{2} \psi_{m,\eta}(y) e(-mz), & 4N \pi \leq y < 4N \pi + \eta \\ C_b(m) y^{-1} + E_b(m, x, y), & 4N \pi + \eta \leq y < \frac{4N}{\pi} + \frac{1}{2} \\ C_b(m) y^{-1}(1 - \chi(y/2)) + E_b(m, x, y), & 4N \pi + \frac{1}{2} \leq y < 2(\frac{4N}{\pi} + \frac{1}{2}) \\ E_b(m, x, y), & y \geq 2(\frac{4N}{\pi} + \frac{1}{2}) \end{cases} \]
(12.1)
By splitting the $y$-integral in II into the different ranges considered in (12.1), we obtain

$$II = \sum_b \int_{4N/\pi}^\infty \int_0^1 |\Delta^A F_{N,m,\eta,T_0}(\sigma_b z)|^2 d\mu(z) = III + O(1),$$

where

$$III := \sum_b \int_{4N/\pi}^{4N/\pi + \eta} \int_0^1 |\Delta^A(F_{N,m}(\sigma_b z) - \delta_{\infty,b} m \psi_{m,\eta}(y)e(-mz))|^2 d\mu(z).$$

By linearity of $\Delta$ and the triangle inequality,

$$III \leq IV + V + O(1),$$

where

$$IV := 2 \sum_b \int_{4N/\pi}^{4N/\pi + \frac{1}{4}} \int_0^1 |\Delta^A F_{N,m}(\sigma_b z)| \cdot |\delta_{\infty,b} m \Delta^A(\psi_{m,\eta}(y)e(-mz))| d\mu(z)$$

and

$$V := \sum_b \int_{4N/\pi}^{4N/\pi + \frac{1}{4}} \int_0^1 |\delta_{\infty,b} m \Delta^A(\psi_{m,\eta}(y)e(-mz))|^2 d\mu(z).$$

Using the estimate

$$\max_{(x,y) \in [0,1] \times \left(\frac{4N/\pi}{4N/\pi + \frac{1}{4}}\right]} |\Delta^A(\psi_{m,\eta}(y)e(-mz))| \ll \eta^{-A},$$

we obtain

$$IV \ll \eta^{-A} \quad \text{and} \quad V \ll \eta^{-2A}.$$

We conclude that

$$II \ll \eta^{-2A}.$$

Next we estimate I. Observe that $F(4N/\pi)$ can be contained in a rectangle

$$R_N := [-B_N, B_N] \times [C_N, \frac{4N}{\pi}]$$

for some $B_N \geq 1$ and $0 < C_N \leq \sqrt{3}/2$. Since $\psi_{m,\eta}(t) = 0$ for $t \leq 4N/\pi$, by (8.1) we have

$$F_{N,m,\eta,T_0}(z) = (F_{N,m}(z) - f_{m,\eta}(x,y)) - \eta r_0(z),$$

where

$$f_{m,\eta}(x,y) := \sum_{n \in \mathbb{Z}} e(nx) \sum_{1 \leq c \leq C_N^{-1}} S(-m,n; c)A_{m,\eta}(n,c,y).$$

By linearity of $\Delta$ and three applications of the triangle inequality, we have

$$I \leq \int_{R_N} |\Delta^A((F_{N,m}(z) - f_{m,\eta}(x,y)) - \eta r_0(z))|^2 d\mu(z) \leq VI + VII + O(1),$$

where

$$VI := 2 \int_{R_N} (|\Delta^A F_{N,m}(z)| + |\Delta^A \eta r_0(z)|) |\Delta^A f_{m,\eta}(x,y)| d\mu(z)$$
and
\[ \text{VII} := \int_{\mathbb{R}^N} |\Delta^A f_{m,\eta}(x, y)|^2 d\mu(z). \]

Using the estimate
\[ \max_{(x, y) \in \mathbb{R}^N} |\Delta^A f_{m,\eta}(x, y)| \ll \eta^{-4A+2}, \tag{12.2} \]
we have
\[ \text{VI} \ll \eta^{-(4A+2)} \quad \text{and} \quad \text{VII} \ll \eta^{-2(4A+2)}. \]

We conclude that
\[ \text{I} \ll \eta^{-2(4A+2)}. \]

It remains to establish the estimate (12.2). Define
\[ \Phi_{m,\eta,c,y}(u) := \psi_{m,\eta} \left( \frac{y}{c^2(u^2 + y^2)} \right) e \left( \frac{m}{c^2(u + iy)} \right), \]
so that
\[ A_{m,\eta}(n, c, y) = \int_{\mathbb{R}} \Phi_{m,\eta,c,y}(u)e(-nu)du. \]

Since
\[ \Phi_{m,\eta,c,y}(u) = 0 \quad \text{for} \quad |u| \geq \sqrt{1 - C_N^2}, \]
integrating by parts $(2A + 2)$-times yields
\[ A_{m,\eta}(n, c, y) = \frac{1}{(2\pi in)^{2A+2}} \int_{-\sqrt{1-C_N^2}}^{\sqrt{1-C_N^2}} \Phi_{m,\eta,c,y}^{(2A+2)}(u)e(-nu)du. \]

In particular, we have
\[ \Delta^A f_{m,\eta}(x, y) = \sum_{n \in \mathbb{Z}} \sum_{1 \leq c \leq C_N^{-1}} S(-m, n; c) \Delta^A I_{m,\eta,n,c}(x, y), \]
where
\[ I_{m,\eta,n,c}(x, y) := \frac{e(nx)}{(2\pi in)^{2A+2}} \int_{-\sqrt{1-C_N^2}}^{\sqrt{1-C_N^2}} \Phi_{m,\eta,c,y}^{(2A+2)}(u)e(-nu)du. \]

For clarity, we first assume that $A = 1$. Then
\[ \Delta I_{m,\eta,n,c}(x, y) = \]
\[ -y^2 \frac{e(nx)}{(2\pi in)^3} \int_{-\sqrt{1-C_N^2}}^{\sqrt{1-C_N^2}} \Phi_{m,\eta,c,y}^{(4)}(u)e(-nu)du - y^2 \frac{e(nx)}{(2\pi in)^3} \int_{-\sqrt{1-C_N^2}}^{\sqrt{1-C_N^2}} \partial_y^2 \Phi_{m,\eta,c,y}^{(4)}(u)e(-nu)du. \]

Using the trivial estimate ([I, (2.37)])
\[ S(-m, n; c) \ll c^2, \]
and the estimates
\[ \max_{u \in [-\sqrt{1-C_N^2}, \sqrt{1-C_N^2}]} |\Phi_{m,\eta,c,y}^{(4)}(u)| \ll_{m,c,y} \eta^{-4}. \]
and
\[
\max_{u \in [-\sqrt{1-C_N}, \sqrt{1-C_N}]} |\partial_y^2 \Phi_{m,\eta,y}^{(4)}(u)| \ll_{m,c,y} \eta^{-6},
\]
we obtain
\[
\max_{(x,y) \in \mathbb{R}^2} |\Delta f_{m,\eta}(x,y)| \ll \eta^{-6}.
\]
The preceding argument generalizes in a straightforward way to \(A \geq 1\), and the estimate (12.2) follows.

\end{proof}

13. Regularized Integrals

First we recall the notion of a regularized integral in the sense of Borcherds [Bo] and Harvey-Moore [HMo]. Let \(F\) be the standard fundamental domain for \(\text{SL}_2(\mathbb{Z})\). Then a fundamental domain for \(\Gamma_0(N)\) is given by
\[
F(N) := \bigcup_{\sigma \in \Gamma(\mathbb{Z}) / \Gamma_0(N)} \sigma F.
\]
For a fixed \(Y > 1\), define the truncated domains
\[
F_Y := \{ z \in F : \text{Im}(z) \leq Y \}
\]
and
\[
F_Y(N) := \bigcup_{\sigma \in \Gamma_0(N) \setminus \text{SL}_2(\mathbb{Z})} \sigma F_Y.
\]
We then define the regularized integral
\[
\int_{\text{reg}} F_{N,m}(z) d\mu(z) := \lim_{Y \to \infty} \int_{F_Y(N)} F_{N,m}(z) d\mu(z).
\]
For each \(\eta > 0\) define the function
\[
\psi_{m,\eta,Y}(t) := \begin{cases} 
\psi_{m,\eta}(t), & t \leq Y \\
0, & t > Y,
\end{cases}
\]
and the associated Poincaré series
\[
\mathcal{P}_{m,\eta,Y}(z) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} \psi_{m,\eta,Y}(\text{Im}(\gamma z)) e(-m\gamma z).
\]

**Lemma 13.1.** For \(z \in F_Y(N)\) we have \(\mathcal{P}_{m,\eta,Y}(z) = \mathcal{P}_{m,\eta}(z)\).

**Proof.** By definition of \(\psi_{m,\eta,Y}\) we have
\[
\mathcal{P}_{m,\eta}(z) = \mathcal{P}_{m,\eta,Y}(z) + \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N) \atop \text{Im}(\gamma z) > Y} \psi_{m,\eta}(\text{Im}(\gamma z)) e(-m\gamma z).
\]
Let \(\gamma \in \Gamma_\infty \setminus \Gamma_0(N)\) and \(z \in F_Y(N)\). Then \(\gamma z = Az\) for some \(A \in \text{SL}_2(\mathbb{Z})\) and \(z' \in F_Y\), so that \(\text{Im}(\gamma z) = \text{Im}(Az) < \text{Im}(z') < Y\). It follows that
\[
\#\{ \gamma \in \Gamma_\infty \setminus \Gamma_0(N) : \text{Im}(\gamma z) > Y \} = 0,
\]
and thus \(\mathcal{P}_{m,\eta,Y}(z) = \mathcal{P}_{m,\eta}(z)\).\(\square\)
Lemma 13.2. We have
\[ \int_{Y_0(N)} F_{N,m,\eta}(z) d\mu(z) = \lim_{Y \to \infty} \int_{F_Y(N)} F_{N,m}(z) d\mu(z). \]

Proof. Since \( F_{N,m,\eta} := F_{N,m} - P_{m,\eta} \in L^1(Y_0(N)), \)
\[ \int_{Y_0(N)} F_{N,m,\eta}(z) d\mu(z) = \lim_{Y \to \infty} \int_{F_Y(N)} (F_{N,m}(z) - P_{m,\eta}(z)) d\mu(z). \]
By Lemma 13.1 we have
\[ \int_{F_Y(N)} P_{m,\eta,Y}(z) d\mu(z) = \int_{F(N)} P_{m,\eta,Y}(z) d\mu(z). \]
We claim that if \( z \in F(N) \setminus F_Y(N), \) then \( P_{m,\eta,Y}(z) = 0. \) Let \( \gamma \in \Gamma_\infty \setminus \Gamma_0(N) \) and \( z \in F(N) \setminus F_Y(N). \) Then \( \gamma z = Az' \) for some \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \SL_2(\mathbb{Z}) \) and \( z' = x'+iy' \in F \setminus F_Y, \) i.e., \( z' \in F \) with \( y' > Y. \) Now, we have
\[ \text{Im}(\gamma z) = \text{Im}(Az') = \frac{y'}{|cz'+d|^2}. \]
If \( c = 0 \) then \( d = 1, \) so that \( \text{Im}(\gamma z) = y' > Y. \) Since \( \psi_{m,\eta,Y}(t) = 0 \) for \( t > Y, \) it follows that \( P_{m,\eta,Y}(z) = 0. \) On the other hand, if \( c \neq 0 \) then \( c^2 \geq 1, \) so that
\[ \text{Im}(\gamma z) = \frac{y'}{(cx'+d)^2 + c^2(y')^2} \leq \frac{1}{y'} < \frac{1}{Y} < 1 \]
(recall \( Y > 1). \) Since \( \psi_{m,\eta,Y}(t) = \psi_{m,\eta}(t) = 0 \) for \( t < 1, \) it follows that \( P_{m,\eta,Y}(z) = 0, \) and we complete the proof of the claim.

By the claim we have
\[ \int_{F_Y(N)} P_{m,\eta,Y}(z) d\mu(z) = \int_{F(N)} P_{m,\eta,Y}(z) d\mu(z). \]
A standard unfolding argument yields
\[ \int_{F(N)} P_{m,\eta,Y}(z) d\mu(z) = 0. \]
Thus we conclude that
\[ \int_{Y_0(N)} F_{N,m,\eta}(z) d\mu(z) = \lim_{Y \to \infty} \int_{F_Y(N)} F_{N,m}(z) d\mu(z). \]
\[ \square \]
We conclude by noting that if \( F_{N,m} = R_{-2}f \) for a harmonic weak Maass form \( f \in H_{-2}(N), \)
then\(^1\)
\[ \int_{\text{reg}} F_{N,m}(z) d\mu(z) = \sum_{\alpha} \alpha \alpha c_+^{f,a}(0), \]
\(^1\)We thank Jan Bruinier for a very helpful correspondence regarding this fact.
where \( \alpha_a \) is the width of the cusp \( a \) and \( c_{f,a}^+(0) \) is the constant term of the holomorphic part of the Fourier expansion of \( f \) in the cusp \( a \) of \( \Gamma_0(N) \). In particular, for the constants \( C_{f,N} \) and \( C_{f,p,6}^* \) in Theorems 1.2 and 1.7, respectively, we have

\[
C_{f,N} = -\frac{N}{\pi} \sum_a \alpha_a c_{f,a}(0)
\]

and

\[
C_{f,p,6}^* = \frac{1}{4\pi} \sum_a \alpha_a c_{f,p,a}(0),
\]

where the last sum is over the 4 cusps of \( \Gamma_0(6) \).

References


