FOURIER COEFFICIENTS OF HARMONIC WEAK MAASS FORMS AND THE PARTITION FUNCTION

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Abstract. In a recent paper, Bruinier and Ono proved that certain harmonic weak Maass forms have the property that the Fourier coefficients of their holomorphic parts are algebraic traces of weak Maass forms evaluated on Heegner points. As a special case they obtained a remarkable finite algebraic formula for the Hardy-Ramanujan partition function $p(n)$, which counts the number of partitions of a positive integer n. We establish an asymptotic formula with a power saving error term for the Fourier coefficients in the Bruinier-Ono formula. As a consequence, we obtain a new asymptotic formula for $p(n)$. One interesting feature of this formula is that the main term contains essentially $3 \cdot h(-24n+1)$ fewer terms than the truncated main term in Rademacher's exact formula for $p(n)$, where $h(-24n+1)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-24n+1})$.

1. Introduction and statement of results

During the last ten years there have been remarkable advances in the study of q-series, modular forms, and L–functions through their connection to *harmonic weak Maass forms* (see for example the excellent survey articles of Ono [O] and Zagier [Za]). Roughly speaking, a harmonic weak Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ is a real-analytic function on the complex upper half-plane $\mathbb{H}$ which transforms like a modular form with respect to some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, vanishes under the weight k hyperbolic Laplacian, and has at most linear exponential growth in the cusps of the group. A detailed discussion of harmonic weak Maass forms can be found in the fundamental paper of Bruinier and Funke [BF] (see also Section 3).

A harmonic weak Maass form has a Fourier expansion consisting of a non-holomorphic and holomorphic part. There is a great amount of interest in understanding the arithmetic meaning of these Fourier coefficients. For example, Bruinier and Ono [BO] related the Fourier coefficients of the non-holomorphic and holomorphic parts of weight 1/2 harmonic weak Maass forms to central values and central derivatives of modular L-functions, respectively. Bruinier [B2] has since related the Fourier coefficients of the holomorphic parts of weight 1/2 harmonic weak Maass forms to periods of algebraic differentials of the third kind on modular and elliptic curves.

In a recent paper, Bruinier and Ono [BO2] constructed a theta lift from the space of harmonic weak Maass forms of weight -2 to the space of vector-valued harmonic weak Maass forms of weight -1/2. They used this lift to prove that certain vector-valued harmonic weak Maass forms of weight -1/2 have the property that the Fourier coefficients of their holomorphic parts are algebraic traces of fixed, weight 0 weak Maass forms evaluated on Heegner points. As a special case they established a finite algebraic formula for the Hardy-Ramanujan partition function $p(n)$, which counts the number of partitions of a positive integer n. In this paper we will establish an asymptotic formula with a power saving error term for the Fourier coefficients in the Bruinier-Ono formula. As a consequence, we will obtain a new asymptotic formula for $p(n)$ (see Theorem 1.6). One interesting feature of this formula is that the main term contains essentially $3 \cdot h(-24n+1)$ fewer terms than the truncated main term in Rademacher's exact formula for $p(n)$, where $h(-24n+1)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-24n+1})$ (see the discussion following Theorem 1.6).

In order to state our results we briefly review the Bruinier-Ono formula. Bruinier and Ono [BO2, Section 3.1] defined a theta lift

$$\Lambda : H_{-2}(N) \to \mathbf{H}_{-\frac{1}{2},\rho}$$

from the space $H_{-2}(N)$ of harmonic weak Maass forms of weight -2 and level N to the space $\mathbf{H}_{-\frac{1}{2},\rho}$ of vector-valued harmonic weak Maass forms of weight $-1/2$ and level N with Weil representation ρ (see Sections 3 and 4 for the precise definitions). The image of $f \in H_{-2}(N)$ under the theta lift Λ has a Fourier expansion

$$\Lambda(f,w) = \Lambda^{-}(f,w) + \Lambda^{+}(f,w), \quad w = u+iv \in \mathbb{H}$$

with non-holomorphic part

$$\Lambda^{-}(f,w) := \sum_{h \mod 2N} \sum_{\substack{D\in\mathbb{Z} \\ D<0}} c^{-}_{\Lambda_f}(D,h)\Gamma(1-k,\pi|D|v/N)q^{D/4N}\mathfrak{e}_h, \qquad q := e(w) = e^{2\pi i w}$$

and holomorphic part

$$\Lambda^{+}(f,w) := \sum_{h \mod 2N} \sum_{D=1}^{N_\infty} c^{+}_{\Lambda_f}(-D,h)q^{-D/4N}\mathfrak{e}_h + \sum_{h \mod 2N} \sum_{D=0}^{\infty} c^{+}_{\Lambda_f}(D,h)q^{D/4N}\mathfrak{e}_h$$

where $N_\infty \geq 0$ is an integer, $\Gamma(a,t)$ is the incomplete Gamma function, and $\{\mathfrak{e}_h\}$ is the standard basis for the group algebra $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$. Note that $c^{\pm}_{\Lambda_f}(D,h) = 0$ unless $D \equiv h^2 \mod 4N$.

Assume now that N is squarefree and

(H) $-D < -4$ is an odd fundamental discriminant coprime to N such that every prime divisor of N splits in $K = \mathbb{Q}(\sqrt{-D})$.

Fix a solution $h \in \mathbb{Z}/2N\mathbb{Z}$ of $h^2 \equiv -D \mod 4N$, and for a primitive integral ideal $\mathfrak{A}$ of K write

$$\mathfrak{A} = \mathbb{Z}a + \mathbb{Z}\frac{b+\sqrt{-D}}{2}, \qquad a = N_{K/\mathbb{Q}}(\mathfrak{A}), \quad b \in \mathbb{Z}$$

with $b \equiv h \mod 2N$ and $b^2 \equiv -D \mod 4Na$. Then

$$\tau^{(h)}_{\mathfrak{A}} = \frac{b+\sqrt{-D}}{2Na}$$

is a Heegner point on the modular curve $X_0(N)$. It is known that $\tau^{(h)}_{\mathfrak{A}}$ depends only on the ideal class of $\mathfrak{A}$ and on $h \mod 2N$, so we denote it by $\tau^{(h)}_{[\mathfrak{A}]}$. For details concerning these facts, see [GZ, part II, Section 1].

Let $\mathrm{CL}(K)$ be the ideal class group of K and $h(-D) = \#\mathrm{CL}(K)$ be the class number. By Minkowski's theorem we may choose a primitive integral ideal $\mathfrak{A}$ in each ideal class $[\mathfrak{A}] \in \mathrm{CL}(K)$ such that

$$N_{K/\mathbb{Q}}(\mathfrak{A}) \leq \frac{2}{\pi}\sqrt{D}.$$

Having fixed such a choice $\mathfrak{A}$ for each ideal class, we define

$$\mathcal{O}_{D,N,h} := \{\tau_{[\mathfrak{A}]}^{(h)} : [\mathfrak{A}] \in \mathrm{CL}(K)\}.$$

Let $\overline{\Gamma}_\tau$ be the image of the stabilizer of $\tau = \tau_{[\mathfrak{A}]}^{(h)}$ in $\mathrm{PSL}_2(\mathbb{Z})$.

The Maass level-raising operator

$$R_{-2} := 2i\frac{\partial}{\partial z} - 2\mathrm{Im}(z)^{-1}, \quad z \in \mathbb{H}$$

maps weak Maass forms of weight -2 to weak Maass forms of weight 0. Define the operator

$$\partial := \frac{1}{4\pi}R_{-2}.$$

Bruinier and Ono [BO2, Theorem 3.6] established a formula for the Fourier coefficients of holomorphic parts of weight -1/2 vector-valued harmonic weak Maass forms in the image $\Lambda(H_{-2}(N))$. With the setup in this paper, their formula can be stated as follows.

Theorem 1.1 (Bruinier-Ono). *For each $f \in H_{-2}(N)$, the (D, h)-th Fourier coefficient of the holomorphic part of $\Lambda(f, w) \in \mathbf{H}_{-\frac{1}{2},\rho}$ is given by*

$$c_{\Lambda_f}^{+}(D, h) = -\frac{4N}{D} \sum_{\tau \in \mathcal{O}_{D,N,h}} \frac{\partial f(\tau)}{\#\overline{\Gamma}_\tau}. \tag{1.1}$$

Let $M_{-2}^{!}(N) \subset H_{-2}(N)$ be the space of weakly holomorphic modular forms of weight -2 and level N, and $\mathbf{M}^{!}_{-\frac{1}{2},\rho} \subset \mathbf{H}_{-\frac{1}{2},\rho}$ be the space of vector-valued weakly holomorphic modular forms of weight -1/2 and level N with Weil representation ρ (see Section 3). Each form $f \in M_{-2}^{!}(N)$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{N_\infty} c_f(-n)q^{-n} + \sum_{n=0}^{\infty} c_f(n)q^n.$$

Bruinier and Ono [BO2, Theorem 3.5] proved that the theta lift Λ restricts to a map

$$\Lambda : M_{-2}^{!}(N) \to \mathbf{M}^{!}_{-\frac{1}{2},\rho}.$$

We will establish the following asymptotic formula with a power saving error term for the Fourier coefficients of holomorphic parts of weight -1/2 vector-valued weakly holomorphic modular forms in the image $\Lambda(M_{-2}^{!}(N))$.

Theorem 1.2. *For each $f \in M_{-2}^{!}(N)$, the (D, h)-th Fourier coefficient of the holomorphic part of $\Lambda(f, w) \in \mathbf{M}^{!}_{-\frac{1}{2},\rho}$ satisfies*

$$c_{\Lambda_f}^{+}(D, h) = \frac{1}{D} \sum_{\substack{\tau \in \mathcal{O}_{D,N,h} \\ \mathrm{Im}(\tau) > \frac{4N}{\pi} + D^{-\frac{1}{176}}}} M_{f,N}(\tau) + C_{f,N}\frac{h(-D)}{D} + O_{\epsilon,N}(D^{-\frac{89}{176}+\epsilon})$$

as $D \to \infty$ through all D satisfying the hypothesis (H). *Here*

$$M_{f,N}(z) := -4N \sum_{n=1}^{N_\infty} c_f(-n)n \left(1 - \frac{1}{2\pi n \mathrm{Im}(z)}\right) e(-nz)$$

and

$$C_{f,N} := -\frac{N}{\pi}\int_{\mathrm{reg}} R_{-2}f(z)d\mu(z)$$

is a Borcherds-type regularized integral where $d\mu(z)$ *is the normalized hyperbolic measure on* $\Gamma_0(N)\backslash\mathbb{H}$ *(see Section 13).*

Remark 1.3. We briefly explain how Theorem 1.2 can be generalized to give an asymptotic formula for the Fourier coefficients of holomorphic parts of weight -1/2 vector-valued harmonic weak Maass forms in the image $\Lambda(H_{-2}(N))$. A key step in the proof of Theorem 1.2 is to express the image $\partial(f)$ of a weakly holomorphic modular form $f \in M^!_{-2}(N)$ under the operator $\partial = R_{-2}/4\pi$ as a finite linear combination of certain weight 0 Maass-Poincaré series (see Section 5). By (1.1) it follows that the asymptotic distribution of $c^+_{\Lambda_f}(D,h)$ is determined by the asymptotic distribution of traces of these Maass-Poincaré series. Using [BO2, Corollary 3.4] and [BO2, Proposition 2.2], one can obtain a similar (but more complicated) expression for the image $\partial(f)$ of any harmonic weak Maass form $f \in H_{-2}(N)$. Theorem 1.2 can then be generalized by using this expression and modifying the proof accordingly. We focus here on the case $f \in M^!_{-2}(N)$ since this is what is needed for our application to the partition function.

Remark 1.4. Let $\ell | N$ with $(\ell, N/\ell) = 1$, and let $f \mapsto f|_{-2}W^N_\ell$ be the Atkin-Lehner involution of $M^!_{-2}(N)$ defined in [BO2, Section 4.3]. Then assuming that $f|_{-2}W^N_\ell$ has coefficients in $\mathbb{Z}$ for every such ℓ, Bruinier and Ono [BO2, Theorem 4.5] proved that $6D \cdot \partial f(\tau)$ is an algebraic integer in the ring class field for the order $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{-D})$.

Remark 1.5. The Fourier coefficients $c^+_{\Lambda_f}(D,h)$ are likely related to periods of differentials of the third kind in a manner similar to [B2].

We now discuss our application to the partition function $p(n)$. Define the weakly holomorphic modular form $f_p \in M^!_{-2}(6)$ by

$$f_p(z) := \frac{1}{2}\frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2} = q^{-1} - 10 - 29q - \cdots \tag{1.2}$$

where

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

is the Dedekind eta function and

$$E_2(z) := 1 - 24\sum_{n=1}^{\infty}(\sum_{d|n} d)q^n$$

is the usual weight 2 Eisenstein series. Bruinier and Ono [BO2, Section 3.2] proved that

$$p(n) = -\frac{1}{24}c^+_{\Lambda_{f_p}}(24n-1, 1). \tag{1.3}$$

By combining (1.3) with (1.1), they obtained the following formula for $p(n)$ (see [BO2, Theorem 1.1])

$$p(n) = \frac{1}{24n-1}\sum_{\tau \in \mathcal{Q}_{24n-1,6,1}} \partial f_p(\tau).$$

Bruinier and Ono [BO2, Theorem 4.2] also proved that $f_p(z)$ satisfies the assumption on the Atkin-Lehner involutions in Remark 1.4, and hence $6(24n-1)\partial f_p(\tau)$ is an algebraic integer in the ring class field of discriminant $-24n+1$. Note that Larson and Rolen [LR] later showed this result holds without the 6, answering a question of Bruinier and Ono [BO2, Section 5, (1)].

By combining Theorem 1.2 with (1.3), we will obtain the following new asymptotic formula for $p(n)$.

Theorem 1.6. *Let n be a positive integer with $24n-1$ squarefree. Then*

$$p(n) = \frac{1}{24n-1} \sum_{\substack{\tau \in \mathcal{O}_{24n-1,6,1} \\ \mathrm{Im}(\tau) > \frac{24}{\pi} + (24n-1)^{-\frac{1}{176}}}} \left(1 - \frac{1}{2\pi \mathrm{Im}(\tau)}\right) e(-\tau) + C^*_{f_p,6} \frac{h(-24n+1)}{24n-1} + O_\epsilon(n^{-\frac{89}{176}+\epsilon})$$

as $n \to \infty$ where

$$C^*_{f_p,6} := \frac{1}{4\pi} \int_{\mathrm{reg}} R_{-2} f_p(z) d\mu(z).$$

Remark 1.7. If n is a positive integer with $24n-1$ squarefree, then $-24n+1$ is an odd fundamental discriminant which satisfies the hypothesis (H) for $N=6$.

The asymptotic distribution of $p(n)$ has been studied extensively since the early part of the 20th century. Hardy and Ramanujan [HR] invented the "circle method" and used it to establish the well-known asymptotic

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(2\pi\sqrt{\frac{n}{6}}\right).$$

Rademacher [R] later used a refinement of the circle method to establish the exact formula

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{c=1}^{\infty} \sqrt{c} A_c(n) \frac{d}{dn} \frac{\exp\left(\frac{\pi \lambda_n \sqrt{2/3}}{c}\right)}{\lambda_n}.$$

Here $\lambda_n := \sqrt{n - 1/24}$ and $A_c(n)$ is the exponential sum

$$A_c(n) := \sum_{\substack{b \mod c \\ (b,c)=1}} \omega_{b,c} e\left(-\frac{nb}{c}\right),$$

where $\omega_{b,c}$ is a certain root of unity. By truncating the series at $N := \lfloor \sqrt{n/6} \rfloor$ and estimating the remainder, Rademacher [R2] obtained the asymptotic formula

$$p(n) = \mathrm{MT}(n) + O(n^{-\frac{3}{8}}) \tag{1.4}$$

with main term

$$\mathrm{MT}(n) := \frac{1}{2\pi\sqrt{2}} \sum_{c=1}^{N} \sqrt{c} A_c(n) \frac{d}{dn} \frac{\exp\left(\frac{\pi \lambda_n \sqrt{2/3}}{c}\right)}{\lambda_n}.$$

Lehmer [L] improved the error term in (1.4) to $O(n^{-\frac{1}{2}+\epsilon})$. Using an arithmetic reformulation of Rademacher's exact formula due to Bringmann and Ono [BrO], the author and Folsom

[FM] established an asymptotic formula for $p(n)$ with an error term which is $O(n^{-\delta})$ for some absolute $\delta > 1/2$.

We now compare the main term in Theorem 1.6 with the main term $\mathrm{MT}(n)$ in Rademacher's asymptotic formula (1.4), which reveals some interesting new features in the asymptotic distribution of $p(n)$. Suppose that $n \neq 6\ell^2$ for any $\ell \in \mathbb{Z}$. Then using an analysis with exponential sums (see [FM, Proposition 6.1]), one can show that

$$\mathrm{MT}(n) = \frac{1}{24n-1} \sum_{\substack{\tau \in \Lambda_{24n-1,6} \\ \mathrm{Im}(\tau)>1}} \chi_{12}(\tau) \left(1 - \frac{1}{2\pi \mathrm{Im}(\tau)}\right) e(-\tau) \tag{1.5}$$

where $\Lambda_{24n-1,6}$ is the set of Heegner points of discriminant $-24n+1$ on $X_0(6)$ and $\chi_{12}(\tau) = (\frac{12}{-b})$ where $-b$ is the middle coefficient of the quadratic form $Q(X,Y) = 6aX^2 - bXY + cY^2$ corresponding to the Heegner point τ. Let H denote the Hilbert class field of $K = \mathbb{Q}(\sqrt{-24n+1})$. The set of Heegner points decomposes as (see e.g. [GKZ, p. 507] and [GZ, pp. 235-236])

$$\Lambda_{24n-1,6} = \bigcup_{\substack{h \mod 12 \\ h^2 \equiv -24n+1 \mod 24}} \mathcal{O}_{24n-1,6,h} \tag{1.6}$$

where each set $\mathcal{O}_{24n-1,6,h}$ is a simple, transitive $\mathrm{Gal}(H/K)$-orbit. Now, recall that the main term in Theorem 1.6 contains no character twist and is summed over *one* Galois orbit $\{\tau \in \mathcal{O}_{24n-1,6,1}\}$ subject to $\mathrm{Im}(\tau) > 24/\pi + (24n-1)^{-1/176}$. On the other hand, from (1.5) we see that $\mathrm{MT}(n)$ is summed over *all* Heegner points $\{\tau \in \Lambda_{24n-1,6}\}$ subject to $\mathrm{Im}(\tau) > 1$. The number of elements in each Galois orbit is $h(-24n+1)$ since $\mathrm{Gal}(H/K) \cong \mathrm{CL}(K)$, and there are 4 orbits corresponding to the solutions $h = 1, 5, 7$ and 11 of the congruence in (1.6). Hence the main term in Theorem 1.6 contains essentially $3 \cdot h(-24n+1)$ fewer terms than the main term $\mathrm{MT}(n)$. It would be very interesting to know whether the shorter main term in Theorem 1.6, or the form of the main term itself, offers any computational advantages.

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2. Outline of the proof of Theorem 1.2

In this section we give a brief outline of the proof of Theorem 1.2. We want to establish an asymptotic formula as $D \to \infty$ for the trace

$$\sum_{\tau \in \mathcal{O}_{D,N,h}} \frac{\partial f(\tau)}{\#\overline{\Gamma}_\tau}$$

appearing in the Bruinier-Ono formula (1.1). The image of a weakly holomorphic modular form $f \in M^!_{-2}(N)$ under the operator $\partial = R_{-2}/4\pi$ is a weight 0 weak Maass form on $\Gamma_0(N)$. As a consequence, the function $\partial(f)$ can be expressed as a finite linear combination of certain Maass-Poincaré series $\{F_{N,n}\}_{n=1}^{N_\infty}$, so it suffices to study the trace of $F_{N,n}$. We will compute the Fourier expansion of $F_{N,n}$ in the cusps of $\Gamma_0(N)$ and find that it has a part with linear exponential growth and a part with polynomial growth. Now, the Galois orbit of Heegner points $\mathcal{O}_{D,N,h}$ becomes (quantitatively) equidistributed with respect to the normalized hyperbolic measure on $\Gamma_0(N)\backslash\mathbb{H}$ as $D \to \infty$. However, we cannot *directly* use this fact to obtain an asymptotic formula for the trace of $F_{N,n}$ because the "test function" $F_{N,n}$ grows very rapidly in the cusps and hence is not admissible. We will overcome this difficulty using two different regularizations. We first prove an equidistribution theorem for Galois orbits of Heegner points in which the test functions are allowed to grow polynomially in the cusps (see Theorem 6.3). We then construct for each $\eta > 0$ a certain smooth Poincaré series $\mathscr{P}_{n,\eta}$ which regularizes the linear exponential growth of $F_{N,n}$ in the cusps of $\Gamma_0(N)$. This is inspired by a construction of Duke [D] to regularize the pole at ∞ of the j-function. Upon substituting the regularized function $F_{N,n} - \mathscr{P}_{n,\eta}$ into the equidistribution theorem, we will obtain a "smooth" asymptotic formula for the trace of $F_{N,\eta}$. Finally, using a Borcherds-type integration along with a delicate analysis to un-smooth the main term and bound (as a function of η) the Sobolev norm of $F_{N,n} - \mathscr{P}_{n,\eta}$ appearing in the error term, we will obtain the desired asymptotic formula for the trace of $F_{N,n}$ (see Theorem 9.1). Note that our proof has some elements in common with [FM2], though considerable new difficulties arise because of the presence of level in all of our arguments (which is *crucial* for our application to the partition function).

3. harmonic weak Maass forms

In this section we review some facts concerning harmonic weak Maass forms. For more details, see [BF]. Let $k \in \frac{1}{2}\mathbb{Z}$ and $z = x + iy \in \mathbb{H}$. The weight k hyperbolic Laplacian is defined by

$$\Delta_k := -y^2\left(\partial_x^2 + \partial_y^2\right) + iky\left(\partial_x + i\partial_y\right).$$

Let N be a positive integer. A *weak Maass form* of weight k on $\Gamma_0(N)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying:

(1) $f|_k M = f$ for all $M \in \Gamma_0(N)$.
(2) There is a complex number λ such that $\Delta_k f = \lambda f$.
(3) There is a constant $C > 0$ such that $f(z) = O(e^{Cy})$ as $y \to \infty$. An analogous condition is required at all cusps.

Note. The slash operator $|_k$ is defined as in Shimura's theory of half-integral weight forms.

A weak Maass form is *harmonic* if $\Delta_k f = 0$. Every harmonic weak Maass form has a Fourier expansion

$$f(z) = f^-(z) + f^+(z)$$

with non-holomorphic part

$$f^-(z) := \sum_{n<0} c_f^-(n)\Gamma(1-k, 4\pi\,|n|\,y)q^n$$

and holomorphic part

$$f^+(z) := \sum_{n=1}^{N_\infty} c_f^+(-n)q^{-n} + \sum_{n=0}^{\infty} c_f^+(n)q^n,$$

where $N_\infty \geq 0$ is an integer and $\Gamma(a,t)$ is the incomplete Gamma function. A harmonic weak Maass form with trivial non-holomorphic part is called a *weakly holomorphic modular form.* Let

$$M_k^!(N) \subset H_k(N)$$

denote the spaces of weakly holomorphic modular forms and harmonic weak Maass forms, respectively.

We also require the notion of a vector-valued weak Maass form. Let $w = u + iv \in \mathbb{H}$, and let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic two-fold cover of $\mathrm{SL}_2(\mathbb{R})$ realized as the group of pairs $(M, \phi(w))$ where $M \in \mathrm{SL}_2(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi(w)^2 = cw+d$. The multiplication is defined by

$$(M, \phi(w))(M', \phi'(w)) = (MM', \phi(M'w)\phi'(w)).$$

Let $\widetilde{\Gamma} = \mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map. The group $\widetilde{\Gamma}$ is generated by $T = ((\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}), 1)$ and $S = ((\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}), \sqrt{w})$. Given $h \in \mathbb{Z}/2N\mathbb{Z}$ let $\mathfrak{e}_h$ be the corresponding standard basis vector for the group algebra $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$. The *Weil representation* ρ is the unitary representation of $\widetilde{\Gamma}$ on $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$ defined in terms of the generators T and S of $\widetilde{\Gamma}$ by

$$\rho(T)(\mathfrak{e}_h) = e\left(\frac{h^2}{4N}\right)\mathfrak{e}_h,$$

$$\rho(S)(\mathfrak{e}_h) = \frac{1}{\sqrt{2iN}} \sum_{h' \mod 2N} e\left(-\frac{hh'}{2N}\right)\mathfrak{e}_{h'}.$$

A *vector-valued weak Maass form* of weight k (with respect to $\widetilde{\Gamma}$ and ρ) is a smooth function $g : \mathbb{H} \to \mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$ satisfying:

(1) $g(Mw) = \phi(w)^{2k}\rho(M, \phi)g(w)$ for all $(M, \phi) \in \widetilde{\Gamma}$.
(2) There is a complex number λ such that $\Delta_k g = \lambda g$.
(3) There is a constant $C > 0$ such that $g(w) = O(e^{Cv})$ as $v \to \infty$. An analogous condition is required at all cusps.

A vector-valued weak Maass form is *harmonic* if $\Delta_k g = 0$. Every vector-valued harmonic weak Maass form has a Fourier expansion

$$g(w) = g^-(w) + g^+(w)$$

with non-holomorphic part

$$g^-(w) := \sum_{h \mod 2N} \sum_{D<0} c_g^-(D, h)\Gamma(1 - k, \pi|D|v/N)q^{D/4N}\mathfrak{e}_h$$

and holomorphic part

$$g^+(w) := \sum_{h \mod 2N} \sum_{D=1}^{N_\infty} c_g^+(-D, h)q^{-D/4N}\mathfrak{e}_h + \sum_{h \mod 2N} \sum_{D=0}^{\infty} c_g^+(D, h)q^{D/4N}\mathfrak{e}_h.$$

Note that $c_g^\pm(D, h) = 0$ unless $D \equiv h^2 \mod 4N$. Let

$$\mathbf{M}^!_{k,\rho} \subset \mathbf{H}_{k,\rho}$$

denote the spaces of vector-valued weakly holomorphic modular forms and vector-valued harmonic weak Maass forms, respectively.

4. The Bruinier-Ono theta lift

Let $k \in \frac{1}{2}\mathbb{Z}$ and $z \in \mathbb{H}$. The Maass level raising and lowering differential operators are defined by

$$R_{k,z} := 2i\frac{\partial}{\partial z} + k\mathrm{Im}(z)^{-1},$$
$$L_{k,z} := -2i\mathrm{Im}(z)^2\frac{\partial}{\partial \bar{z}}.$$

Bruinier and Ono (see Sections 2.3 and 3.1, and Corollary 3.4 of [BO2]) defined a theta lift

$$\Lambda : H_{-2}(N) \to \mathbf{H}_{-\frac{1}{2},\rho}$$

by

$$\Lambda(f,w) := L_{3/2,w}\int_{\Gamma_0(N)\backslash\mathbb{H}} (R_{-2,z}f(z))\Theta(w,z,\phi_{KM}),$$

where $\Theta(w,z,\phi_{KM})$ is a $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$-valued theta function constructed from the Kudla-Millson Schwartz function ϕ_{KM} (see [KM] and [BO2, Section 2.3]). As a function of z, the theta kernel $\Theta(w,z,\phi_{KM})$ is a $\Gamma_0(N)$-invariant harmonic $(1,1)$-form on $\mathbb{H}$. When restricting Λ to weakly holomorphic modular forms, one has (see [BO2, Theorem 3.5])

$$\Lambda : M^!_{-2}(N) \to \mathbf{M}^!_{-\frac{1}{2},\rho}.$$

5. Differential operators and weak Maass forms

Define the operator

$$\partial_{-2} := \frac{1}{2\pi i}\frac{\partial}{\partial z} + \frac{1}{2\pi y}.$$

Then ∂_{-2} maps weak Maass forms of weight -2 on $\Gamma_0(N)$ to weak Maass forms of weight 0 on $\Gamma_0(N)$. It turns out that the image $\partial_{-2}(f)$ of a harmonic weak Maass form $f \in H_{-2}(N)$ under the operator ∂_{-2} can be expressed as a finite linear combination of certain Maass-Poincaré series. Let $m \in \mathbb{Z}^+$, and define the *Maass-Poincaré series* (see e.g. [B, Section 1.3])

$$F_{N,m}(z) := \pi m \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \mathrm{Im}(m\gamma z)^{1/2} I_{3/2}(2\pi m\mathrm{Im}(\gamma z))e(-m\mathrm{Re}(\gamma z)), \quad z = x+iy \in \mathbb{H}$$

where $I_{3/2}$ is the I-Bessel function of order $3/2$. It is known that $F_{N,m}$ is a weak Maass form of weight 0 on $\Gamma_0(N)$. Now, suppose that $f \in M^!_{-2}(N)$ has the Fourier expansion

$$f(z) = \sum_{n=1}^{N_\infty} c_f(-n)q^{-n} + \sum_{n=0}^{\infty} c_f(n)e(nz).$$

Then by [MP, Theorem 1.2],

$$\partial_{-2}(f) = -2\sum_{n=1}^{N_\infty} c_f(-n)F_{N,n}.$$

In particular, the identity

$$\frac{1}{4\pi}R_{-2} = -\partial_{-2}$$

implies that

$$\frac{1}{4\pi}R_{-2}(f) = 2\sum_{n=1}^{N_\infty} c_f(-n)F_{N,n}. \tag{5.1}$$

As explained in Remark 1.3, a similar (but more complicated) expression exists for any harmonic weak Maass form $f \in H_{-2}(N)$, but we will not need this here.

6. QUANTITATIVE EQUIDISTRIBUTION

Let $f_1, f_2 : \mathbb{H} \to \mathbb{C}$ be $\Gamma_0(N)$-invariant functions and $\mathcal{F}(N)$ be a fundamental domain for $\Gamma_0(N)$. Define the Petersson inner product

$$\langle f_1, f_2\rangle_2 := \int_{\mathcal{F}(N)} f_1(z)\overline{f_2(z)}d\mu(z)$$

where

$$d\mu(z) := \frac{1}{\mathrm{vol}(\mathcal{F}(N))}\frac{dxdy}{y^2}$$

is the normalized hyperbolic measure on $\mathcal{F}(N)$. Let

$$\Delta := -y^2(\partial_x^2 + \partial_y^2)$$

be the weight 0 hyperbolic Laplacian and Δ^A be the composition of Δ with itself A-times where $A \in \mathbb{Z}_{\geq 0}$. For a cusp $\mathfrak{a}$ of $\Gamma_0(N)$, let $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ be a scaling matrix such that $\sigma_{\mathfrak{a}}(\infty) = \mathfrak{a}$ (see [I, p. 47]).

Proposition 6.1. *Let $g : \mathbb{H} \to \mathbb{C}$ be a C^∞, $\Gamma_0(N)$-invariant function, and suppose that for each cusp $\mathfrak{a}$ we have*

$$\Delta^A g(\sigma_{\mathfrak{a}} z) = O(e^{-Cy}), \quad A = 0, 1, 2, \ldots \tag{6.1}$$

for some constant $C > 0$ (depending on $\mathfrak{a}$ and A). Then

$$\sum_{\tau\in\mathcal{O}_{D,N,h}} g(\tau) = h(-D)\int_{\mathcal{F}(N)} g(z)d\mu(z) + O_{\epsilon,N}(||\Delta^2 g||_2 D^{\frac{1}{2}-\frac{1}{16}+\epsilon}).$$

Proof. Let $\{u_m\}$ be an orthonormal basis of Hecke-Maass cusp forms of weight 0 for $\Gamma_0(N)$ with Δ-eigenvalues $\lambda_m = \frac{1}{4} + t_m^2$ for $m \in \mathbb{Z}^+$. For each cusp $\mathfrak{a}$ define the real-analytic Eisenstein series

$$E_{\mathfrak{a}}(z, s) := \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma_0(N)} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s, \quad z \in \mathbb{H}, \quad \mathrm{Re}(s) > 1.$$

Because g satisfies the bound (6.1) we have the spectral expansion

$$g(z) = \langle g, 1\rangle_2 + \sum_{m=1}^{\infty}\langle g, u_m\rangle_2 u_m(z) + \sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{\mathbb{R}}\langle g, E_{\mathfrak{a}}(\cdot, \tfrac{1}{2} + it)\rangle_2 E_{\mathfrak{a}}(z, \tfrac{1}{2} + it)dt,$$

which converges pointwise absolutely and uniformly on compact subsets of $\mathcal{F}(N)$. Summing over the spectral expansion yields

$$\sum_{\tau\in\mathcal{O}_{D,N,h}} g(\tau) = h(-D)\langle g,1\rangle_2 + \sum_{m=1}^{\infty}\langle g,u_m\rangle_2 W_m(D) + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{\mathbb{R}}\langle g, E_{\mathfrak{a}}(\cdot,\tfrac{1}{2}+it)\rangle_2 W_{\mathfrak{a}}(D,t)dt, \tag{6.2}$$

where the hyperbolic Weyl sums are defined by

$$W_m(D) := \sum_{\tau\in\mathcal{O}_{D,N,h}} u_m(\tau) \qquad \text{and} \qquad W_{\mathfrak{a}}(D,t) := \sum_{\tau\in\mathcal{O}_{D,N,h}} E_{\mathfrak{a}}(\tau,\tfrac{1}{2}+it).$$

To estimate the contribution of the discrete spectrum, it suffices to consider only L^2-normalized Hecke-Maass *newforms* for $\Gamma_0(N)$ (see e.g. the proof of [HM, Theorem 6]). By a formula of Waldspurger/Zhang (see e.g. [W, W2] and [Z]), for such a newform u_m one has

$$\left|\sum_{\tau\in\mathcal{O}_{D,N,h}} u_m(\tau)\right|^2 = c_D\sqrt{D}\,|a_m(1)|^2\,\Lambda(u_m,\tfrac{1}{2})\Lambda(u_m\times\chi_{-D},\tfrac{1}{2}), \tag{6.3}$$

where c_D is a positive constant which takes only finitely many different values, $a_m(1)$ is the first Fourier coefficient of u_m, and

$$\Lambda(\Pi,s) := L_\infty(\Pi,s)L(\Pi,s)$$

is the completed L–function. Note that the term $|a_m(1)|^2$ appears on the right hand side because in the Waldspurger/Zhang formula the Hecke-Maass newform is arithmetically normalized.

By a subconvexity bound of Jutila and Motohashi [JM] we have

$$L(u_m,\tfrac{1}{2}) \ll_{\epsilon,N} \lambda_m^{\frac{1}{6}+\epsilon},$$

and by a hybrid subconvexity bound of Blomer and Harcos [BH, Theorem 2] we have

$$L(u_m\times\chi_{-D},\tfrac{1}{2}) \ll_{\epsilon,N} \lambda_m^{\frac{7}{4}+\epsilon} D^{\frac{1}{2}-\frac{1}{8}+\epsilon}.$$

Then using the following estimate of Hoffstein and Lockhart [HL]

$$|a_m(1)|^2 \ll_{\epsilon,N} \lambda_m^{\epsilon} e^{\pi|t|},$$

and the fact that the contribution from the infinite parts of the L–functions in (6.3) is $\ll e^{-\pi|t_m|}$ by Stirling's formula, we obtain the estimate

$$W_m(D) \ll_{\epsilon,N} \lambda_m^{\frac{23}{24}+\epsilon} D^{\frac{1}{2}-\frac{1}{16}+\epsilon}. \tag{6.4}$$

Following the argument in [HM, section 6.4], one can reduce the estimate of $W_{\mathfrak{a}}(D,t)$ to an analogous estimate for

$$\sum_{\tau\in\mathcal{O}_{D,1,1}} E(\tau,\tfrac{1}{2}+it)$$

where $E(z,s)$ is the real-analytic Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$. One has the identity (see e.g. [GZ, p. 248])

$$\sum_{\tau\in\mathcal{O}_{D,1,1}} E(\tau,s) = 2^{-s}D^{s/2}\zeta(2s)^{-1}\zeta(s)L(\chi_{-D},s).$$

Then using a standard lower bound for $\zeta(2s)$, the subconvexity bound

$$\zeta(\tfrac{1}{2}+it) \ll (\tfrac{1}{4}+t^2)^{\frac{1}{12}+\epsilon},$$

and the following hybrid subconvexity bound of Heath-Brown [HB]

$$L(\chi_{-D},\tfrac{1}{2}+it) \ll_\epsilon (\tfrac{1}{4}+t^2)^{\frac{3}{32}+\epsilon} D^{\frac{1}{4}-\frac{1}{16}+\epsilon},$$

we obtain

$$W_{\mathfrak{a}}(D,t) \ll_\epsilon (\tfrac{1}{4}+t^2)^{\frac{17}{96}+\epsilon} D^{\frac{1}{2}-\frac{1}{16}+\epsilon}. \tag{6.5}$$

Using the bound (6.1), a repeated application of Stokes' theorem (see e.g. [I, Lemma 1.18]) yields the identities

$$\langle g, u_m\rangle_2 = \lambda_m^{-A}\langle \Delta^A g, u_m\rangle_2$$

and

$$\langle g, E_{\mathfrak{a}}(\cdot,\tfrac{1}{2}+it)\rangle_2 = (\tfrac{1}{4}+t^2)^{-A}\langle \Delta^A g, E_{\mathfrak{a}}(\cdot,\tfrac{1}{2}+it)\rangle_2.$$

By Parseval's formula (see [IK, (15.17)]),

$$\sum_{m=1}^{\infty} |\langle \Delta^A g, u_m\rangle_2|^2 + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{\mathbb{R}} |\langle \Delta^A g, E_{\mathfrak{a}}(\cdot,\tfrac{1}{2}+it)\rangle_2|^2 dt = ||\Delta^A g||_2^2.$$

Then by the Cauchy-Schwarz inequality we have

$$\sum_{m=1}^{\infty} |\langle g, u_m\rangle_2|\, \lambda_m^{\frac{23}{24}+\epsilon} \ll ||\Delta^2 g||_2 \tag{6.6}$$

and

$$\int_{\mathbb{R}} |\langle g, E_{\mathfrak{a}}(\cdot,\tfrac{1}{2}+it)\rangle_2|(\tfrac{1}{4}+t^2)^{\frac{17}{96}+\epsilon} dt \ll ||\Delta^2 g||_2. \tag{6.7}$$

Finally, the proposition follows by combining (6.2) with the estimates (6.4)–(6.5) and (6.6)–(6.7). □

Definition 6.2. Let $g : \mathbb{H} \to \mathbb{C}$ be a C^∞, $\Gamma_0(N)$-invariant function and α be a real number. We say that g has *cuspidal growth of power* α if for each cusp $\mathfrak{a}$ of $\Gamma_0(N)$ there exists a constant $c_{\mathfrak{a}} \in \mathbb{C}$ such that

$$\Delta^A(g(\sigma_{\mathfrak{a}} z) - c_{\mathfrak{a}} y^\alpha) = O(e^{-Cy}), \quad A = 0, 1, 2, \ldots$$

for some constant $C > 0$ (depending on $\mathfrak{a}$ and A).

We now combine Proposition 6.1 with a suitable regularization to establish the following

Theorem 6.3. *Suppose that F has cuspidal growth of power $\alpha < 5/8$. Then*

$$\sum_{\tau \in \mathcal{O}_{D,N,h}} F(\tau) = h(-D)\int_{\mathcal{F}(N)} F(z)d\mu(z)$$
$$+ O_{\epsilon,N}(||\Delta^2 F_{T_0}||_2 D^{\frac{1}{2}-\frac{1}{16}+\epsilon}) + O_{\epsilon,N}(D^{\frac{1}{2}-c(\alpha)+\epsilon}) + O_N(D^{\frac{1}{2}-\frac{(1-\alpha)}{2}+\epsilon}),$$

where F_{T_0} is a regularized version of F for a certain constant $T_0 > 1$ independent of D (see (6.9)) and

$$c(\alpha) := \begin{cases} \frac{1}{16}, & \alpha \le \frac{1}{2} \\ \frac{1}{16} - (\frac{\alpha}{2} - \frac{1}{4}), & \frac{1}{2} < \alpha < \frac{5}{8}. \end{cases}$$

Proof. For $T > 1$, define

$$\psi_T(t) := t^\alpha \chi(t/T)$$

where $\chi : \mathbb{R}^+ \to [0,1]$ is a C^∞ function such that

$$\chi(t) = \begin{cases} 0, & t < 1 \\ 1, & t > 2. \end{cases}$$

Define the incomplete Eisenstein series

$$E_{\mathfrak{b}}(\psi_T|z) := \sum_{\gamma \in \Gamma_{\mathfrak{b}} \backslash \Gamma_0(N)} \psi_T\left(\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}\gamma z)\right)$$

and let

$$\eta_T(z) := \sum_{\mathfrak{b}} c_{\mathfrak{b}} E_{\mathfrak{b}}(\psi_T|z).$$

Then by [I, (3.17)] and (8.2) we have

$$\eta_T(\sigma_{\mathfrak{a}} z) = \begin{cases} 0, & 1 < y < T \\ c_{\mathfrak{a}} y^\alpha \chi(y/T), & T \le y \le 2T \\ c_{\mathfrak{a}} y^\alpha, & y > 2T. \end{cases}$$

It follows that for $y > 2T$ the regularized function

$$F_T(z) := F(z) - \eta_T(z)$$

satisfies the bound (6.1).

Now, let

$$T = T_D := 1 + \max\left\{\frac{\sqrt{D}}{2N}, \frac{4N}{\pi}\right\}.$$

Then by Lemma 6.4 we have $\eta_T(\tau) = 0$ for all $\tau \in \mathcal{O}_{D,N,h}$, and thus

$$\sum_{\tau \in \mathcal{O}_{D,N,h}} F(\tau) = \sum_{\tau \in \mathcal{O}_{D,N,h}} F_T(\tau). \tag{6.8}$$

Let T_0 be a constant independent of D with $T > T_0 > 1$ and decompose the regularized function as

$$F_T(z) = F_{T_0}(z) + \tilde{\eta}_T(z) \tag{6.9}$$

where

$$\tilde{\eta}_T(z) := \eta_{T_0}(z) - \eta_T(z).$$

Since F_{T_0} satisfies the bound (6.1), it follows from Proposition 6.1 that

$$\sum_{\tau\in\mathcal{O}_{D,N,h}} F_{T_0}(\tau) = h(-D)\int_{\mathcal{F}(N)} F_{T_0}(z)d\mu(z) + O_{\epsilon,N}(||\Delta^2 F_{T_0}||_2 D^{\frac{1}{2}-\frac{1}{16}+\epsilon}).$$

Then by (6.8)–(6.9), to complete the proof it suffices to show that

$$\sum_{\tau\in\mathcal{O}_{D,N,h}} \tilde{\eta}_T(\tau) = h(-D)\int_{\mathcal{F}(N)} \eta_{T_0}(z)d\mu(z) + O_{\epsilon,N}(D^{\frac{1}{2}-c(\alpha)+\epsilon}) + O_N(D^{\frac{1}{2}-\frac{(1-\alpha)}{2}+\epsilon}).$$

By [I, Theorem 11.3 and (7.12)–(7.13)] we have

$$\tilde{\eta}_T(z) = \langle \tilde{\eta}_T, 1\rangle_2 + \frac{1}{2\pi i}\sum_{\mathfrak{b}} c_{\mathfrak{b}}\int_{\mathbb{R}} \big(\widehat{\psi}_{T_0}(\tfrac{1}{2}+it) - \widehat{\psi}_T(\tfrac{1}{2}+it)\big)E_{\mathfrak{b}}(z,\tfrac{1}{2}+it)dt,$$

where

$$\widehat{\psi}(t) := \int_0^\infty \psi(t)t^{-(s+1)}dt.$$

It follows that

$$\sum_{\tau\in\mathcal{O}_{D,N,h}} \tilde{\eta}_T(\tau) = h(-D)\int_{\mathcal{F}(N)} \eta_{T_0}(z)d\mu(z) - h(-D)\int_{\mathcal{F}(N)} \eta_T(z)d\mu(z) + E(D,T),$$

where

$$E(D,T) := \frac{1}{2\pi i}\sum_{\mathfrak{b}} c_{\mathfrak{b}}\int_{\mathbb{R}} \big(\widehat{\psi}_{T_0}(\tfrac{1}{2}+it) - \widehat{\psi}_T(\tfrac{1}{2}+it)\big)W_{\mathfrak{b}}(D,t)dt.$$

By [KMY, Lemma 5.6], for all $B>0$ we have

$$\int_{\mathbb{R}} |\widehat{\psi}_{T_0}(\tfrac{1}{2}+it) - \widehat{\psi}_T(\tfrac{1}{2}+it)|(1+|t|)^B dt \ll C(\alpha,T), \tag{6.10}$$

where

$$C(\alpha,T) := \begin{cases} \log(T), & \alpha \le \frac{1}{2} \\ T^{\alpha-\frac{1}{2}}, & \alpha > \frac{1}{2}. \end{cases}$$

Since $T < (1+\frac{4N}{\pi})\sqrt{D}$, we combine (6.5) and (6.10) to obtain

$$E(D,T) = \begin{cases} O_{\epsilon,N}(D^{\frac{1}{2}-\frac{1}{16}+\epsilon}), & \alpha \le \frac{1}{2} \\ O_{\epsilon,N}(D^{\frac{1}{2}-(\frac{1}{16}-(\frac{\alpha}{2}-\frac{1}{4}))+\epsilon}), & \frac{1}{2} < \alpha < \frac{5}{8}. \end{cases}$$

Finally, a straightforward estimate yields

$$h(-D)\int_{\mathcal{F}(N)} \eta_T(z)d\mu(z) = O(h(-D)T^{\alpha-1}) = O_{\epsilon,N}(D^{\frac{1}{2}-\frac{(1-\alpha)}{2}+\epsilon}),$$

where we used

$$h(-D) \ll_\epsilon D^{\frac{1}{2}+\epsilon}.$$

□

Lemma 6.4. *For $\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma_0(N)$ and $\tau \in \mathcal{O}_{D,N,h}$ we have*

$$\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\tau) \le \max\left\{\mathrm{Im}(\tau), \frac{4N}{\pi}\right\} \le \max\left\{\frac{\sqrt{D}}{2N}, \frac{4N}{\pi}\right\}.$$

Proof. Recall that we chose $\mathcal{O}_{D,N,h}$ so that a Heegner point $\tau \in \mathcal{O}_{D,N,h}$ has the form

$$\tau = \tau_{\mathfrak{A}}^{(h)} = \frac{b+\sqrt{-D}}{2Na}$$

with

$$a \le \frac{2}{\pi}\sqrt{D}. \tag{6.11}$$

Write $\sigma_{\mathfrak{a}}^{-1}\gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Then we have

$$\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c'\tau+d'|^2} = \frac{1}{|c'\tau+d'|^2}\frac{\sqrt{D}}{2Na}.$$

If $c'=0$, then $d'=1$ (see [I, (2.15)-(2.17)]) so that

$$\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\tau) \le \frac{\sqrt{D}}{2N}$$

(recall that $a = N_{K/\mathbb{Q}}(\mathfrak{A}) \ge 1$). On the other hand, if $c' \ne 0$ then by (8.2) we have $(c')^2 \ge 1$ so that

$$|c'\tau+d'|^2 = \left(\frac{c'b}{2Na}+d'\right)^2 + \left(\frac{c'\sqrt{D}}{2Na}\right)^2 \ge \frac{D}{4N^2a^2}.$$

Then using (6.11) we obtain

$$\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\tau) \le \frac{4N^2a^2}{D}\frac{\sqrt{D}}{2Na} = \frac{2Na}{\sqrt{D}} \le \frac{4N}{\pi}.$$

□

7. Fourier expansion of $F_{N,m}(z)$

Write

$$F_{N,m}(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} p_m(\gamma z)$$

where

$$p_m(z) := \psi_m(\mathrm{Im}(z))e(-mz)$$

and

$$\psi_m(t) := \pi m\sqrt{mt}I_{3/2}(2\pi mt)e(mit), \quad t \in \mathbb{R}^+.$$

Then by [I, p. 60] the Fourier expansion of $F_{N,m}$ in the cusp $\mathfrak{b}$ is given by

$$F_{N,m}(\sigma_{\mathfrak{b}}z) = \delta_{\infty,\mathfrak{b}}\psi_m(y)e(-mz) + \sum_{n\in\mathbb{Z}} e(nx) \sum_{c\in\mathbb{R}^+} S_{\infty,\mathfrak{b}}(-m,n;c)A_m(n,c,y),$$

where $S_{\infty,\mathfrak{b}}(-m,n;c)$ is the Kloosterman sum

$$S_{\infty,\mathfrak{b}}(-m,n;c) := \sum_{\begin{pmatrix} a & * \\ c & d \end{pmatrix} \in B\backslash\sigma_\infty^{-1}\Gamma_0(N)\sigma_{\mathfrak{b}}/B} e\left(\frac{-md+na}{c}\right)$$

with group of integral translations

$$B := \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} : \ b \in \mathbb{Z} \right\}$$

and

$$A_m(n,c,y) := \int_{\mathbb{R}} \psi_m\left(\frac{y}{c^2(t^2+y^2)}\right) e\left(\frac{m}{c^2(t+iy)} - nt\right) dt.$$

In Lemma 7.1 we will evaluate the integral $A_m(n,c,y)$. Then inserting this evaluation into the Fourier expansion yields

$$F_{N,m}(\sigma_{\mathfrak{b}} z) = \delta_{\infty,\mathfrak{b}} \pi m^{3/2} \sqrt{y} I_{3/2}(2\pi m y) e(-mx) + C_{\mathfrak{b}}(m) y^{-1} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} C_{\mathfrak{b}}(m,n)\sqrt{y} K_{3/2}(2\pi |n| y) e(nx)$$

where

$$C_{\mathfrak{b}}(m) := \frac{2\pi^3 m^3}{3} \sum_{c\in\mathbb{R}^+} \frac{S_{\infty,\mathfrak{b}}(-m,0;c)}{c^4}$$

and

$$C_{\mathfrak{b}}(m,n) := \begin{cases} 2\pi m^{3/2} \sum_{c\in\mathbb{R}^+} \frac{S_{\infty,\mathfrak{b}}(-m,n;c)}{c} J_3\left(\frac{4\pi\sqrt{m|n|}}{c}\right), & n<0 \\ 2\pi m^{3/2} \sum_{c\in\mathbb{R}^+} \frac{S_{\infty,\mathfrak{b}}(-m,n;c)}{c} I_3\left(\frac{4\pi\sqrt{mn}}{c}\right), & n>0. \end{cases}$$

Finally, using the identities

$$\sqrt{t} I_{3/2}(t) = \frac{1}{\sqrt{2\pi}}\left(e^t\left(1-\frac{1}{t}\right) + e^{-t}\left(1+\frac{1}{t}\right)\right)$$

and

$$\sqrt{t} K_{3/2}(t) = \sqrt{\frac{\pi}{2}} e^{-t}\left(1+\frac{1}{t}\right),$$

we obtain

$$F_{N,m}(\sigma_{\mathfrak{b}} z) = C_{\mathfrak{b}}(m) y^{-1} + \delta_{\infty,\mathfrak{b}} \frac{m}{2}\left(1-\frac{1}{2\pi m y}\right) e(-mz) + E_{\mathfrak{b}}(m,x,y) \tag{7.1}$$

where

$$E_{\mathfrak{b}}(m,x,y) := \delta_{\infty,\mathfrak{b}}\frac{m}{2}e^{-2\pi my}\left(1+\frac{1}{2\pi my}\right)e(-mx) + \frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} C_{\mathfrak{b}}(m,n)\,|n|^{-1/2}\,e^{-2\pi|n|y}\left(1+\frac{1}{2\pi\,|n|\,y}\right)e(nx).$$

Lemma 7.1. *We have*

$$A_m(n,c,y) = \begin{cases} \dfrac{2\pi m^{3/2}}{c}\sqrt{y}K_{3/2}(2\pi|n|y)J_3\left(\dfrac{4\pi\sqrt{m\,|n|}}{c}\right), & n<0\\ \dfrac{2\pi^3 m^3}{3c^4}y^{-1}, & n=0\\ \dfrac{2\pi m^{3/2}}{c}\sqrt{y}K_{3/2}(2\pi ny)I_3\left(\dfrac{4\pi\sqrt{mn}}{c}\right), & n>0. \end{cases}$$

Proof. Using the identity

$$M_{0,3/2}(2u) = 2^{7/2}\Gamma(5/2)\sqrt{u}I_{3/2}(u)$$

where $M_{0,3/2}$ is the usual M-Whittaker function of order $(0,3/2)$, we find that

$$\psi_m(t) = C_1 m M_{0,3/2}(4\pi mt)e(mit)$$

where

$$C_1 := \frac{\pi}{\sqrt{2\pi}2^{7/2}\Gamma(5/2)} = \frac{1}{12}.$$

Therefore

$$A_m(n,c,y) = \frac{m}{12}I_m(n,c,y),$$

where

$$I_m(n,c,y) := \int_{\mathbb{R}} M_{0,3/2}\left(\frac{4\pi my}{c^2(t^2+y^2)}\right)e\left(\frac{mt}{c^2(t^2+y^2)}-nt\right)dt.$$

By a simple change of variables, one can show that $I_m(n,c,y)$ equals the integral I in [B, p. 33] with the choices $k=0$ and $s=2$. Then using the evaluation of the integral I given there, we have

$$I_m(n,c,y) = \begin{cases} C_2\dfrac{\sqrt{m/\,|n|}}{c}W_{0,3/2}(4\pi\,|n|\,y)J_3\left(\dfrac{4\pi\sqrt{m\,|n|}}{c}\right), & n<0\\ C_3\dfrac{m^2}{c^4}y^{-1}, & n=0\\ C_2\dfrac{\sqrt{m/n}}{c}W_{0,3/2}(4\pi ny)I_3\left(\dfrac{4\pi\sqrt{mn}}{c}\right), & n>0, \end{cases}$$

where

$$C_2 := \frac{2\pi\Gamma(4)}{\Gamma(2)} = 12\pi, \qquad C_3 := \frac{4\pi^3\Gamma(4)}{3\Gamma(2)^2} = 8\pi^3,$$

K_3 and J_3 are the usual K and J-Bessel functions of order 3, respectively, and $W_{0,3/2}$ is the usual W-Whittaker function of order $(0, 3/2)$. Using the identity

$$W_{0,3/2}(2u) = \sqrt{2u/\pi} K_{3/2}(u)$$

where $K_{3/2}$ is the usual K-Bessel function of order $3/2$, we have

$$W_{0,3/2}(4\pi |n| y) = 2\sqrt{|n| y} K_{3/2}(2\pi |n| y).$$

The result now follows after simplification. $\square$

8. Poincaré series

Let $\lambda : \mathbb{R} \to [0, 1]$ be a C^∞ function such that

$$\lambda(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t \geq 1. \end{cases}$$

Let $\eta > 0$ and define

$$\mathscr{P}_{m,\eta}(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \psi_{m,\eta}(\operatorname{Im}(\gamma z)) e(-m\gamma z),$$

where

$$\psi_{m,\eta}(t) := \lambda\left(\frac{t - \frac{4N}{\pi}}{\eta}\right) \frac{m}{2}\left(1 - \frac{1}{2\pi m t}\right).$$

Then define the regularized function

$$F_{N,m,\eta}(z) := F_{N,m}(z) - \mathscr{P}_{m,\eta}(z).$$

Proposition 8.1. *For $y > \frac{4N}{\pi} + \eta$ we have*

$$F_{N,m,\eta}(\sigma_{\mathfrak{b}} z) = C_{\mathfrak{b}}(m) y^{-1} + E_{\mathfrak{b}}(m, x, y).$$

In particular, the regularized function $F_{N,m,\eta}$ has cuspidal growth of power $\alpha = -1$.

Proof. We have the Fourier expansion

$$\mathscr{P}_{m,\eta}(\sigma_{\mathfrak{b}} z) = \delta_{\infty,\mathfrak{b}} \psi_{m,\eta}(y) e(-mz) + \sum_{n \in \mathbb{Z}} e(nx) \sum_{c \in \mathbb{R}^+} S_{\infty,\mathfrak{b}}(-m, n; c) A_{m,\eta}(n, c, y), \tag{8.1}$$

where

$$A_{m,\eta}(n, c, y) := \int_{\mathbb{R}} \psi_{m,\eta}\left(\frac{y}{c^2(t^2 + y^2)}\right) e\left(\frac{m}{c^2(t + iy)} - nt\right) dt.$$

The function $\psi_{m,\eta} : \mathbb{R} \to [0, m]$ is C^∞ and satisfies

$$\psi_{m,\eta}(t) = \begin{cases} 0, & t \leq \frac{4N}{\pi}, \\ \frac{m}{2}\left(1 - \frac{1}{2\pi m t}\right), & t \geq \frac{4N}{\pi} + \eta. \end{cases}$$

Moreover, since

$$\min\left\{c \in \mathbb{R}^+ : \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma_0(N) \sigma_{\mathfrak{b}}\right\} \geq 1 \tag{8.2}$$

for all cusps $\mathfrak{a}, \mathfrak{b}$ of $\Gamma_0(N)$ (see [I, eqs. (2.28)–(2.31)]), we have

$$\frac{y}{c^2(t^2+y^2)} \leq \frac{4N}{\pi}$$

for $y \geq \frac{4N}{\pi}$. It follows from (8.1) that

$$\mathscr{P}_{m,\eta}(\sigma_{\mathfrak{b}} z) = \begin{cases} \delta_{\infty,\mathfrak{b}} \psi_{m,\eta}(y) e(-mz), & y \geq \frac{4N}{\pi} \\ \delta_{\infty,\mathfrak{b}} \frac{m}{2}\left(1 - \frac{1}{2\pi m y}\right) e(-mz), & y \geq \frac{4N}{\pi} + \eta. \end{cases} \tag{8.3}$$

The proposition now follows from the Fourier expansion (7.1). □

9. TRACES OF WEAK MAASS FORMS

Define the trace

$$\mathrm{Tr}_D(F_{N,m}) := \sum_{\tau \in \mathcal{O}_{D,N,h}} F_{N,m}(\tau).$$

Theorem 9.1. *We have*

$$\mathrm{Tr}_D(F_{N,m}) = \frac{m}{2} \sum_{\substack{\tau \in \mathcal{O}_{D,N,h} \\ \mathrm{Im}(\tau) > \frac{4N}{\pi} + D^{-\frac{1}{176}}}} \left(1 - \frac{1}{2\pi m \mathrm{Im}(\tau)}\right) e(-m\tau) + h(-D) c_{N,m} + O_{\epsilon,N}(D^{\frac{1}{2} - \frac{1}{176} + \epsilon})$$

as $D \to \infty$ through all D satisfying the hypothesis (H). *Here*

$$c_{N,m} := \int_{\mathrm{reg}} F_{N,m}(z) d\mu(z)$$

is a Borcherds-type regularized integral (see Section 13).

Proof. Assume that $0 < \eta < \frac{998}{1000}$ (we will eventually choose η to be very small as a function of D). By Proposition 8.1, the regularized function $F_{N,m,\eta}$ has cuspidal growth of power $\alpha = -1$. Moreover, this growth is uniform in η for $y > \frac{4N}{\pi} + \frac{998}{1000}$, hence the same choice of constant $T_0 := \frac{4N}{\pi} + \frac{999}{1000}$ and corresponding function η_{T_0} given by

$$\eta_{T_0}(\sigma_{\mathfrak{b}} z) = \begin{cases} 0, & 1 < y < \frac{4N}{\pi} + \frac{999}{1000} \\ C_{\mathfrak{b}}(m) y^{-1} \chi(y/2), & \frac{4N}{\pi} + \frac{999}{1000} \leq y \leq 2(\frac{4N}{\pi} + \frac{999}{1000}) \\ C_{\mathfrak{b}}(m) y^{-1}, & y > 2(\frac{4N}{\pi} + \frac{999}{1000}) \end{cases} \tag{9.1}$$

can be used to regularize each function $F_{N,m,\eta}$ as in the proof of Theorem 6.3. Upon substituting $F_{N,m,\eta}$ into Theorem 6.3, we obtain the asymptotic formula

$$\begin{aligned} \mathrm{Tr}_D(F_{N,m}) = \mathrm{Tr}_D(\mathscr{P}_{m,\eta}) + h(-D) \int_{\mathcal{F}(N)} F_{N,m,\eta}(z) d\mu(z) \\ + O_{\epsilon,N}(||\Delta^2 F_{N,m,\eta,T_0}||_2 D^{\frac{1}{2} - \frac{1}{16} + \epsilon}) + O_{\epsilon,N}(D^{\frac{1}{2} - \frac{1}{16} + \epsilon}) + O_N(D^{-\frac{1}{2} + \epsilon}), \end{aligned} \tag{9.2}$$

where

$$F_{N,m,\eta,T_0}(z) := F_{N,m,\eta}(z) - \eta_{T_0}(z).$$

By Lemma 9.2 we have

$$\mathrm{Tr}_D(\mathscr{P}_{m,\eta}) = \sum_{\mathrm{Im}(\tau) > \frac{4N}{\pi}} \psi_{m,\eta}(\mathrm{Im}(\tau)) e(-m\tau).$$

Split the sum on the right hand side into the ranges $\mathrm{Im}(\tau) \leq \frac{4N}{\pi} + \eta$ and $\mathrm{Im}(\tau) > \frac{4N}{\pi} + \eta$, and define

$$R_{N,m,\eta}(D) := \mathrm{Tr}_D(F_{N,m}) - \frac{m}{2} \sum_{\mathrm{Im}(\tau) > \frac{4N}{\pi} + \eta} \left(1 - \frac{1}{2\pi m \mathrm{Im}(\tau)}\right) e(-m\tau).$$

Then (9.2) can be written as

$$\begin{aligned} R_{N,m,\eta}(D) = h(-D) &\int_{\mathcal{F}(N)} F_{N,m,\eta}(z) d\mu(z) \\ &+ O_{\epsilon,N}(||\Delta^2 F_{N,m,\eta,T_0}||_2 D^{\frac{1}{2} - \frac{1}{16} + \epsilon}) + O_{\epsilon,N}(D^{\frac{1}{2} - \frac{1}{16} + \epsilon}) + O_N(D^{-\frac{1}{2} + \epsilon}) \\ &+ \sum_{\frac{4N}{\pi} < \mathrm{Im}(\tau) \leq \frac{4N}{\pi} + \eta} \psi_{m,\eta}(\mathrm{Im}(\tau)) e(-m\tau). \end{aligned}$$

In Lemma 13.2 we will show that

$$\int_{\mathcal{F}(N)} F_{N,m,\eta}(z) d\mu(z) = \int_{\mathrm{reg}} F_{N,m}(z) d\mu(z) =: c_{N,m}$$

where the right hand side is a Borcherds-type regularized integral (note that the right hand side is independent of η).

A straightforward estimate yields

$$\sum_{\frac{4N}{\pi} < \mathrm{Im}(\tau) \leq \frac{4N}{\pi} + \eta} \psi_{m,\eta}(\mathrm{Im}(\tau)) e(-m\tau) \ll m \cdot e^{2\pi m (\frac{4N}{\pi} + \frac{998}{1000})} \# \Lambda_{N,h,\eta}(D),$$

where

$$\Lambda_{N,h,\eta}(D) := \{\tau \in \mathcal{O}_{D,N,h} : \ \frac{4N}{\pi} < \mathrm{Im}(\tau) \leq \frac{4N}{\pi} + \eta\}.$$

By Lemma 11.1 we have the estimate

$$\# \Lambda_{N,h,\eta}(D) = O_N(\eta h(-D)) + O_{\epsilon,N}(\eta^{-1} D^{\frac{1}{2} - \frac{1}{16} + \epsilon}).$$

Moreover, by Lemma 12.1 we have the estimate

$$||\Delta^2 F_{N,m,\eta,T_0}||_2 = O_{N,m}(\eta^{-10}).$$

Combining the preceding estimates yields

$$R_{N,m,\eta}(D) = h(-D) c_{N,m} + O_{\epsilon,N}(\eta^{-10} D^{\frac{1}{2} - \frac{1}{16} + \epsilon}) + O_N(\eta h(-D)).$$

If we let $\eta = D^{-b}$ for some $b > 0$, then

$$R_{N,m,\eta}(D) = h(-D) c_{N,m} + O_{\epsilon,N}(D^{\frac{1}{2} - (\frac{1}{16} - 10b) + \epsilon}) + O_{\epsilon,N}(D^{\frac{1}{2} - b + \epsilon})$$

where we used

$$h(-D) \ll_\epsilon D^{\frac{1}{2} + \epsilon}.$$

The exponent is optimized when $\frac{1}{16} - 10b = b$, or $b = 1/176$, thus[1]

$$R_{N,m,\eta}(D) = h(-D)c_{N,m} + O_{\epsilon,N}(D^{\frac{1}{2}-\frac{1}{176}+\epsilon}).$$

□

Lemma 9.2. *We have*

$$\mathrm{Tr}_D(\mathscr{P}_{m,\eta}) = \sum_{\mathrm{Im}(\tau)>\frac{4N}{\pi}} \psi_{m,\eta}(\mathrm{Im}(\tau))e(-m\tau).$$

Proof. Write

$$\mathrm{Tr}_D(\mathscr{P}_{m,\eta}) = \sum_{\mathrm{Im}(\tau)\leq\frac{4N}{\pi}} \mathscr{P}_{m,\eta}(\tau) + \sum_{\mathrm{Im}(\tau)>\frac{4N}{\pi}} \mathscr{P}_{m,\eta}(\tau) =: \mathrm{I} + \mathrm{II}.$$

Let $\gamma \in \Gamma_\infty\backslash\Gamma_0(N)$ and fix a Heegner point $\tau \in \mathcal{O}_{D,N,h}$ with $\mathrm{Im}(\tau) \leq 4N/\pi$. Then by Lemma 6.4, $\mathrm{Im}(\gamma\tau) \leq 4N/\pi$. Since $\psi_{m,\eta}(t) = 0$ for $t \leq \frac{4N}{\pi}$, it follows that

$$\mathscr{P}_{m,\eta}(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \psi_{m,\eta}(\mathrm{Im}(\gamma\tau))e(-m\gamma\tau) = 0,$$

and thus $\mathrm{I} = 0$. On the other hand, if we fix a Heegner point $\tau \in \mathcal{O}_{D,N,h}$ with $\mathrm{Im}(\tau) > 4N/\pi$, then by (8.3) we have

$$\mathscr{P}_{m,\eta}(\tau) = \psi_{m,\eta}(\mathrm{Im}(\tau))e(-m\tau),$$

and thus

$$\mathrm{II} = \sum_{\mathrm{Im}(\tau)>\frac{4N}{\pi}} \psi_{m,\eta}(\mathrm{Im}(\tau))e(-m\tau).$$

□

10. Proof of Theorems 1.2 and 1.6

Proof of Theorem 1.2. By combining the Bruinier-Ono period formula (1.1), the identity (5.1), and Theorem 9.1 we obtain

$$\begin{aligned} c^+_{\Lambda_f}(D,h) &= -\frac{4N}{D}\mathrm{Tr}_D(\frac{1}{4\pi}R_{-2}f) \\ &= -\frac{8N}{D}\sum_{n=1}^{N_\infty} c_f(-n)\mathrm{Tr}_D(F_{N,n}) \\ &= \frac{1}{D}\sum_{\mathrm{Im}(\tau)>\frac{4N}{\pi}+D^{-\frac{1}{176}}} M_{f,N}(\tau) + C_{f,N}\frac{h(-D)}{D} + O_{\epsilon,N}(D^{-\frac{89}{176}+\epsilon}), \end{aligned}$$

where

$$M_{f,N}(z) := -4N\sum_{n=1}^{N_\infty} c_f(-n)n\left(1 - \frac{1}{2\pi n\mathrm{Im}(z)}\right)e(-nz)$$

[1]Recall that we assumed $\eta \leq \frac{998}{1000}$, which is equivalent to $D \geq \left(\frac{1000}{998}\right)^{176} \approx 1.4224$ for the choice $\eta = D^{-1/176}$. Since $D \geq 4$ by assumption, the latter inequality is satisfied.

and

$$C_{f,N} := -8N \sum_{n=1}^{N_\infty} c_f(-n) c_{N,n} = -8N \int_{\text{reg}} \sum_{n=1}^{N_\infty} c_f(-n) F_{N,n}(z) d\mu(z) = -\frac{N}{\pi} \int_{\text{reg}} R_{-2} f(z) d\mu(z),$$

where for the last equality we again used (5.1). □

Proof of Theorem 1.6. Recall the Bruinier-Ono formula (1.3) for the partition function,

$$p(n) = -\frac{1}{24} c^{+}_{\Lambda_{f_p}}(24n-1, 1),$$

where for the weakly holomorphic modular form $f_p \in M^{!}_{-2}(6)$ defined by (1.2) we have $N_\infty = 1$ and $c_{f_p}(-1) = 1$. Then by Theorem 1.2 we obtain

$$c^{+}_{\Lambda_{f_p}}(24n-1,1) = \frac{1}{24n-1} \sum_{\substack{\tau \in \mathcal{O}_{24n-1,6,1} \\ \text{Im}(\tau) > \frac{24}{\pi} + (24n-1)^{-\frac{1}{176}}}} M_{f_p,6}(\tau) + C_{f_p,6} \frac{h(-24n+1)}{24n-1} + O_\epsilon(n^{-\frac{89}{176}+\epsilon}),$$

where

$$M_{f_p,6}(z) = -24 \left(1 - \frac{1}{2\pi \text{Im}(z)}\right) e(-z)$$

and

$$C_{f_p,6} = -\frac{6}{\pi} \int_{\text{reg}} R_{-2} f_p(z) d\mu(z).$$

The theorem now follows after multiplying by $-1/24$. □

11. Proof of Lemma 11.1

In this section we establish the following estimate by modifying the argument in [D, p. 248-249].

Lemma 11.1. *For each number $0 < \eta \leq 1$ we have*

$$\#\Lambda_{N,h,\eta}(D) = O_N(\eta h(-D)) + O_\epsilon(\eta^{-1} D^{\frac{1}{2} - \frac{1}{16} + \epsilon}).$$

Proof. Let $0 < \eta \leq 1$ and $\phi_\eta : \mathbb{R} \to [0,1]$ be a C^∞ function such that

(1) ϕ_η is supported on $(\frac{4N}{\pi} - \eta, \frac{4N}{\pi} + 2\eta)$.
(2) $\phi_\eta = 1$ on $[\frac{4N}{\pi}, \frac{4N}{\pi} + \eta]$.
(3) ϕ_η satisfies the bound

$$\phi_\eta^{(A)} \ll \eta^{-A}, \quad A = 0, 1, 2. \tag{11.1}$$

Define

$$g_\eta(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \phi_\eta(\text{Im}(\gamma z)).$$

Then we have

$$\begin{aligned}\sum_{\tau\in\mathcal{O}_{D,N,h}} g_\eta(\tau) &= \sum_{\tau\in\mathcal{O}_{D,N,h}} \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)\\ \gamma\neq\bar{I}}} \phi_\eta(\mathrm{Im}(\gamma\tau)) + \sum_{\tau\in\mathcal{O}_{D,N,h}} \phi_\eta(\mathrm{Im}(\tau))\\ &\geq \sum_{\tau\in\mathcal{O}_{D,N,h}} \phi_\eta(\mathrm{Im}(\tau))\\ &= \sum_{\substack{\tau\in\mathcal{O}_{D,N,h}\\ \tau\in\Lambda_{N,h,\eta}(D)}} 1 + \sum_{\substack{\tau\in\mathcal{O}_{D,N,h}\\ \tau\notin\Lambda_{N,h,\eta}(D)}} \phi_\eta(\mathrm{Im}(\tau))\\ &\geq \#\Lambda_{N,h,\eta}(D).\end{aligned}$$

The real-analytic Eisenstein series $E_\infty(z,s)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s=1$ with residue $1/\mathrm{vol}(\mathcal{F}(N))$ (see [I, Theorem 11.3 and Proposition 6.13]). Then by [I, eq. (7.12)] we have

$$g_\eta(z) = \frac{1}{\mathrm{vol}(\mathcal{F}(N))}\widehat{\phi}_\eta(1) + \frac{1}{2\pi i}\int_{\mathbb{R}} \widehat{\phi}_\eta(\tfrac{1}{2}+it)E_\infty(z,\tfrac{1}{2}+it)dt,$$

where

$$\widehat{\phi}_\eta(s) := \int_0^\infty \phi_\eta(u)u^{-(s+1)}du.$$

Thus

$$\sum_{\tau\in\mathcal{O}_{D,N,h}} g_\eta(\tau) = \frac{1}{\mathrm{vol}(\mathcal{F}(N))}\widehat{\phi}_\eta(1)h(-D) + \frac{1}{2\pi i}\int_{\mathbb{R}} \widehat{\phi}_\eta(\tfrac{1}{2}+it)W_\infty(D,t)dt.$$

Now, a straightforward estimate yields

$$\widehat{\phi}_\eta(1) \ll \eta.$$

Moreover, by (6.5) we have

$$W_\infty(D,t) \ll_\epsilon (\tfrac{1}{4}+t^2)^{\frac{17}{96}+\epsilon} D^{\frac{1}{2}-\frac{1}{16}+\epsilon}.$$

It follows that

$$\sum_{\tau\in\mathcal{O}_{D,N,h}} g_\eta(\tau) = O_N(\eta h(-D)) + O_\epsilon(C_\eta D^{\frac{1}{2}-\frac{1}{16}+\epsilon}),$$

where

$$C_\eta := \int_{\mathbb{R}} |\widehat{\phi}_\eta(\tfrac{1}{2}+it)|(\tfrac{1}{4}+t^2)^{\frac{17}{96}+\epsilon}dt.$$

We integrate by parts 2 times and use the bound (11.1) to obtain

$$\widehat{\phi}_\eta(\tfrac{1}{2}+it) \ll \eta^{-1}\prod_{j=1}^{2} |\tfrac{3}{2}+it-j|^{-1},$$

which yields $C_\eta \ll \eta^{-1}$.

□

12. Proof of Lemma 12.1

Lemma 12.1. *For each number* $0 < \eta < \frac{998}{1000}$ *we have*

$$||\Delta^A F_{N,m,\eta,T_0}||_2 \ll_{N,m} \eta^{-(4A+2)}, \quad A = 1, 2, \ldots.$$

Proof. Fix $Y > 0$ and define

$$P(Y) := \{z = x + iy : \ 0 < x < 1, \ y \geq Y\}$$

and

$$L(Y) := \{z = x + iY : \ 0 < x < 1\}.$$

One can choose a fundamental domain $\mathscr{D}$ for $\Gamma_0(N)$ such that

$$\mathscr{D} = \mathscr{D}(Y) \cup \bigcup_{\mathfrak{a}} \mathscr{D}_{\mathfrak{a}}(Y)$$

where $\mathscr{D}_{\mathfrak{a}}(Y) := \sigma_{\mathfrak{a}} P(Y)$ and

$$\mathscr{D}(Y) := \mathscr{D} \backslash \bigcup_{\mathfrak{a}} \mathscr{D}_{\mathfrak{a}}(Y)$$

has compact closure and is adjacent to each cuspidal zone $\mathscr{D}_{\mathfrak{a}}(Y)$ along the horocycle $\sigma_{\mathfrak{a}} L(Y)$ (see [I, Section 2.2]). Using the $\mathrm{SL}_2(\mathbb{R})$-invariance of the measure $d\mu(z)$ we find that

$$||\Delta^A F_{N,m,\eta,T_0}||_2^2 := \int_{\mathscr{D}} |\Delta^A F_{N,m,\eta,T_0}(z)|^2 d\mu(z) \leq \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} := \int_{\mathscr{D}(\frac{4N}{\pi})} |\Delta^A F_{N,m,\eta,T_0}(z)|^2 d\mu(z)$$

and

$$\mathrm{II} := \sum_{\mathfrak{b}} \int_{P(\frac{4N}{\pi})} |\Delta^A F_{N,m,\eta,T_0}(\sigma_{\mathfrak{b}} z)|^2 d\mu(z).$$

First we estimate II. By (7.1), (8.3) and (9.1) we have (recall that $\eta < \frac{998}{1000}$)

$$F_{N,m,\eta,T_0}(\sigma_{\mathfrak{b}} z) = \begin{cases} F_{N,m}(\sigma_{\mathfrak{b}} z) - \delta_{\infty,\mathfrak{b}} \dfrac{m}{2} \psi_{m,\eta}(y) e(-mz), & \frac{4N}{\pi} \leq y < \frac{4N}{\pi} + \eta \\ C_{\mathfrak{b}}(m) y^{-1} + E_{\mathfrak{b}}(m,x,y), & \frac{4N}{\pi} + \eta \leq y < \frac{4N}{\pi} + \frac{999}{1000} \\ C_{\mathfrak{b}}(m) y^{-1}(1 - \chi(y/2)) + E_{\mathfrak{b}}(m,x,y), & \frac{4N}{\pi} + \frac{999}{1000} \leq y < 2(\frac{4N}{\pi} + \frac{999}{1000}) \\ E_{\mathfrak{b}}(m,x,y), & y \geq 2(\frac{4N}{\pi} + \frac{999}{1000}). \end{cases} \tag{12.1}$$

By splitting the y-integral in II into the different ranges considered in (12.1), we obtain

$$\mathrm{II} = \sum_{\mathfrak{b}} \int_{\frac{4N}{\pi}}^{\infty} \int_0^1 |\Delta^A F_{N,m,\eta,T_0}(\sigma_{\mathfrak{b}} z)|^2 d\mu(z) = \mathrm{III} + O(1),$$

where

$$\mathrm{III} := \sum_{\mathfrak{b}} \int_{\frac{4N}{\pi}}^{\frac{4N}{\pi}+\eta} \int_0^1 |\Delta^A (F_{N,m}(\sigma_{\mathfrak{b}} z) - \delta_{\infty,\mathfrak{b}} \frac{m}{2} \psi_{m,\eta}(y) e(-mz))|^2 d\mu(z).$$

By linearity of Δ and the triangle inequality,

$$\mathrm{III} \le \mathrm{IV} + \mathrm{V} + O(1),$$

where

$$\mathrm{IV} := 2\sum_{\mathfrak{b}} \int_{\frac{4N}{\pi}}^{\frac{4N}{\pi}+\frac{998}{1000}} \int_0^1 |\Delta^A F_{N,m}(\sigma_{\mathfrak{b}} z)| \cdot |\delta_{\infty,\mathfrak{b}} \frac{m}{2} \Delta^A(\psi_{m,\eta}(y)e(-mz))| d\mu(z)$$

and

$$\mathrm{V} := \sum_{\mathfrak{b}} \int_{\frac{4N}{\pi}}^{\frac{4N}{\pi}+\frac{998}{1000}} \int_0^1 |\delta_{\infty,\mathfrak{b}} \frac{m}{2} \Delta^A(\psi_{m,\eta}(y)e(-mz))|^2 d\mu(z).$$

Using the estimate

$$\max_{(x,y)\in[0,1]\times[\frac{4N}{\pi},\frac{4N}{\pi}+\frac{998}{1000}]} |\Delta^A(\psi_{m,\eta}(y)e(-mz))| \ll \eta^{-A},$$

we obtain

$$\mathrm{IV} \ll \eta^{-A} \qquad \text{and} \qquad \mathrm{V} \ll \eta^{-2A}.$$

We conclude that

$$\mathrm{II} \ll \eta^{-2A}.$$

Next we estimate I. Observe that $\mathscr{D}(4N/\pi)$ can be contained in a rectangle

$$R_N := [-B_N, B_N] \times [C_N, \frac{4N}{\pi}]$$

for some $B_N \ge 1$ and $0 < C_N \le \sqrt{3}/2$. Since $\psi_{m,\eta}(t) = 0$ for $t \le 4N/\pi$, by (8.1) we have

$$F_{N,m,\eta,T_0}(z) = (F_{N,m}(z) - f_{m,\eta}(x,y)) - \eta_{T_0}(z),$$

where

$$f_{m,\eta}(x,y) := \sum_{n\in\mathbb{Z}} e(nx) \sum_{1\le c\le C_N^{-1}} S(-m,n;c) A_{m,\eta}(n,c,y).$$

By linearity of Δ and three applications of the triangle inequality, we have

$$\mathrm{I} \le \int_{R_N} |\Delta^A((F_{N,m}(z) - f_{m,\eta}(x,y)) - \eta_{T_0}(z))|^2 d\mu(z) \le \mathrm{VI} + \mathrm{VII} + O(1),$$

where

$$\mathrm{VI} := 2\int_{R_N} \left(|\Delta^A F_{N,m}(z)| + |\Delta^A \eta_{T_0}(z)|\right) |\Delta^A f_{m,\eta}(x,y)| d\mu(z)$$

and

$$\mathrm{VII} := \int_{R_N} |\Delta^A f_{m,\eta}(x,y)|^2 d\mu(z).$$

Using the estimate

$$\max_{(x,y)\in R_N} |\Delta^A f_{m,\eta}(x,y)| \ll \eta^{-(4A+2)}, \tag{12.2}$$

we have

$$\mathrm{VI} \ll \eta^{-(4A+2)} \qquad \text{and} \qquad \mathrm{VII} \ll \eta^{-2(4A+2)}.$$

We conclude that

$$\mathrm{I} \ll \eta^{-2(4A+2)}.$$

It remains to establish the estimate (12.2). Define

$$\Phi_{m,\eta,c,y}(u) := \psi_{m,\eta}\left(\frac{y}{c^2(u^2+y^2)}\right) e\left(\frac{m}{c^2(u+iy)}\right),$$

so that

$$A_{m,\eta}(n,c,y) = \int_{\mathbb{R}} \Phi_{m,\eta,c,y}(u) e(-nu) du.$$

Since

$$\Phi_{m,\eta,c,y}(u) = 0 \quad \text{for} \quad |u| \geq \sqrt{1-C_N^2},$$

integrating by parts $(2A+2)$-times yields

$$A_{m,\eta}(n,c,y) = \frac{1}{(2\pi i n)^{2A+2}} \int_{-\sqrt{1-C_N^2}}^{\sqrt{1-C_N^2}} \Phi^{(2A+2)}_{m,\eta,c,y}(u) e(-nu) du.$$

In particular, we have

$$\Delta^A f_{m,\eta}(x,y) = \sum_{n\in\mathbb{Z}} \sum_{1\leq c\leq C_N^{-1}} S(-m,n;c) \Delta^A I_{m,\eta,n,c}(x,y),$$

where

$$I_{m,\eta,n,c}(x,y) := \frac{e(nx)}{(2\pi i n)^{2A+2}} \int_{-\sqrt{1-C_N^2}}^{\sqrt{1-C_N^2}} \Phi^{(2A+2)}_{m,\eta,c,y}(u) e(-nu) du.$$

For clarity, we first assume that $A = 1$. Then

$$\Delta I_{m,\eta,n,c}(x,y) =$$
$$- y^2 \frac{e(nx)}{(2\pi i n)^2} \int_{-\sqrt{1-C_N^2}}^{\sqrt{1-C_N^2}} \Phi^{(4)}_{m,\eta,c,y}(u) e(-nu) du - y^2 \frac{e(nx)}{(2\pi i n)^4} \int_{-\sqrt{1-C_N^2}}^{\sqrt{1-C_N^2}} \partial_y^2 \Phi^{(4)}_{m,\eta,c,y}(u) e(-nu) du.$$

Using the estimate ([I, (2.37)])

$$S(-m,n;c) \ll c^2,$$

and the estimates

$$\max_{u\in[-\sqrt{1-C_N^2},\sqrt{1-C_N^2}]} |\Phi^{(4)}_{m,\eta,c,y}(u)| \ll_{m,c,y} \eta^{-4}$$

and

$$\max_{u\in[-\sqrt{1-C_N^2},\sqrt{1-C_N^2}]} |\partial_y^2 \Phi^{(4)}_{m,\eta,c,y}(u)| \ll_{m,c,y} \eta^{-6},$$

we obtain

$$\max_{(x,y)\in R_N} |\Delta f_{m,\eta}(x,y)| \ll \eta^{-6}.$$

The preceding argument generalizes in a straightforward way to $A \geq 1$, and the estimate (12.2) follows.

□

13. Regularized integrals

First we recall the notion of a regularized integral in the sense of Borcherds [Bo] and Harvey-Moore [HMo]. Let $\mathcal{F}$ be the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. Then a fundamental domain for $\Gamma_0(N)$ is given by

$$\mathcal{F}(N) := \bigcup_{\sigma\in\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} \sigma\mathcal{F}.$$

For a fixed $Y > 1$, define the truncated domains

$$\mathcal{F}_Y := \{z \in \mathcal{F} : \ \mathrm{Im}(z) \le Y\}$$

and

$$\mathcal{F}_Y(N) := \bigcup_{\sigma\in\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} \sigma\mathcal{F}_Y.$$

We then define the regularized integral

$$\int_{\mathrm{reg}} F_{N,m}(z)d\mu(z) := \lim_{Y\to\infty}\int_{\mathcal{F}_Y(N)} F_{N,m}(z)d\mu(z).$$

For each $\eta > 0$ define the function

$$\psi_{m,\eta,Y}(t) := \begin{cases}\psi_{m,\eta}(t), & t \le Y\\ 0, & t > Y,\end{cases}$$

and the associated Poincaré series

$$\mathscr{P}_{m,\eta,Y}(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \psi_{m,\eta,Y}(\mathrm{Im}(\gamma z))e(-m\gamma z).$$

Lemma 13.1. *For $z \in \mathcal{F}_Y(N)$ we have $\mathscr{P}_{m,\eta,Y}(z) = \mathscr{P}_{m,\eta}(z)$.*

Proof. By definition of $\psi_{m,\eta,Y}$ we have

$$\mathscr{P}_{m,\eta}(z) = \mathscr{P}_{m,\eta,Y}(z) + \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)\\ \mathrm{Im}(\gamma z)>Y}} \psi_{m,\eta}(\mathrm{Im}(\gamma z))e(-m\gamma z).$$

Let $\gamma \in \Gamma_\infty\backslash\Gamma_0(N)$ and $z \in \mathcal{F}_Y(N)$. Then $\gamma z = Az'$ for some $A \in \mathrm{SL}_2(\mathbb{Z})$ and $z' \in \mathcal{F}_Y$, thus $\mathrm{Im}(\gamma z) = \mathrm{Im}(Az') < \mathrm{Im}(z') < Y$. It follows that

$$\#\{\gamma \in \Gamma_\infty\backslash\Gamma_0(N) : \ \mathrm{Im}(\gamma z) > Y\} = 0,$$

hence $\mathscr{P}_{m,\eta,Y}(z) = \mathscr{P}_{m,\eta}(z)$. □

Lemma 13.2. *We have*

$$\int_{\mathcal{F}(N)} F_{N,m,\eta}(z)d\mu(z) = \lim_{Y\to\infty}\int_{\mathcal{F}_Y(N)} F_{N,m}(z)d\mu(z).$$

Proof. Since $F_{N,m,\eta} := F_{N,m} - \mathscr{P}_{m,\eta} \in L^1(\mathcal{F}(N))$,

$$\int_{\mathcal{F}(N)} F_{N,m,\eta}(z)d\mu(z) = \lim_{Y\to\infty}\int_{\mathcal{F}_Y(N)} \left(F_{N,m}(z) - \mathscr{P}_{m,\eta}(z)\right) d\mu(z).$$

Now, by Lemma 13.1 we have

$$\int_{\mathcal{F}_Y(N)} \mathscr{P}_{m,\eta}(z)d\mu(z) = \int_{\mathcal{F}_Y(N)} \mathscr{P}_{m,\eta,Y}(z)d\mu(z).$$

We claim that if $z \in \mathcal{F}(N) \setminus \mathcal{F}_Y(N)$, then $\mathscr{P}_{m,\eta,Y}(z) = 0$. Let $\gamma \in \Gamma_\infty \backslash \Gamma_0(N)$ and $z \in \mathcal{F}(N) \setminus \mathcal{F}_Y(N)$. Then $\gamma z = Az'$ for some $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z' = x' + iy' \in \mathcal{F} \setminus \mathcal{F}_Y$, i.e., $z' \in \mathcal{F}$ with $y' > Y$. We have

$$\mathrm{Im}(\gamma z) = \mathrm{Im}(Az') = \frac{y'}{|cz' + d|^2}.$$

If $c = 0$ then $d = 1$, so that $\mathrm{Im}(\gamma z) = y' > Y$. Since $\psi_{m,\eta,Y}(t) = 0$ for $t > Y$, it follows that $\mathscr{P}_{m,\eta,Y}(z) = 0$. On the other hand, if $c \neq 0$ then $c^2 \geq 1$, so that

$$\mathrm{Im}(\gamma z) = \frac{y'}{(cx' + d)^2 + c^2(y')^2} \leq \frac{1}{y'} < \frac{1}{Y} < 1$$

(recall $Y > 1$). Since $\psi_{m,\eta,Y}(t) = \psi_{m,\eta}(t) = 0$ for $t < 1$, it follows that $\mathscr{P}_{m,\eta,Y}(z) = 0$, which completes the proof of the claim. By the claim we have

$$\int_{\mathcal{F}_Y(N)} \mathscr{P}_{m,\eta,Y}(z)d\mu(z) = \int_{\mathcal{F}(N)} \mathscr{P}_{m,\eta,Y}(z)d\mu(z).$$

Moreover, unfolding yields

$$\int_{\mathcal{F}(N)} \mathscr{P}_{m,\eta,Y}(z)d\mu(z) = 0.$$

Thus we conclude that

$$\int_{\mathcal{F}(N)} F_{N,m,\eta}(z)d\mu(z) = \lim_{Y \to \infty} \int_{\mathcal{F}_Y(N)} F_{N,m}(z)d\mu(z).$$

□

Remark 13.3. If $F_{N,m} = R_{-2}f$ for a harmonic weak Maass form $f \in H_{-2}(N)$, then

$$\int_{\mathrm{reg}} F_{N,m}(z)d\mu(z) = \sum_{\mathfrak{a}} \alpha_{\mathfrak{a}} c_{f,\mathfrak{a}}^{+}(0),$$

where $\alpha_{\mathfrak{a}}$ is the width of the cusp $\mathfrak{a}$ and $c_{f,\mathfrak{a}}^{+}(0)$ is the constant term of the holomorphic part of the Fourier expansion of f in the cusp $\mathfrak{a}$ of $\Gamma_0(N)$.[2] In particular, for the constants $C_{f,N}$ and $C_{f_p,6}^{*}$ in Theorems 1.2 and 1.6, respectively, we have

$$C_{f,N} = -\frac{N}{\pi} \sum_{\mathfrak{a}} \alpha_{\mathfrak{a}} c_{f,\mathfrak{a}}(0)$$

and

$$C_{f_p,6}^{*} = \frac{1}{4\pi} \sum_{\mathfrak{a}} \alpha_{\mathfrak{a}} c_{f_p,\mathfrak{a}}(0),$$

where the last sum is over the 4 cusps of $\Gamma_0(6)$.

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References

[BH] V. Blomer and G. Harcos, *Hybrid bounds for twisted L-functions.* J. Reine Angew. Math. **621** (2008), 53–79.

[Bo] R. Borcherds, *Automorphic forms with singularities on Grassmannians*. Inv. Math. **132** (1998), 491–562.

[BrO] K. Bringmann and K. Ono, *An arithmetic formula for the partition function.* Proc. Amer. Math. Soc. **135** (2007), 3507–3514.

[B] J. H. Bruinier, *Borcherds products on O(2, l) and Chern classes of Heegner divisors.* Lecture Notes in Mathematics, 1780. Springer-Verlag, Berlin, 2002. viii+152 pp.

[B2] J. H. Bruinier, *Harmonic Maass forms and periods*. Math. Ann. **357** (2013), 1363–1387.

[BF] J. H. Bruinier and J. Funke, *On two geometric theta lifts*. Duke Math. Journal **125** (2004), 45–90.

[BO] J. H. Bruinier and K. Ono, *Heegner divisors, L-functions and harmonic weak Maass forms.* Ann. of Math. **172** (2010), 2135–2181.

[BO2] J. H. Bruinier and K. Ono, *Algebraic formulas for the coefficients of half-integral weight harmonic weak Maass forms*. Adv. Math. **246** (2013), 198–219.

[D] W. Duke, *Modular functions and the uniform distribution of CM points.* Math. Ann. **334** (2006), 241–252.

[FM] A. Folsom and R. Masri, *Equidistribution of Heegner points and the partition function*. Math. Ann. **348** (2010), 289–317.

[FM2] A. Folsom and R. Masri, *The asymptotic distribution of traces of Maass-Poincaré series*. Adv. Math. **226** (2011), 3724–3759.

[GKZ] B. Gross, W. Kohnen, and D. Zagier, *Heegner points and derivatives of L–series. II.* Math. Ann. **278** (1987), 497–562.

[GZ] B. Gross and D. Zagier, *Heegner points and derivatives of L–series*. Invent. Math. **84** (1986), 225–320.

[HM] G. Harcos and P. Michel, *The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points. II.* Invent. Math. **163** (2006), 581–655.

[HR] G. H. Hardy and S. Ramanujan, *Une formule asymptotique pour le nombre des partitions de n.* Comptes Rendus **2**, (1917), found in *Collected papers of Srinivasa Ramanujan,* AMS Chelsea Publ., Providence, RI, (2000), 239–241.

[HMo] J. Harvey and G. Moore, *Algebras, BPS states, and strings*. Nuclear Phys. B **463** (1996), no. 2-3, 315–368.

[HB] D. R. Heath-Brown, *Hybrid bounds for Dirichlet L-functions. II.* Quart. J. Math. Oxford Ser. **31** (1980), 157–167.

[HL] J. Hoffstein and P. Lockhart, *Coefficients of Maass forms and the Siegel zero. With an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman.* Ann. of Math. **140** (1994), 161–181.

[I] H. Iwaniec, *Introduction to the spectral theory of automorphic forms*. Biblioteca de la Revista Matemática Iberoamericana. Revista Matematica Iberoamericana, Madrid, 1995. xiv+247 pp.

[IK] H. Iwaniec and E. Kowalski, *Analytic number theory.* American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004. xii+615 pp.

[JM] M. Jutila and Y. Motohashi, *Uniform bound for Hecke L-functions*. Acta Math. **195** (2005), 61–115.

[KM] S. Kudla and J. Millson, *The theta correspondence and harmonic forms I.* Math. Ann. **274** (1986), 353–378.

[KMY] B. D. Kim, R. Masri, and T. H. Yang, *Nonvanishing of Hecke L–functions and the Bloch-Kato conjecture*. Math. Ann. **349** (2011), 301–343.

[LR] E. Larson and L. Rolen, *Integrality properties of the CM-values of certain weak Maass forms*. Forum Mathematicum, DOI: 10.1515/forum-2012-0112, March 2013.

[L] D. H. Lehmer, *On the remainders and convergence of the series for the partition function*. Trans. Amer. Math. Soc. **46** (1939), 362–373.

[MP] A. Miller and A. Pixton, *Arithmetic traces of non-holomorphic modular invariants.* Int. J. Number Theory **6** (2010), 69–87.

[O] K. Ono, *Unearthing the visions of a master: harmonic Maass forms and number theory.* Current developments in mathematics, 2008, 347–454, Int. Press, Somerville, MA, 2009.

[R] H. Rademacher, *On the expansion of the partition function in a series*. Ann. of Math. **44** (1943), 416–422.

[R2] H. Rademacher, *Topics in analytic number theory.* Edited by E. Grosswald, J. Lehner and M. Newman. Die Grundlehren der mathematischen Wissenschaften, Band 169. Springer-Verlag, New York-Heidelberg, 1973. ix+320 pp.

[W] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*. J. Math. Pures Appl. **60** (1981), 375–484.

[W2] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*. Compos. Math. **54** (1985), 173–242.

[Za] D. Zagier, *Ramanujan's mock theta functions and their applications* (after Zwegers and Bringmann-Ono). Séminaire Bourbaki. Vol. 2007/2008. Astérisque No. 326 (2009), Exp. No. 986, viiviii, 143–164 (2010).

[Z] S. Zhang, *Gross–Zagier formula for GL_2*. Asian J. Math. **5** (2001), 183–290.

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