KRONECKER’S SOLUTION OF PELL’S EQUATION FOR CM FIELDS

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ABSTRACT. We generalize Kronecker’s solution of Pell’s equation to CM fields $K$ whose Galois group over $\mathbb{Q}$ is an elementary abelian 2-group. This is an identity which relates CM values of a certain Hilbert modular function to products of logarithms of fundamental units. When $K$ is imaginary quadratic, these CM values are algebraic numbers related to elliptic units in the Hilbert class field of $K$. Assuming Schanuel’s conjecture, we show that when $K$ has degree greater than 2 over $\mathbb{Q}$ these CM values are transcendental.

1. Introduction and statement of results

The analytic construction of solutions of certain natural Diophantine equations is a problem of central importance in number theory. One of the most remarkable examples of this is Kronecker’s “solution” of Pell’s equation

$$x^2 - dy^2 = \pm 1. \tag{1.1}$$

The fundamental unit $\varepsilon_d$ in the real quadratic field $\mathbb{Q}(\sqrt{d})$ satisfies (1.1). Kronecker expressed $\varepsilon_d$ in terms of values of the Dedekind eta function $\eta(z)$ at CM points on the modular curve $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ (see the discussion below, and in particular, equation (1.5)).

In this paper we will generalize Kronecker’s solution of Pell’s equation to CM fields $K$ whose Galois group over $\mathbb{Q}$ is an elementary abelian 2-group (see Theorem 1.3). This is an identity which relates values of a certain Hilbert modular function at CM points on a Hilbert modular variety to products of logarithms of fundamental units. When $K$ is imaginary quadratic, these CM values are algebraic numbers which can be expressed as absolute values of Galois conjugates of elliptic units in the Hilbert class field of $K$ (see [S, p. 103]). In contrast, when $K$ has degree greater than 2 over $\mathbb{Q}$ we will show, assuming Schanuel’s conjecture, that these CM values are transcendental (see Theorem 1.6). This result is related to interesting recent work of Murty and Murty [MM1, MM2] on transcendental values of class group $L$–functions for imaginary quadratic fields.

We begin by reviewing Kronecker’s solution of Pell’s equation. For a quadratic field $\mathbb{Q}(\sqrt{\Delta})$ of discriminant $\Delta$, let $\chi_{\Delta}$ be the Kronecker symbol, $L(\chi_{\Delta}, s)$ be the Dirichlet $L$–function, $h(\Delta)$ be the class number, $\varepsilon_{\Delta}$ be the fundamental unit, and $w_{\Delta}$ be the number of roots of unity. Let $K = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field of discriminant $D < -4$ (so $w_D = 2$). For an ideal class $C$ of $K$, let $\tau_a \in \mathbb{H}$ be the CM point of discriminant $D$ on the modular curve $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ corresponding to $[a] = C^{-1}$ (here $\mathbb{H}$ is the complex upper half-plane). More precisely, if $Q(X,Y) = N(a)X^2 + bXY + cY^2$ is the reduced, primitive, integral binary quadratic form of discriminant $b^2 - 4N(a)c = D$ corresponding to the class.
\[ C^{-1}, \text{ then} \]
\[ \tau_a = \frac{-b + \sqrt{D}}{2N(a)} \]
is the unique root in \( \mathbb{H} \) of the dehomogenized form \( Q(X, 1) \) (here \( N(a) \) is the norm of \( a \)).

Kronecker established the following “limit formula” for the constant term in the Laurent expansion of the partial Dedekind zeta function \( \zeta_K(s, C) \) at \( s = 1 \),
\[
\lim_{s \to 1} \left[ \zeta_K(s, C) - \frac{\pi}{\sqrt{|D|}} \frac{1}{s - 1} \right] = \frac{\pi}{\sqrt{|D|}} (2\gamma - \log |D| - 2\log g(\tau_a)) , \quad (1.2)
\]
where \( \gamma \) is Euler’s constant and \( g : \mathbb{H} \to \mathbb{R}^+ \) is the \( \text{SL}_2(\mathbb{Z}) \)-invariant function
\[
g(z) := \sqrt{\frac{2}{\sqrt{|D|}}} \text{Im}(z) |\eta(z)|^2 ,
\]
where
\[
\eta(z) = e(z/24) \prod_{n=1}^{\infty} (1 - e(nz)) , \quad e(z) := e^{2\pi iz}
\]
is Dedekind’s weight 1/2 modular form for \( \text{SL}_2(\mathbb{Z}) \).

Let \( D = D_1D_2 \) be a nontrivial factorization of \( D \) into coprime fundamental discriminants \( D_1 > 0 \) and \( D_2 < 0 \). Let \( \chi \) be the genus character of \( K \) corresponding to the decomposition \( D = D_1D_2 \) and let
\[
L_K(\chi, s) = \sum_{C \in \text{CL}(K)} \chi(C) \zeta_K(s, C)
\]
be the \( L \)-function of \( \chi \) where \( \text{CL}(K) \) is the ideal class group of \( K \). Kronecker established the factorization
\[
L_K(\chi, s) = L(\chi_{D_1}, s)L(\chi_{D_2}, s) . \quad (1.3)
\]

By orthogonality of group characters, one obtains from (1.2) the formula
\[
L_K(\chi, 1) = -\frac{2\pi}{\sqrt{|D|}} \sum_{C \in \text{CL}(K)} \chi(C) \log F(\tau_a) .
\]
On the other hand, by Dirichlet’s class number formula for quadratic fields one has
\[
L(\chi_{\Delta}, 1) = \begin{cases} 
2\log(\varepsilon_{\Delta})h(\Delta), & \text{if } \Delta > 0, \\
2\pi h(\sqrt{\Delta}) & \text{if } \Delta < 0.
\end{cases} \quad (1.4)
\]

Equating both sides of Kronecker’s factorization (1.3) at \( s = 1 \) yields the beautiful identity
\[
- \sum_{C \in \text{CL}(K)} \chi(C) \log F(\tau_a) = \frac{2h(D_1)h(D_2)}{w_{D_2}} \log(\varepsilon_{D_1}) ,
\]
or equivalently
\[
\prod_{C \in \text{CL}(K)} F(\tau_a)^{-\chi(C)} = \varepsilon_{D_1}^{2h(D_1)h(D_2)/w_{D_2}} . \quad (1.5)
\]
The fundamental unit $\varepsilon_{D_1}$ satisfies Pell’s equation
\[ x^2 - D_1 y^2 = \pm 1, \]
thus one has a “solution” of this equation in terms of the CM values $F(\tau_a)$.

Recall that a CM field is a totally imaginary quadratic extension of a totally real number field. In order to generalize Kronecker’s identity (1.5) to CM fields we proceed as follows. First, we evaluate the special value $L_K (\chi, 1)$ where $\chi$ is a nontrivial class group character of a CM field $K$ (see Theorem 1.1). To do this we establish a suitable version of the Kronecker limit formula for CM fields, which relates the constant term in the Laurent expansion at $s = 1$ of $\zeta_K (s, C)$ to values of a Hilbert modular function at CM points on a Hilbert modular variety (see Theorem 4.1). Second, we identify the CM fields which possess a genus character $\chi$ whose $L$–function $L_K (\chi, s)$ factors as a product of quadratic Dirichlet $L$–functions. These are the CM fields whose Galois group over $\mathbb{Q}$ is an elementary abelian 2-group. Given such a factorization, we can evaluate $L_K (\chi, 1)$ using Dirichlet’s class number formula for quadratic fields. By equating the two different evaluations of $L_K (\chi, 1)$ we will generalize (1.5).

Note that a limit formula for CM fields was established by Konno in [K]. See also the work of Asai [A], who calculated the constant term in the Laurent expansion at $s = 1$ of the real-analytic Eisenstein series associated to any number field of class number 1. Our approach to the limit formula for CM fields differs from [K]. In particular, we proceed via the Fourier expansion of the Hilbert modular Eisenstein series, which enables us to use periods of this Eisenstein series to explicitly determine the CM zero-cycles along which we evaluate the Hilbert modular function.

In order to state our results we fix the following notation. Let $F$ be a totally real number field of degree $n$ over $\mathbb{Q}$ with embeddings $\sigma_1, \ldots, \sigma_n$ and ring of integers $\mathcal{O}_F$. Let $K$ be a CM extension of $F$ with a CM type $\Phi$, and let
\[ \mathcal{CM}(K, \Phi, \mathcal{O}_F) = \{ z_a \in \mathbb{H}^n : [a] \in \text{CL}(K) \} \]
be the zero-cycle of CM points on the Hilbert modular variety $X_F = \text{SL}_2(\mathcal{O}_F) \backslash \mathbb{H}^n$ (see Section 3). Let $R_K$, $w_K$ and $d_K$ be the regulator, number of roots of unity, and absolute discriminant of $K$, respectively.

In the following theorem we give a formula for the special value $L_K (\chi, 1)$.

**Theorem 1.1.** Let $F$ be a totally real number field of degree $n$ over $\mathbb{Q}$ with narrow class number 1. Let $K$ be a CM extension of $F$ with a CM type $\Phi$. For each class $C \in \text{CL}(K)$, let $z_a$ be the CM point in $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$ corresponding to $C^{-1}$. Then for each nontrivial class group character $\chi$ of $K$,
\[ L_K (\chi, 1) = -\frac{2^{n+1} \pi^n R_K}{w_K \sqrt{d_K}} \sum_{C \in \text{CL}(K)} \chi(C) \log G(z_a), \]
where $G : \mathbb{H}^n \to \mathbb{R}^+$ is the $\text{SL}_2(\mathcal{O}_F)$-invariant function
\[ G(z) := \sqrt{\left( \frac{2^n d_F}{\sqrt{d_K}} \right) \prod_{i=1}^n \text{Im}(z_i) \cdot \phi(z)^2}, \quad z = (z_1, \ldots, z_n) \in \mathbb{H}^n \]
and $\phi(z)$ is the positive, real-analytic function generalizing $|\eta(z)|$ defined by (1.6).
Remark 1.2. The narrow class number 1 assumption in Theorem 1.1 can be removed by working adelically. We have worked classically throughout the paper to emphasize the parallels with Kronecker’s original work.

The function $\phi(z)$ in Theorem 1.1 is defined by

$$\phi(z) := f(z)^{-\sqrt{d_F}/2\pi n r_F},$$

where $r_F$ is the residue of $\zeta_F(2s - 1)$ at $s = 1/2$ and

$$f(z) := \exp \left( \zeta_F(2) \prod_{i=1}^{n} y_i + \frac{\pi^n}{\sqrt{d_F}} \sum_{\hat{a} = ab} \sum_{\hat{b} \in \mathcal{O}_F^\times} \frac{e^{-2\pi i S(ab)}}{N_{F/Q}((b))} e^{2\pi i T(abx)} \right),$$

where $z = x + iy \in \mathbb{H}^n$, $\mathcal{O}_F^\times$ is the dual lattice, $\mathcal{O}_F^\times$ is the unit group,

$$S(ab) = \sum_{i=1}^{n} |\sigma_i(ab)| y_i,$$

$$T(abx) = \sum_{i=1}^{n} \sigma_i(ab) x_i,$$

and the prime means the sum is over nonzero elements. In Proposition 4.3 we will show that $\phi(z)$ transforms like

$$\phi(M z) = \left| \prod_{i=1}^{n} (\sigma_i(\gamma) z_i + \sigma_i(\delta)) \right|^{i \frac{2}{2}} \phi(z)$$

for $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathcal{O}_F)$.

Our main result is the following theorem generalizing Kronecker’s identity (1.5).

**Theorem 1.3.** Let $F$ be a totally real number field with narrow class number 1. Let $K$ be a CM extension of $F$ with $\text{Gal}(K/Q) \cong (\mathbb{Z}/2\mathbb{Z})^r$ for some integer $r \geq 2$, and let $E$ be an unramified quadratic extension of $K$ with $\text{Gal}(E/Q) \cong (\mathbb{Z}/2\mathbb{Z})^{r+1}$. Let $\chi$ be the genus character of $K$ arising from the extension $E/K$. Let $\Delta_i$ for $1 \leq i \leq 2^r$ be the discriminants of the quadratic subfields $\mathbb{Q}(\sqrt{\Delta_i})$ of $E$ which are not contained in $K$ and define $S_R := \{ \Delta_i : \Delta_i > 0 \}$ and $S_I := \{ \Delta_i : \Delta_i < 0 \}$. Then

$$\prod_{C \in \text{Cl}(K)} G(z_a)^{-\chi(C)} = \exp \left( \frac{\alpha}{\prod_{i=1}^{2^r} \sqrt{\Delta_i}} \frac{\sqrt{d_K}}{R_F} \prod_{\Delta_i \in S_R} \log(\varepsilon_{\Delta_i}) \right),$$

where

$$\alpha := \frac{w_K \prod_{i=1}^{2^r} h(\Delta_i)}{\prod_{\Delta_i \in S_I} w_{\Delta_i}} \in \mathbb{Q}.$$

In the following theorem we give an explicit example of Theorem 1.3 for CM biquadratic fields.
Theorem 1.4. Let $F = \mathbb{Q}(\sqrt{p})$ where $p \equiv 1 \pmod{4}$ is a prime such that $F$ has narrow class number 1. Let $D = D_1 D_2 < 0$ be a composite fundamental discriminant with $D_1 > 0$ and $D_2 < 0$ fundamental discriminants. Let $K = \mathbb{Q}(\sqrt{D}, \sqrt{pD})$ and $E = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}, \sqrt{p})$. Let $\chi$ be the genus character of $K$ arising from the extension $E/K$. Then

$$\prod_{C \in \text{CL}(K)} G(z_a)^{-\chi(C)} = \exp \left( \beta \sqrt{d_K} \frac{\log(\varepsilon_{D_1}) \log(\varepsilon_{pD_1}) - \log(\varepsilon_p)}{\log(\varepsilon_{D_1})} \right),$$

where

$$\beta := \frac{w_K h(D_1) h(D_2) h(pD_1) h(pD_2)}{pD_1 D_2 w_{D_2} w_{pD_2}} \in \mathbb{Q}.$$

Kronecker’s identity (1.5) implies that the product of CM values

$$\prod_{C \in \text{CL}(K)} g(\tau_a)^{-\chi(C)}$$

is an algebraic number. This product is also related to elliptic units in the Hilbert class field $H$ of $K = \mathbb{Q}(\sqrt{D})$. Namely, using quotients of powers of $\eta(\tau_a)$ and the theory of complex multiplication, one can construct a sequence $\zeta_\ell$, $\ell = 1, \ldots, h(D) - 1$, of independent units in $H$ (see [S, p. 103]). If $\sigma_k$ is the automorphism of $H/K$ corresponding to the ideal class $C_k$ under the isomorphism

$$\text{Gal}(H/K) \rightarrow \text{CL}(K),$$

one can show that

$$\frac{g(\tau_{ak})}{g(\tau_{ak}^{-1})} = |\zeta_\ell^{(k)}|^{1/12h(D)}, \quad k, \ell = 1, \ldots, h(D) - 1,$$

where $\zeta_\ell^{(k)} := \sigma_k(\zeta_\ell)$. In particular, the quotients $g(\tau_{ak})/g(\tau_{ak}^{-1})$ are algebraic.

More generally, let $H_K$ be the Hilbert class field of a CM field $K$ as in Theorem 1.1 and let $h_K$ be the class number of $K$. In light of the preceding facts, it is natural to ask whether the products of CM values

$$\prod_{C \in \text{CL}(K)} G(z_a)^{-\chi(C)}$$

are algebraic, and if so, whether they are related to analogs of elliptic units in $H_K$. We will show, assuming Schanuel’s conjecture, that these products are transcendental.

Recall the following well-known conjecture of Schanuel from transcendental number theory (see e.g. [Wa, Conjecture 1.14]).

Conjecture 1.5 (Schanuel). Given complex numbers $x_1, \ldots, x_n$ that are linearly independent over $\mathbb{Q}$, the field

$$\overline{\mathbb{Q}}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n))$$

has transcendence degree at least $n$ over $\overline{\mathbb{Q}}$.

We will prove the following theorem.
**Theorem 1.6.** Let notation and assumptions be as in Theorem 1.3. Then assuming Schanuel’s conjecture, the numbers

\[ \prod_{C \in \text{Cl}(K)} G(z_0)^{-\chi(C)} \]

are transcendental.

Theorem 1.6 indicates that one cannot in general expect the quotients

\[ \frac{G(z_{a_k})}{G(z_{a_\ell})}, \quad k, \ell = 1, \ldots, h_K - 1, \]

to be related to analogs of elliptic units in \( H_K \). For example, if we assume in Theorem 1.6 that \( K \) has class number 2, then Schanuel’s conjecture implies that the quotients \( G(z_0)/G(z_{O_K}) \) are transcendental. Note that there are more than 150 CM biquadratic fields with class number 2 (see [BWW]).

**Organization.** The paper is organized as follows. In Section 2 we calculate the Laurent expansion at \( s = 1 \) of the Hilbert modular Eisenstein series. In Section 3 we review some facts regarding CM zero-cycles on Hilbert modular varieties. Finally, in Sections 4, 5, 6, and 7, we prove Theorems 1.1, 1.3, 1.4, and 1.6, respectively.

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2. **Laurent expansion of the Hilbert modular Eisenstein series**

Let \( F \) be a totally real number field with class number 1. Let \( F \) have degree \( n \) over \( \mathbb{Q} \) with embeddings \( \sigma_1, \ldots, \sigma_n \) and let

\[ z = x + iy = (z_1, \ldots, z_n) \in \mathbb{H}^n. \]

Let \( \mathcal{O}_F \) be the ring of integers of \( F \) and \( \text{SL}_2(\mathcal{O}_F) \) be the Hilbert modular group. Then \( \text{SL}_2(\mathcal{O}_F) \) acts componentwise on \( \mathbb{H}^n \) by linear fractional transformations,

\[ Mz = (\sigma_1(M)z_1, \ldots, \sigma_n(M)z_n), \quad M \in \text{SL}_2(\mathcal{O}_F). \]

Let

\[ N(y(z)) = \prod_{j=1}^{n} \text{Im}(z_j) = \prod_{j=1}^{n} y_j \]

denote the product of the imaginary parts of the components of \( z \in \mathbb{H}^n \). Define the real-analytic Hilbert modular Eisenstein series

\[ \mathcal{E}(z, s) := \sum_{M \in \Gamma_\infty \backslash \text{SL}_2(\mathcal{O}_F)} N(y(Mz))^s, \quad z \in \mathbb{H}^n, \quad \text{Re}(s) > 1, \]

where

\[ \Gamma_\infty = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \text{SL}_2(\mathcal{O}_F) \right\}. \]
Furthermore, let

\[ N(a + bz) = \prod_{j=1}^{n} (\sigma_j(a) + \sigma_j(b)z_j) \]

for \((a, b) \in \mathcal{O}_F \times \mathcal{O}_F\) and define the Eisenstein series

\[ E(z, s) := \sum' \frac{N(y(z))^s}{|N(a + bz)|^{2s}}; \quad z \in \mathbb{H}^n, \quad \text{Re}(s) > 1, \]

where the sum is over a complete set of nonzero, nonassociated representatives of \(\mathcal{O}_F \times \mathcal{O}_F\) (recall that \((a, b)\) and \((a', b')\) are associated if there exists a unit \(\epsilon \in \mathcal{O}_F^\times\) such that \((a, b) = (\epsilon a', \epsilon b')\)). The two Eisenstein series are related by

\[ E(z, s) = \zeta_F(2s)E(z, s), \quad (2.1) \]

where \(\zeta_F(s)\) is the Dedekind zeta function of \(F\).

The Eisenstein series \(E(z, s)\) has the Fourier expansion

\[ E(z, s) = N(y(z))^s \zeta_F(2s) + \frac{N(y(z))^{1-s}}{\sqrt{d_F}} \left[ \frac{\sqrt{\pi} \Gamma \left( s - \frac{1}{2} \right)}{\Gamma(s)} \right]^n \zeta_F(2s - 1) \]

\[ + \frac{2^n N(y(z))^{1/2}}{\sqrt{d_F}} \left[ \frac{\pi}{\Gamma(s)} \right]^n \times \sum'_{\bar{a} \in \mathcal{O}_F^*} \sum_{\bar{b} = ab \in \mathcal{O}_F^*} \left( \frac{N_{F/Q}(a)}{N_{F/Q}(b)} \right)^{s-\frac{1}{2}} e^{2\pi i T(ax)} \prod_{j=1}^{n} K_{s-\frac{1}{2}}(2\pi |\sigma_j(ab)| y_j) \]

\[ =: A(s) + B(s) + C(s), \]

where \(\mathcal{O}_F^*\) is the dual lattice, \(d_F\) is the absolute discriminant, \(T(ax) = \sum_{j=1}^{n} \sigma_j(a)x_j\) is the trace, \(K_s(v)\) is the usual \(K\)-Bessel function of order \(s\), and \(A(s), B(s), C(s)\) are the three functions on the right hand side of (2.2), respectively.

The Fourier expansion provides a meromorphic continuation of \(E(z, s)\) to \(\mathbb{C}\) with a simple pole at \(s = 1\). We now use this to compute the Laurent expansion at \(s = 1\).

The Laurent expansion of \(A(s)\) at \(s = 1\) is

\[ A(s) = N(y(z)) \zeta_F(2) + O(s - 1). \]

Next, observe that

\[ \frac{N(y(z))^{1-s}}{\sqrt{d_F}} = \frac{1}{\sqrt{d_F}} - \frac{\log N(y(z))}{\sqrt{d_F}}(s - 1) + O(s - 1)^2, \]

\[ \left[ \frac{\sqrt{\pi} \Gamma \left( s - \frac{1}{2} \right)}{\Gamma(s)} \right]^n = \pi^n - 2n \pi^n \log(2)(s - 1) + O(s - 1)^2, \]

and

\[ \zeta_F(2s - 1) = \frac{r_F}{2(s - 1)} + A_F + O(s - 1). \]
After a calculation, we find that the Laurent expansion of $B(s)$ at $s = 1$ is

$$B(s) = \frac{\pi^n r_F}{2\sqrt{d_F}} \frac{1}{(s-1)} + \frac{\pi^n}{\sqrt{d_F}} A_F - \frac{\pi^n r_F}{2\sqrt{d_F}} \log N(y(z)) + 2n \log(2) + O(s-1).$$

Using

$$K_{1/2}(v) = \sqrt{\pi/2ve^{-v}}$$

we compute

$$\prod_{j=1}^n K_{1/2}(2\pi |\sigma_j(ab)| y_j) = \frac{N(y(z))^{-1/2}}{2^n} \frac{1}{N_{F/Q}((ab))^{-1/2}e^{-2\pi S(aby)}},$$

where

$$S(aby) = \sum_{j=1}^n |\sigma_j(ab)| y_j.$$

Thus the Laurent expansion of $C(s)$ at $s = 1$ is

$$C(s) = \frac{\pi^n}{\sqrt{d_F}} \sum_{a \in \mathcal{O}_F^+} \sum_{\tilde{a} = ab \in \mathcal{O}_F^\times} e^{-2\pi i S(aby)} \frac{1}{e^{2\pi i T(abx)}} + O(s-1).$$

Putting things together, we find that the Laurent expansion of $E(z,s)$ at $s = 1$ is

$$E(z, s) = \frac{E_{-1}}{s-1} + E_0(z) + O(s-1), \quad (2.3)$$

where the residue

$$E_{-1} = \frac{\pi^n r_F}{2\sqrt{d_F}},$$

and

$$E_0(z) = \frac{\pi^n}{\sqrt{d_F}} A_F - \frac{\pi^n r_F}{2\sqrt{d_F}} 2n \log(2) + \log \left( N(y(z))^{-E_{-1}} f(z) \right), \quad (2.4)$$

where

$$\log f(z) = N(y(z)) \zeta_F(2) + \frac{\pi^n}{\sqrt{d_F}} \sum_{a \in \mathcal{O}_F^+} \sum_{\tilde{a} = ab \in \mathcal{O}_F^\times} e^{-2\pi i S(aby)} \frac{1}{e^{2\pi i T(abx)}}.$$

3. CM zero-cycles on Hilbert modular varieties

In this section we review some facts we will need regarding CM zero-cycles on Hilbert modular varieties following Bruinier and Yang [BY, Section 3]. See also the recent book of Howard and Yang [HY]. Let $F$ be a totally real number field of degree $n$ over $\mathbb{Q}$. For $S \subset F$, let $S^+$ be the subset of $S$ consisting of totally positive elements. For a fractional ideal $f_0$ of $F$, let

$$\Gamma(f_0) = \text{SL}(\mathcal{O}_F \oplus f_0) = \{ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(F) : \alpha, \delta \in \mathcal{O}_F, \beta \in f_0, \gamma \in f_0^{-1} \}.$$
Recall that $\Gamma(f_0)$ acts on $\mathbb{H}^n$ by
$$Mz = (\sigma_1(M)z_1, \ldots, \sigma_n(M)z_n).$$
The quotient space
$$X(f_0) = \Gamma(f_0)\backslash \mathbb{H}^n$$
is the (open) Hilbert modular variety associated to $f_0$. The variety $X(f_0)$ parameterizes isomorphism classes of triples $(A,i,m)$ where $(A,i)$ is an abelian variety with real multiplication $i: \mathcal{O}_F \hookrightarrow \text{End}(A)$ and
$$m: (\mathcal{M}_A, \mathcal{M}_A^+) \to ((\partial_{F}f_0)^{-1}, (\partial_{F}f_0)^{-1, +})$$
is an $\mathcal{O}_F$-isomorphism from $\mathcal{M}_A$ to $(\partial_{F}f_0)^{-1}$ which maps $\mathcal{M}_A^+$ to $(\partial_{F}f_0)^{-1, +}$. Here $\mathcal{M}_A$ is the polarization module of $A$ and $\mathcal{M}_A^+$ is its positive cone.

Let $K$ be a CM extension of $F$ and $\Phi = (\sigma_1, \ldots, \sigma_n)$ be a CM type of $K$. A point $z = (A,i,m) \in X(f_0)$ is a CM point of type $(K,\Phi)$ if one of the following equivalent definitions holds:

1. As a point $z \in \mathbb{H}^n$, there is a point $\tau \in K$ such that
   $$\Phi(\tau) = (\sigma_1(\tau), \ldots, \sigma_n(\tau)) = z$$
   and
   $$\Lambda_\tau = f_0 + \mathcal{O}_F \tau$$
is a fractional ideal of $K$.

2. $(A,i')$ is a CM abelian variety of type $(K,\Phi)$ with complex multiplication $i: \mathcal{O}_K \hookrightarrow \text{End}(A)$ such that $i = i'|_{\mathcal{O}_F}$.

Fix $\varepsilon_0 \in K^\times$ such that $\overline{\varepsilon_0} = -\varepsilon_0$ and $\Phi(\varepsilon_0) = (\sigma_1(\varepsilon_0), \ldots, \sigma_n(\varepsilon_0)) \in \mathbb{H}^n$. Let $\mathfrak{a}$ be a fractional ideal of $K$ and $f_0 = \varepsilon_0 \partial_K/F\mathfrak{a}\mathfrak{a} \cap F$. By [BY, Lemma 3.1], the CM abelian variety $(A_\mathfrak{a} = \mathbb{C}^n/\Phi(\mathfrak{a}), i)$ defines a CM point on $X(f_0)$ if there exists an $r \in F^\times$ such that $f_0 = rf_0$. Thus any pair $(a,r)$ with $\mathfrak{a}$ a fractional ideal of $K$ and $r \in F^\times$ with $f_0 = rf_0$ defines a CM point $(A_{a},i,m) \in X(f_0)$ (we refer the reader to [BY] for a discussion of how the $\mathcal{O}_F$-isomorphism $m$ depends on $r$). Two such pairs $(a_1,r_1)$ and $(a_2,r_2)$ are equivalent if there exists an $\alpha \in K^\times$ such that $a_2 = \alpha a_1$ and $r_2 = r_1\alpha \alpha$. Write $[a,r]$ for the class of $(a,r)$ and identify it with its associated CM point $(A_{a},i,m) \in X(f_0)$.

By [BY, Lemma 3.2], given a CM point $[a,r] \in X(f_0)$ there is a decomposition
$$\mathfrak{a} = \mathcal{O}_F \alpha + f_0 \beta$$
with $z = \alpha/\beta \in K^\times \cap \mathbb{H}^n = \{z \in K^\times: \Phi(z) \in \mathbb{H}^n\}$. Moreover, $z$ represents the CM point $[a,r] \in X(f_0)$.

Let $\mathcal{CM}(K,\Phi,f_0)$ be the set of CM points $[a,r] \in X(f_0)$, which we view as a CM zero-cycle in $X(f_0)$. Let
$$\mathcal{CM}(K,\Phi) = \sum_{[f_0] \in \text{CL}(F)^+} \mathcal{CM}(K,\Phi,f_0),$$
where $\text{CL}(F)^+$ is the narrow ideal class group of $F$. The forgetful map
$$\mathcal{CM}(K,\Phi) \to \text{CL}(K),$$
$$[a,r] \mapsto [a]$$
is surjective. Each fiber is indexed by \( \epsilon \in \mathcal{O}_F^×/N_{K/F}\mathcal{O}_K^× \). Here \( \#(\mathcal{O}_F^×/N_{K/F}\mathcal{O}_K^×) \) equals 1 or 2; in particular, it equals 1 if \( \epsilon \in N_{K/F}\mathcal{O}_K^× \).

Assume now that \( F \) has narrow class number 1. Then
\[
\mathcal{CM}(K, \Phi) = \mathcal{CM}(K, \Phi, \mathcal{O}_F),
\]
and the forgetful map
\[
\mathcal{CM}(K, \Phi) \to \text{CL}(K)
\]
is injective (hence bijective) since \( N_{K/F}\mathcal{O}_K^× = \mathcal{O}_F^× \). We will repeatedly use this bijection to identify the zero-cycle of CM points \( \mathcal{CM}(K, \Phi, \mathcal{O}_F) \subset X_F := X(\mathcal{O}_F) \) with the set
\[
\{ z_a \in K^× \cap \mathbb{H}^n : [a] \in \text{CL}(K) \},
\]
where \( z_a \) represents \([a, r] \in X_F \) as above. The reader should keep in mind that the later set depends on \( \Phi \).

4. **Proof of Theorem 1.1**

We first establish the following version of the Kronecker limit formula for CM fields.

**Theorem 4.1.** Let \( F \) be a totally real number field of degree \( n \) over \( \mathbb{Q} \) with narrow class number 1. Let \( K \) be a CM extension of \( F \) with a CM type \( \Phi \). For each class \( C \in \text{CL}(K) \), let \( z_a \) be the CM point in \( \mathcal{CM}(K, \Phi, \mathcal{O}_F) \) corresponding to \( C^{-1} \). Then we have
\[
\lim_{s \to 1} \left[ \zeta_K(s, C) - \frac{(2\pi)^n R_K}{w_K \sqrt{d_K}} \frac{1}{s - 1} \right] = \frac{(2\pi)^n R_K}{w_K \sqrt{d_K}} \left( \frac{\pi^n A_F}{E_1 \sqrt{d_F}} + 2 \log(d_F) - \log(d_K) - 2 \log G(z_a) \right),
\]
where
\[
G(z) := \sqrt{\left( \frac{2^n d_F / \sqrt{d_K}}{N(y(z)) \cdot \phi(z)^2} \right)}
\]
and
\[
\phi(z) := f(z)^{-1/4E_1}.
\]

**Proof.** Fix a CM type \( \Phi \) for \( K \). Let \( C \in \text{CL}(K) \), and fix an integral ideal \( a \in C^{-1} \). Then the partial Dedekind zeta function equals
\[
\zeta_K(s, C) = \sum_{b \in C} N_{K/\mathbb{Q}}(b)^{-s} = \sum_{(\omega) \subset a} N_{K/\mathbb{Q}}(a^{-1}(\omega))^{-s} = N_{K/\mathbb{Q}}(a)^s \sum_{\omega \in a/\mathcal{O}_K^×} N_{K/\mathbb{Q}}((\omega))^{-s}.
\]
Notice that
\[
\sum_{\omega \in a/\mathcal{O}_K^×} N_{K/\mathbb{Q}}((\omega))^{-s} = \frac{1}{|\mathcal{O}_K^× : \mathcal{O}_F^×|} \sum_{\omega \in a/\mathcal{O}_F^×} N_{K/\mathbb{Q}}((\omega))^{-s}.
\]
Thus we have
\[ \zeta_K(s, C) = \frac{N_{K/Q}(a)^s}{\lvert \mathcal{O}_K^\times : \mathcal{O}_F^\times \rvert} \sum_{\omega \in a/\mathcal{O}_F^\times}^\prime N_{K/Q}(\omega)^{-s}. \]

By the facts in Section 3 there exists a decomposition
\[ a = \mathcal{O}_F \alpha + \mathcal{O}_F \beta, \]
where \( z_a = \beta/\alpha \in K^\times \cap \mathbb{H}^n \) and \( z_a \) represents the CM point \([a, r] \in X_F\) (here \( f_0 = \mathcal{O}_F \) since \( \# \text{Cl}(F)^+ = 1 \)). Then
\[ \sum_{\omega \in a/\mathcal{O}_F^\times}^\prime N_{K/Q}(\omega)^{-s} = \sum_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F/\mathcal{O}_F^\times}^\prime N_{K/Q}((a\alpha + b\beta))^{-s} \]
\[ = N_{K/Q}((\alpha))^{-s} \sum_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F/\mathcal{O}_F^\times}^\prime N_{K/Q}((a + bz_a)). \]

By a calculation with the CM type \( \Phi \) we obtain
\[ N_{K/Q}(a + bz_a) = \lvert N(a + bz_a) \rvert^2, \]
where we have identified \( z_a \) with \( \Phi(z_a) \in \mathbb{H}^n \). Moreover, one has
\[ N_{K/Q}(a/(\alpha)) = N(y(z_a)) \frac{2^n d_f}{\sqrt{d_K}}. \]

By combining the preceding calculations, we obtain
\[ \zeta_K(s, C) = \left( \frac{2^n d_f}{\sqrt{d_K}} \right)^s \frac{1}{\lvert \mathcal{O}_K^\times : \mathcal{O}_F^\times \rvert} \sum_{(a,b) \in \mathcal{O}_F \times \mathcal{O}_F/\mathcal{O}_F^\times}^\prime \frac{N(y(z_a))^s}{\lvert N(a + bz_a) \rvert^{2s}} \]
\[ = \left( \frac{2^n d_f}{\sqrt{d_K}} \right)^s \frac{1}{\lvert \mathcal{O}_K^\times : \mathcal{O}_F^\times \rvert} E(z_a, s). \]

Observe that
\[ \left( \frac{2^n d_f}{\sqrt{d_K}} \right)^{s-1} = 1 + \log \left( \frac{2^n d_f}{\sqrt{d_K}} \right) (s - 1) + O(s - 1)^2. \]
Then after a calculation using the Laurent expansion
\[ E(z_a, s) = \frac{E_{-1}}{s - 1} + E_0(z_a) + O(s - 1) \]
given by (2.3), we obtain the limit formula in the theorem.

**Remark 4.2.** If \( F = \mathbb{Q} \) in Theorem 4.1, we recover the Kronecker limit formula (1.2).

The function \( \phi(z) \) is positive and real-analytic. In the following proposition, we identify how \( \phi(z) \) transforms with respect to \( \text{SL}_2(\mathcal{O}_F) \) (see also [S, pp. 108-109]).

**Proposition 4.3.** For all \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathcal{O}_F) \), we have
\[ \phi(Mz) = \lvert N(\gamma z + \delta) \rvert^{\frac{1}{2}} \phi(z). \]
Proof. From the relation (2.1) we see that $E(z,s)$ has weight $0$ with respect to $\text{SL}_2(O_F)$. Then the Laurent expansion (2.3) implies that $E_0(Mz) = E_0(z)$, which by (2.4) implies that

$$\log f(Mz) = \log f(z) + E_{-1} \log \left( \frac{N(\text{Im}(Mz))}{N(\text{Im}(z))} \right).$$

A straightforward calculation shows that

$$\frac{N(\text{Im}(Mz))}{N(\text{Im}(z))} = |N(\gamma z + \delta)|^{-2},$$

and thus

$$f(Mz) = |N(\gamma z + \delta)|^{-2E-1} f(z).$$

The result now follows from the definition of $\phi(z)$. □

Remark 4.4. By Proposition 4.3, the function $G : \mathbb{H}^n \to \mathbb{R}^+$ defined by (4.1) has weight $0$ with respect to $\text{SL}_2(O_F)$ and thus is well-defined on CM points.

We can now deduce Theorem 1.1.

Proof of Theorem 1.1: For a class group character $\chi$ of $K$, let

$$L_K(\chi, s) = \sum_{C \in \text{Cl}(K)} \chi(C) \zeta_K(s, C)$$

be its associated $L$-function. By orthogonality for group characters, if $\chi$ is nontrivial we have

$$\sum_{C \in \text{Cl}(K)} \chi(C) = 0.$$ 

The theorem now follows from Theorem 4.1. □

5. Proof of Theorem 1.3

Let $K$ be a CM field with $\text{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^r$ for some integer $r \geq 2$, and let $E$ be an unramified quadratic extension of $K$ with $\text{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^{r+1}$. Then the zeta function $\zeta_E(s)$ (resp. $\zeta_K(s)$) factors as $\zeta(s)$ times the product of the quadratic Dirichlet $L$-functions associated to the quadratic subfields of $E$ (resp. $K$). Note that there are $2^r - 1$ quadratic subfields of $K$, $2^{r+1} - 1$ quadratic subfields of $E$, and $2^r$ quadratic subfields of $E$ that are not contained in $K$. By class field theory, the unramified extension $E/K$ gives rise to a real class group character $\chi$ of $K$ (a genus character) whose $L$-function factors as

$$L_K(\chi, s) = \frac{\zeta_E(s)}{\zeta_K(s)}.$$ 

Then by the preceding facts we obtain the factorization

$$L_K(\chi, s) = \prod_{i=1}^{2^r} L(\chi_{\Delta_i}, s),$$

where $\chi_{\Delta_i}$ for $1 \leq i \leq 2^r$ are the Kronecker symbols associated to the quadratic subfields $\mathbb{Q}(\sqrt{\Delta_i})$ of $E$ which are not contained in $K$. 

Divide the discriminants $\Delta_i$ into two disjoint sets, $S_R := \{\Delta_i : \Delta_i > 0\}$ and $S_I := \{\Delta_i : \Delta_i < 0\}$. Then we obtain the following formula for $L_K(\chi, 1)$ using Dirichlet’s class number formula (1.4) for quadratic fields,

$$L_K(\chi, 1) = \frac{2^{2r} \pi^s \prod_{i=1}^{2^r} h(\Delta_i) \prod_{\Delta_i \in S_R} \log(\epsilon_{\Delta_i})}{\prod_{i=1}^{2^r} \sqrt{|\Delta_i|} \prod_{\Delta_i \in S_I} w_{\Delta_i}}. \tag{5.1}$$

On the other hand, by Theorem 1.1 we have

$$L_K(\chi, 1) = \frac{2^{n+1} \pi^n R_K}{w_K \sqrt{d_K}} \left( - \sum_{C \in \text{Cl}(K)} \chi(C) \log G(z_a) \right). \tag{5.2}$$

Observe that $\#S_R = \#S_I = 2^{r-1} = [F : \mathbb{Q}] = n$, and the regulators of $K$ and $F$ satisfy the relation

$$R_K = 2^{n-1} R_F$$

(see [W, p. 41]). The theorem now follows by equating (5.1) and (5.2) and simplifying the resulting expression. □

6. Proof Theorem 1.4

Let $F = \mathbb{Q}(\sqrt{p})$ where $p \equiv 1 \mod 4$ is a prime such that $F$ has narrow class number 1. Let $D = D_1 D_2 < 0$ be a composite fundamental discriminant with $D_1 > 0$ and $D_2 < 0$ fundamental discriminants. Let $K = \mathbb{Q}(\sqrt{D}, \sqrt{pD})$, which is a CM biquadratic extension of $\mathbb{Q}$ with $\text{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let $E = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}, \sqrt{p})$, which is an unramified quadratic extension of $K$ with $\text{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$. Let $\chi$ be the genus character of $K$ arising from the extension $E/K$, and let $K_\Delta$ denote $\mathbb{Q}(\sqrt{\Delta})$ for a fundamental discriminant $\Delta$. Then we have the following diagram:

$$
\begin{array}{c}
E = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}, \sqrt{p}) \\
| \chi \\
K = \mathbb{Q}(\sqrt{D}, \sqrt{pD}) \\
| \\
K_{D_1} \quad K_{D_2} \quad K_D \\
| \chi_D \\
F = \mathbb{Q}(\sqrt{p}) \\
| \\
K_{pD} \quad K_{pD_1} \quad K_{pD_2} \\
\end{array}
$$

We have

$$L_K(\chi, s) = \frac{\zeta_E(s)}{\zeta_K(s)}.$$

Then the factorizations

$$\zeta_E(s) = \zeta(s) L(\chi_p, s) L(\chi_D, s) L(\chi_{pD}, s) L(\chi_{pD_1}, s) L(\chi_{pD_2}, s)$$
and
\[ \zeta_K(s) = \zeta(s) L(\chi_p, s) L(\chi_D, s) L(\chi_{pD}, s) \]
yield
\[ L_K(\chi, s) = L(\chi_{D_1}, s) L(\chi_{D_2}, s) L(\chi_{pD_1}, s) L(\chi_{pD_2}, s). \]

By Dirichlet’s class number formula (1.4) for quadratic fields, we have
\[ L_K(\chi, 1) = \frac{2 \log(\varepsilon_{D_1}) h(D_1)}{\sqrt{D_1}} \frac{2 \pi h(D_2)}{w_{D_2} \sqrt{|D_2|}} \frac{2 \log(\varepsilon_{pD_1}) h(pD_1)}{\sqrt{pD_1}} \frac{2 \pi h(pD_2)}{w_{pD_2} \sqrt{|pD_2|}}. \]  \hspace{1cm} (6.1)

On the other hand, by Theorem 1.1 we have
\[ L_K(\chi, 1) = \frac{16 \pi^2 \log(\varepsilon_p)}{w_K \sqrt{d_K}} \left( - \sum_{C \in \text{Cl}(K)} \chi(C) \log G(z_C) \right), \]  \hspace{1cm} (6.2)

where we used \( R_K = 2 \log(\varepsilon_p) \) (see [W, Proposition 4.16]). The theorem now follows by equating (6.1) and (6.2) and simplifying the resulting expression. \( \square \)

7. PROOF OF THEOREM 1.6

Assume first that \( r = 2 \). Then \( K \cong (\mathbb{Z}/2\mathbb{Z})^2 \), \( E \cong (\mathbb{Z}/2\mathbb{Z})^3 \), and the maximal totally real subfield \( F \) of \( K \) is real quadratic. Let \( \mathbb{Q}(\sqrt{D_1}) \) and \( \mathbb{Q}(\sqrt{D_2}) \) be the real quadratic subfields of \( E \) which are not contained in \( K \), and let \( F = \mathbb{Q}(\sqrt{D_3}) \). Then because \( R_K = 2 \log(\varepsilon_{D_1}) \), it suffices to show that \( A := \exp(B) \) is transcendent, where
\[ B := Q_1 \sqrt{Q_2} \frac{\log(\varepsilon_{D_1}) \log(\varepsilon_{D_2})}{\log(\varepsilon_{D_3})} \]
for rational numbers \( Q_1, Q_2 \in \mathbb{Q} \).

Let \( x_1 := \log(\varepsilon_{D_1}), x_2 := \log(\varepsilon_{D_2}) \) and \( x_3 := \log(\varepsilon_{D_3}) \). Then
\[ \mathbb{Q}(x_1, x_2, x_3, \exp(x_1), \exp(x_2), \exp(x_3)) = \mathbb{Q}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3})). \]

Because \( \varepsilon_{D_1}, \varepsilon_{D_2} \) and \( \varepsilon_{D_3} \) are multiplicatively independent, \( x_1, x_2 \) and \( x_3 \) are linearly independent over \( \mathbb{Q} \). Then by Schanuel’s conjecture (see Conjecture 1.5), the field
\[ \mathbb{Q}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3})) \]
has transcendence degree at least 3 over \( \mathbb{Q} \), and hence exactly 3 as it is generated by 3 elements. In particular, \( x_1, x_2 \) and \( x_3 \) are algebraically independent over \( \overline{Q} \).

We claim that because \( x_1, x_2 \) and \( x_3 \) are algebraically independent over \( \overline{Q} \), the numbers \( x_1, x_2, x_3 \) and \( x_4 := B \) are linearly independent over \( \mathbb{Q} \). To see this, suppose to the contrary that there exist rational numbers \( \alpha_i \in \mathbb{Q} \), not all zero, such that
\[ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 B = 0. \]  \hspace{1cm} (7.1)

Define the polynomial
\[ q(t_1, t_2, t_3) := \alpha_1 t_1 t_3 + \alpha_2 t_2 t_3 + \alpha_3 t_3^2 + \alpha_4 Q_1 \sqrt{Q_2 t_1 t_2}. \]
Then (7.1) implies that \( q(x_1, x_2, x_3) = 0 \), which contradicts the algebraic independence of \( x_1, x_2 \) and \( x_3 \) over \( \mathbb{Q} \). Thus \( x_1, x_2, x_3 \) and \( x_4 \) are linearly independent over \( \mathbb{Q} \). By another application of Schanuel’s conjecture, the field
\[
\mathbb{Q}(x_1, x_2, x_3, \exp(x_1), \exp(x_2), \exp(x_3), \exp(x_4))
\]
\[
= \mathbb{Q}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3}), B, A)
\]
\[
= \mathbb{Q}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3}), A)
\]
has transcendence degree at least 4 over \( \mathbb{Q} \), hence \( A \) must be transcendental. This completes the proof when \( r = 2 \).

Next assume that \( r \geq 2 \). Then \( K \cong (\mathbb{Z}/2\mathbb{Z})^r \), \( E \cong (\mathbb{Z}/2\mathbb{Z})^{r-1} \), and \( F \cong (\mathbb{Z}/2\mathbb{Z})^{r-1} \). The rank of the unit group \( \mathcal{O}_F^\times \) is \( n - 1 \), where \( n = [F:\mathbb{Q}] \), and recall that the regulators of \( K \) and \( F \) satisfy the relation
\[
R_K = 2^{n-1} R_F.
\]
Let \( \varepsilon_1, \ldots, \varepsilon_{n-1} \) be fundamental units for the \( n - 1 \) real quadratic subfields of \( F \). These units form a set of multiplicatively independent units in \( F \) which are a basis for \( \mathcal{O}_F^\times / \{ \pm 1 \} \), and thus
\[
R_F = |\det(\log |\sigma_i(\varepsilon_j)|)|_{1 \leq i,j \leq n-1}
\]
where the \( \sigma_i \) run through any \( n - 1 \) embeddings of \( F \). The conjugate of a unit in a real quadratic field is, up to a sign, its inverse. Thus for \( \sigma \in \text{Gal}(F/\mathbb{Q}) \), either \( \sigma(\varepsilon_j) = \varepsilon_j \) or \( \sigma(\varepsilon_j) = \varepsilon_j^{-1} \). It follows that the regulator \( R_F \) is a positive integer multiple of the product \( \log(\varepsilon_1) \cdots \log(\varepsilon_{n-1}) \). Therefore it suffices to show that \( \exp(C) \) is transcendental, where
\[
C := Q_3 \sqrt{Q_4} \prod_{\Delta_i \in S_R} \frac{\log(\varepsilon_{\Delta_i})}{\log(\varepsilon_1) \cdots \log(\varepsilon_{n-1})}
\]
for rational numbers \( Q_3, Q_4 \in \mathbb{Q} \). Because the units \( \{ \varepsilon_1, \ldots, \varepsilon_{n-1} \} \cup \{ \varepsilon_{\Delta_i} : \Delta_i \in S_R \} \) are multiplicatively independent, a straightforward modification of the argument for \( r = 2 \) shows that \( \exp(C) \) is transcendental.

\[ \square \]

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