THE CHOWLA-SELBERG FORMULA FOR CM ABELIAN SURFACES

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Abstract. In this paper, we establish an explicit two-dimensional analog of the classical Chowla-Selberg formula. This is an identity which evaluates the Faltings height of a CM abelian surface in terms of the Barnes double Gamma function at algebraic arguments. We use this identity to evaluate Faltings heights of Jacobians of genus 2 curves with complex multiplication by *non-abelian* CM fields.

1. Introduction and statement of results

The Chowla-Selberg formula [CS67] is a remarkable identity which evaluates the Dedekind eta function at CM points in terms of Euler's Gamma function at rational arguments. As observed by Deligne [Del85], the Chowla-Selberg formula can be reformulated "geometrically" as an identity for the Faltings height of a CM elliptic curve (see equation (1.8)). In this paper, we will establish an explicit two-dimensional analog of the Chowla-Selberg formula, which evaluates the Faltings height of a CM abelian surface in terms of the Barnes double Gamma function at algebraic arguments (see Theorem 1.1). To illustrate this, we give several examples which explicitly evaluate Faltings heights of Jacobians of genus 2 curves with complex multiplication by *non-abelian* CM fields (see e.g. Corollaries 2.4 and 2.6). Among other things, our analysis relies crucially on an extensive study of the combinatorial structure of certain finite sets of algebraic numbers which we call "Shintani sets".

1.1. The Chowla-Selberg formula.

In this section, we review the classical Chowla-Selberg formula and explain its geometric reformulation due to Deligne [Del85]. For a very nice discussion of the Chowla-Selberg formula and its applications, we refer the reader to Zagier's article [Zag08].

Let K be an imaginary quadratic field of discriminant $-D < 0$. Let $\zeta_K(s)$ be the Dedekind zeta function, $h(-D)$ be the class number, $w(-D)$ be the number of units, χ_{-D} be the Kronecker symbol, and $L(\chi_{-D}, s)$ be the Dirichlet L–function associated to χ_{-D}. The *Dedekind eta function* is the weight 1/2 modular form for $SL_2(\mathbb{Z})$ defined by the infinite product

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q := e^{2\pi i z}.$$

Using the factorization $\zeta_K(s) = \zeta(s)L(\chi_{-D}, s)$ and Kronecker's first limit formula, one can establish the identity

$$\sum_{C} \log(\sqrt{\mathrm{Im}(\tau_{C^{-1}})}|\eta(\tau_{C^{-1}})|^2) = \frac{h(-D)}{2} \left(\log(\sqrt{D}/2) - \frac{1}{2}\log(2\pi) + \frac{L'(\chi_{-D}, 0)}{L(\chi_{-D}, 0)} \right), \tag{1.1}$$

where the sum is over a complete set of CM points $\tau_{C^{-1}}$ of discriminant $-D$ on the modular surface $SL_2(\mathbb{Z})\backslash\mathbb{H}$. There are $h(-D)$ such points, corresponding to the ideal classes C of K.

Now, the *Hurwitz zeta function* is defined by

$$\zeta(s, z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s}, \quad \mathrm{Re}(s) > 1, \quad \mathrm{Re}(z) > 0.$$

Lerch [Ler97] evaluated the second term in the Taylor expansion at $s = 0$ of the Hurwitz zeta function in terms of Euler's Gamma function $\Gamma(z)$ as follows

$$\frac{d}{ds}\zeta(s,z)\bigg|_{s=0} = \log\left(\Gamma^*(z)\right), \tag{1.2}$$

where $\Gamma^*(z) := \Gamma(z)/\sqrt{2\pi}$. By expressing $L(\chi_{-D}, s)$ as a linear combination of Hurwitz zeta functions

$$L(\chi_{-D}, s) = D^{-s}\sum_{k=1}^{D}\chi_{-D}(k)\zeta(s, k/D), \tag{1.3}$$

one can employ Lerch's identity (1.2) and Dirichlet's class number formula

$$L(\chi_{-D}, 0) = 2h(-D)/w(-D)$$

to relate the logarithmic derivative of $L(\chi_{-D}, s)$ at $s = 0$ to Euler's Gamma function at rational arguments,

$$\frac{L'(\chi_{-D}, 0)}{L(\chi_{-D}, 0)} = -\log(D) + \frac{w(-D)}{2h(-D)}\sum_{k=1}^{D}\chi_{-D}(k)\log\Gamma(k/D). \tag{1.4}$$

Finally, by substituting (1.4) into (1.1) and exponentiating, one arrives at the Chowla-Selberg formula:

$$\prod_{C}\sqrt{\operatorname{Im}(\tau_{C^{-1}})}|\eta(\tau_{C^{-1}})|^2 = \left(\frac{1}{4\pi\sqrt{D}}\right)^{h(-D)/2}\prod_{k=1}^{D}\Gamma(k/D)^{w(-D)\chi_{-D}(k)/4}. \tag{1.5}$$

To explain Deligne's geometric reformulation of (1.5), we require the (stable) Faltings height of a CM abelian variety.

Let F be a totally real number field of degree n. Let E/F be a CM extension of F and Φ be a CM type for E. Let X_Φ be an abelian variety defined over $\overline{\mathbb{Q}}$ with complex multiplication by $\mathcal{O}_E$ and CM type Φ. We call X_Φ a *CM abelian variety of type* $(\mathcal{O}_E, \Phi)$.

Let $L \subseteq \overline{\mathbb{Q}}$ be a number field over which X_Φ has everywhere good reduction and choose a differential $\omega \in H^0(X_\Phi, \Omega^n_{X_\Phi})$. Then the *Faltings height* of X_Φ is defined by

$$h_{\mathrm{Fal}}(X_\Phi) := -\frac{1}{2[L:\mathbb{Q}]}\sum_{\sigma: L\hookrightarrow\mathbb{C}}\log\left|\int_{X_\Phi^\sigma(\mathbb{C})}\omega^\sigma\wedge\overline{\omega^\sigma}\right|.$$

The Faltings height does not depend on the choice of L, ω, or X_Φ. In particular, the Faltings height depends only on the choice of CM type Φ, and hence is often denoted by $h_{\mathrm{Fal}}(\Phi)$.

Assume now that $E = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic field and $X = X_\Phi$ is a CM elliptic curve of type $(\mathcal{O}_E, \Phi)$. Then the Faltings height of X can be calculated directly in terms of the CM values of $\eta(z)$ (see for example [Gro80, Sil86, BSM15, BS+17]),

$$h_{\mathrm{Fal}}(X) = -\log(2^{3/2}\pi) - \frac{1}{h(-D)}\sum_{C}\log(\sqrt{\operatorname{Im}(\tau_{C^{-1}})}|\eta(\tau_{C^{-1}})|^2). \tag{1.6}$$

The identity (1.6) allows us to express the "pre" Chowla-Selberg formula (1.1) in the equivalent form

$$h_{\mathrm{Fal}}(X) = -\frac{1}{2}\frac{L'(\chi_{-D}, 0)}{L(\chi_{-D}, 0)} - \frac{1}{4}\log(D) - \frac{1}{2}\log(2\pi). \tag{1.7}$$

Then by substituting (1.4) into (1.7), we can express the "exact" Chowla-Selberg formula (1.5) in the equivalent geometric form

$$\exp[h_{\mathrm{Fal}}(X)] = \left(\frac{\sqrt{D}}{2\pi}\right)^{1/2} \prod_{\substack{k=1 \\ \gcd(k,D)=1}}^{D} \Gamma(k/D)^{w(-D)\chi_{-D}(k)/4h(-D)}. \tag{1.8}$$

1.2. **The Chowla-Selberg formula for CM abelian surfaces.** In this section, we state our main result, which is an explicit two-dimensional analog of the Chowla-Selberg formula (1.8) for CM abelian surfaces.

We will need the following notation and definitions. Let F be a real quadratic field. Let d_F be the discriminant, $\mathcal{O}_F$ be the ring of integers, and $\mathcal{O}_F^\times$ be the group of units of F, respectively.

Let E be a CM extension of F, and let E^c be the Galois closure of $E/\mathbb{Q}$. Then $E/\mathbb{Q}$ is either biquadratic with $\mathrm{Gal}(E/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, cyclic with $\mathrm{Gal}(E/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$, or non-Galois with $\mathrm{Gal}(E^c/\mathbb{Q}) \cong D_4$. Let d_E be the discriminant, $\mu_{E/F} = [\mathcal{O}_E^\times : \mathcal{O}_F^\times]$ be the index of the unit groups, h_E be the class number, and $\mathfrak{D}_{E/F}$ be the relative discriminant, respectively.

Let $\chi_{E/F}$ be the quadratic narrow ray class character modulo $\mathfrak{D}_{E/F}$ associated to the extension E/F.

Next, we define the Barnes double Gamma function. Let z be a complex number and $\omega = (\omega_1, \omega_2)$ be a pair of complex numbers. Assume that z, ω_1 and ω_2 have positive real part. For a complex number $w \in \mathbb{C} \setminus (-\infty, 0]$, let $w^s = \exp(s \log w)$, where $\log w = \log|w| + i \arg w$ with $|\arg w| < \pi$. Define the *double Hurwitz zeta function*

$$\zeta_H(s, z, \omega) := \sum_{m,n=0}^{\infty} \frac{1}{(z + m\omega_1 + n\omega_2)^s}, \quad \mathrm{Re}(s) > 2.$$

The function $\zeta_H(s, z, \omega)$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at $s = 1, 2$. In analogy with Lerch's identity (1.2), the *normalized Barnes double Gamma function* is defined by

$$\log\left(\Gamma_2^*(z, \omega)\right) := \frac{d}{ds}\zeta_H(s, z, \omega)\bigg|_{s=0}.$$

Now, the function $\Gamma^*(z) = \Gamma(z)/\sqrt{2\pi}$ has a simple pole at $z = 0$ with residue $1/\sqrt{2\pi}$. Hence, if we define $\rho^{-1} := 1/\sqrt{2\pi}$, then we can write $\Gamma(z) = \rho\Gamma^*(z)$. Analogously, the function $\Gamma_2^*(z, \omega)$ has a simple pole at $z = 0$. Then if $\rho_2(\omega)^{-1}$ denotes the residue of $\Gamma_2^*(z, \omega)$ at $z = 0$, the *Barnes double gamma function* is defined by

$$\Gamma_2(z, \omega) := \rho_2(\omega)\Gamma_2^*(z, \omega).$$

More concretely, the Barnes double Gamma function can also be defined by

$$\Gamma_2(z, \omega) := F(z, \omega)^{-1},$$

where $F(z, \omega)$ is given by the Weierstrass product expansion

$$F(z, \omega) := z \exp\left(\gamma_{22}(\omega) z + \frac{z^2}{2}\gamma_{21}(\omega)\right) \prod_{(m,n)} \left(1 + \frac{z}{m\omega_1 + n\omega_2}\right) \exp\left(-\frac{z}{m\omega_1 + n\omega_2} + \frac{z^2}{2(m\omega_1 + n\omega_2)^2}\right),$$

the product being over all pairs of integers $(m, n) \in \mathbb{Z}_{\geq 0}^2$ with $(m, n) \neq (0, 0)$. The function $F(z, \omega)$ is entire, and the constants $\gamma_{22}(\omega), \gamma_{21}(\omega)$ are explicit higher analogs of Euler's constant γ.

Finally, let $\varepsilon > 1$ be a generator of the group of totally positive units $\mathcal{O}_F^{\times,+}$ of F, and let ε' denote the conjugate of ε by the nontrivial automorphism in $\mathrm{Gal}(F/\mathbb{Q})$. Define the *Shintani set* by

$$\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) := \left\{ z = x + y\varepsilon \in \mathfrak{D}_{E/F}^{-1} \,\middle|\, x, y \in \mathbb{Q},\ 0 < x \leq 1,\ 0 \leq y < 1,\ \mathfrak{D}_{E/F}\langle z \rangle \text{ coprime to } \mathfrak{D}_{E/F} \right\},$$

where $\langle z\rangle$ is the ideal generated by z. The Shintani set is finite, and embeds via the map $\alpha \mapsto (\alpha, \alpha')$ as a subset of the Shintani cone $C(\varepsilon)$ generated by the vectors $(1,1)$ and $(\varepsilon, \varepsilon')$, defined by

$$C(\varepsilon) := \left\{t_1(1,1) + t_2\left(\varepsilon, \varepsilon'\right) : \ t_1 > 0, \ t_2 \geq 0\right\} \subset \mathbb{R}_+^2.$$

Let $B_2(x) := x^2 - x + 1/6$ be the second Bernoulli polynomial.

Our main result is the following two-dimensional analog of the Chowla-Selberg formula (1.8).

Theorem 1.1. *Let F be a real quadratic field with narrow class number 1. Let E be a CM extension of F with $E/\mathbb{Q}$ non-biquadratic. Let $X = X_\Phi$ be a CM abelian surface of type $(\mathcal{O}_E, \Phi)$. Then*

$$\exp[h_{\mathrm{Fal}}(X)] = \frac{1}{2\pi}\left(\frac{d_E}{d_F^3}\right)^{1/4} \varepsilon^{c(E/F)/2} \prod_{z\in\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})} \prod_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})} \Gamma_2(z^\sigma, (1, \varepsilon^\sigma))^{-\mu_{E/F}\chi_{E/F}(\mathfrak{D}_{E/F}\langle z\rangle)/4h_E},$$

where

$$c(E/F) := \frac{\varepsilon - \varepsilon'}{4}\frac{\mu_{E/F}}{h_E} \sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})\\ z=x+y\varepsilon}} \chi_{E/F}(\mathfrak{D}_{E/F}\langle z\rangle) B_2(x).$$

Remark 1.2. In Section 2, we give several examples of Theorem 1.1, in which we explicitly evaluate Faltings heights of Jacobians of genus 2 curves with complex multiplication by *non-abelian* CM fields.

Remark 1.3. The first explicit evaluation of the Faltings height of a CM abelian surface is due to Bost, Mestre, and Moret-Bailly [BMM-B], who evaluated the Faltings height of the Jacobian of the genus 2 CM curve $C : y^2 = x^5 + 1$ as[1]

$$h_{\mathrm{Fal}}(J_C) = \log(2\pi) - \frac{1}{2}\log\left(\Gamma(1/5)^5\Gamma(2/5)^3\Gamma(3/5)\Gamma(4/5)^{-1}\right) \approx -3.29038630605499.$$

The Jacobian J_C is a CM abelian surface of type $(\mathcal{O}_E, \Phi)$ where E is the cyclic quartic CM field $E = \mathbb{Q}(\zeta_5)$. Therefore, we can evaluate this Faltings height using Theorem 1.1, and we do so in Example 2.1 (see in particular Corollary 2.2) where we show that

$$\exp[h_{\mathrm{Fal}}(J_C)] = \frac{1}{2\pi}\left(\frac{3+\sqrt{5}}{2}\right)^{\sqrt{5}/10} \prod_{\pm} \frac{\Gamma_2(2\cdot(\frac{5\pm\sqrt{5}}{10}), (1, \frac{3\pm\sqrt{5}}{2}))^{5/4}}{\Gamma_2(1\cdot(\frac{5\pm\sqrt{5}}{10}), (1, \frac{3\pm\sqrt{5}}{2}))^{5/4}} \frac{\Gamma_2(3\cdot(\frac{5\pm\sqrt{5}}{10}), (1, \frac{3\pm\sqrt{5}}{2}))^{5/4}}{\Gamma_2(4\cdot(\frac{5\pm\sqrt{5}}{10}), (1, \frac{3\pm\sqrt{5}}{2}))^{5/4}}.$$

By approximating the values of the Barnes double Gamma function appearing in this evaluation, we find that

$$h_{\mathrm{Fal}}(J_C) \approx -3.29038630605499.$$

This serves as a numerical verification of Theorem 1.1.

The algebraic numbers in the Shintani set $\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$ at which the Barnes double gamma function Γ_2 is evaluated in Theorem 1.1 are analogous to the rational numbers

$$R_D := \left\{\frac{k}{D} \in \frac{1}{D}\mathbb{Z} \;\middle|\; 1 \leq k \leq D, \ \gcd(k, D) = 1\right\}$$

at which Euler's Gamma function Γ is evaluated in the Chowla-Selberg formula (1.8). This analogy can be understood from the following group-theoretic perspective. The set of rational numbers defined by

$$\widetilde{R}_D := \left\{\frac{k}{D} \in \frac{1}{D}\mathbb{Z} \;\middle|\; 1 \leq k \leq D\right\}$$

[1]The value given here differs from [BMM-B, Proposition 12] by addition of the number $\log(2)$, due to a difference in the normalization of the Faltings height.

is a complete set of coset representatives for the quotient group

$$\frac{1}{D}\mathbb{Z}\Big/\mathbb{Z}$$

inside the standard fundamental domain $(0,1]$ for the group $\mathbb{R}/\mathbb{Z}$. Similarly, in Proposition 4.3 we will show that the set of algebraic numbers defined by

$$\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{D}_{E/F}^{-1}) := \{z = x + y\varepsilon \in \mathfrak{D}_{E/F}^{-1} \mid x, y \in \mathbb{Q},\ 0 < x \leq 1,\ 0 \leq y < 1\}$$

is a complete set of coset representatives for the quotient group

$$\mathfrak{D}_{E/F}^{-1}\Big/(\mathbb{Z}+\mathbb{Z}\varepsilon)$$

whose images under the embedding $\iota : F \hookrightarrow \mathbb{R}^2$ given by $\alpha \mapsto (\alpha, \alpha')$ lie inside the standard fundamental parallelogram

$$P_F := \{t_1(1,1) + t_2(\varepsilon,\varepsilon') \mid t_1, t_2 \in \mathbb{R},\ 0 < t_1 \leq 1,\ 0 \leq t_2 < 1\}$$

for the group $\mathbb{R}^2/\iota(\mathbb{Z}+\mathbb{Z}\varepsilon)$. Hence, after removing from the set $\widetilde{R}_D$ (resp. $\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{D}_{E/F}^{-1})$) the numbers for which the character values $\chi_{-D}(k)$ (resp. $\chi_{E/F}(\mathfrak{D}_{E/F}\langle z\rangle)$) are zero by enforcing the coprimality conditions $\gcd(k, D) = 1$ (resp. $\mathfrak{D}_{E/F}\langle z\rangle$ coprime to $\mathfrak{D}_{E/F}$), we see that $\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})$ is analogous to R_D.

Now, observe that the size of the set R_D can be expressed as

$$\#R_D = \frac{\mathrm{vol}(\mathbb{R}/\mathbb{Z})}{\mathrm{vol}(\mathbb{Z})} \cdot \varphi(D), \tag{1.9}$$

where φ is the Euler φ-function. By a combinatorial analysis which involves a remarkable theorem of Pick which expresses the area of a lattice polygon in $\mathbb{R}^2$ in terms of the number of lattice points inside and on the boundary of the polygon, we will prove the following analog of (1.9) for the Shintani set $\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})$.

Theorem 1.4. *Let F be a real quadratic field with narrow class number 1. Then the size of the Shintani set $\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})$ is given by*

$$\#\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1}) = \frac{\mathrm{vol}(\mathbb{R}^2/\iota(\mathbb{Z}+\mathbb{Z}\varepsilon))}{\mathrm{vol}(\mathcal{O}_F)} \cdot \varphi(\mathfrak{D}_{E/F}) = \frac{\varepsilon-\varepsilon'}{\sqrt{d_F}} \cdot \varphi(\mathfrak{D}_{E/F}),$$

where

$$\varphi(\mathfrak{D}_{E/F}) := N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F}) \cdot \prod_{\mathfrak{p}\mid\mathfrak{D}_{E/F}} \left(1 - \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{p})}\right)$$

is the generalized Euler φ-function for number fields.

Remark 1.5. The quantity $(\varepsilon-\varepsilon')/\sqrt{d_F}$ is always a positive integer. In fact, if one writes ε in terms of the integral basis $\{1, \frac{d_F+\sqrt{d_F}}{2}\}$ for $\mathcal{O}_F$ as $\varepsilon = a + b\left(\frac{d_F+\sqrt{d_F}}{2}\right)$ for some $a, b \in \mathbb{Z}$ with $b \geq 1$, then $(\varepsilon-\varepsilon')/\sqrt{d_F} = b$.

In the following table we display a list of the non-biquadratic quartic CM fields E of discriminant $|d_E| \leq 4000$ such that the corresponding totally real subfield F has narrow class number 1. In the table, we also list all of the quantities needed to use the formula from Theorem 1.4 to give the size of the Shintani set $\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})$. We have placed a ★ in each entry where the Galois group is $\mathrm{Gal}(E^c/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$. All other entries correspond to non-abelian quartic CM fields with Galois group $\mathrm{Gal}(E^c/\mathbb{Q}) \cong D_4$. The first three entries of the table correspond to the CM fields that we chose for the examples that we give in Section 2. Moreover, as the reader can observe from the table, these were chosen so that we have the least number of elements in the Shintani sets.

E	F	d_E	d_F	ε	$\mathfrak{D}_{E/F}$	$\frac{\varepsilon-\varepsilon'}{\sqrt{d_F}}$	$\varphi(\mathfrak{D}_{E/F})$	$\#\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$
$\bigstar\mathbb{Q}(\zeta_5)$	$\mathbb{Q}(\sqrt{5})$	125	5	$\frac{3+\sqrt{5}}{2}$	$\langle\sqrt{5}\rangle$	1	4	4
$\mathbb{Q}\left(\sqrt{\frac{-13+\sqrt{5}}{2}}\right)$	$\mathbb{Q}(\sqrt{5})$	1025	5	$\frac{3+\sqrt{5}}{2}$	$\left\langle\frac{-13+\sqrt{5}}{2}\right\rangle$	1	40	40
$\mathbb{Q}\left(\sqrt{-5-2\sqrt{2}}\right)$	$\mathbb{Q}(\sqrt{2})$	1088	8	$3+2\sqrt{2}$	$\langle -5-2\sqrt{2}\rangle$	2	16	32
$\mathbb{Q}\left(\sqrt{-9+2\sqrt{5}}\right)$	$\mathbb{Q}(\sqrt{5})$	1525	5	$\frac{3+\sqrt{5}}{2}$	$\left\langle\frac{-1+7\sqrt{5}}{2}\right\rangle$	1	60	60
$\bigstar\mathbb{Q}\left(\sqrt{-2+\sqrt{2}}\right)$	$\mathbb{Q}(\sqrt{2})$	2048	8	$3+2\sqrt{2}$	$\langle 4\sqrt{2}\rangle$	2	16	32
$\bigstar\mathbb{Q}\left(\sqrt{\frac{-13+3\sqrt{13}}{2}}\right)$	$\mathbb{Q}(\sqrt{13})$	2197	13	$\frac{11+3\sqrt{13}}{2}$	$\langle\sqrt{13}\rangle$	3	12	36
$\mathbb{Q}\left(\sqrt{\frac{-7+\sqrt{17}}{2}}\right)$	$\mathbb{Q}(\sqrt{17})$	2312	17	$33+8\sqrt{17}$	$\left\langle\frac{-7+\sqrt{17}}{2}\right\rangle$	16	4	64
$\mathbb{Q}\left(\sqrt{\frac{-21+\sqrt{5}}{2}}\right)$	$\mathbb{Q}(\sqrt{5})$	2725	5	$\frac{3+\sqrt{5}}{2}$	$\left\langle\frac{-21+\sqrt{5}}{2}\right\rangle$	1	108	108
$\mathbb{Q}\left(\sqrt{\frac{-9+\sqrt{13}}{2}}\right)$	$\mathbb{Q}(\sqrt{13})$	2873	13	$\frac{11+3\sqrt{13}}{2}$	$\left\langle\frac{-9+\sqrt{13}}{2}\right\rangle$	3	16	48
$\mathbb{Q}\left(\sqrt{\frac{-25+3\sqrt{5}}{2}}\right)$	$\mathbb{Q}(\sqrt{5})$	3625	5	$\frac{3+\sqrt{5}}{2}$	$\left\langle\frac{5+11\sqrt{5}}{2}\right\rangle$	1	112	112
$\mathbb{Q}\left(\sqrt{-13+2\sqrt{5}}\right)$	$\mathbb{Q}(\sqrt{5})$	3725	5	$\frac{3+\sqrt{5}}{2}$	$\left\langle\frac{3+11\sqrt{5}}{2}\right\rangle$	1	148	148
$\mathbb{Q}\left(\sqrt{-9+2\sqrt{17}}\right)$	$\mathbb{Q}(\sqrt{17})$	3757	17	$33+8\sqrt{17}$	$\langle 2+\sqrt{17}\rangle$	16	12	192

TABLE 1. The number of elements in the Shintani set $\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$ for $|d_E| \leq 4000$.

1.3. **Discussion.** The periods of CM abelian varieties have been studied extensively by many authors; see for example the works of Gross [Gro78], Shimura [Shi79], Anderson [And82], Deligne et al. [Del82], and Yoshida [Yos99]. In this direction, Colmez [Col93] gave a conjectural generalization of the identity (1.7) which relates the Faltings height of a CM abelian variety $X = X_\Phi$ of type $(\mathcal{O}_E, \Phi)$ to logarithmic derivatives of Artin L–functions at $s = 0$.

In [Yan10a, Yan10b, Yan13], Tonghai Yang proved the Colmez conjecture for a large class of non-biquadratic quartic CM fields, including the first *non-abelian* cases. As explained in Section 8, the Colmez conjecture is now known to be true for quartic CM fields. In particular, if E is a non-biquadratic quartic CM field, then the Faltings height of X is given by

$$h_{\mathrm{Fal}}(X) = -\frac{1}{2}\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} - \frac{1}{4}\log\left(\frac{|d_E|}{d_F}\right) - \log(2\pi), \tag{1.10}$$

where $L(\chi_{E/F}, s)$ is the (incomplete) L–function of the Hecke character $\chi_{E/F}$ associated to the quadratic extension E/F. Our primary objective in this paper is to explicitly evaluate the right hand side of (1.10). To do this, we will establish a two-dimensional analog of the identity (1.4) for the logarithmic derivative of $L(\chi_{E/F}, s)$ at $s = 0$ by employing work of Shintani [Shi77a, Shi77b] on special values of ray class L–functions for totally real fields. As an important part of our analysis, we will undertake an extensive study of the combinatorial structure of certain finite sets of algebraic numbers which we call "Shintani sets". These sets arise in Shintani's decomposition of ray class L–functions as linear combinations of Shintani zeta functions. As discussed, one of our discoveries is an exact formula for the number of elements in a Shintani set (see Theorems 1.4 and 4.1). To prove this formula, we introduce a group structure on the Shintani sets and use this to give a combinatorial analysis which reduces the problem to an application of Pick's theorem. The formula for the size of the Shintani set given in Theorem 1.4 plays an important role in our examples which evaluate Faltings heights of Jacobians of genus 2 CM curves (see Section 2). More precisely, for a non-abelian CM field, the number of elements in the corresponding Shintani set will usually be very large. Using our formula, we can choose the CM fields so as to control the number of elements in the corresponding Shintani sets and therefore give more manageable evaluations (see Table 1).

2. Faltings heights of Jacobians of CM curves of genus 2

In this section, we use Theorem 1.1 to give explicit evaluations of Faltings heights of CM abelian surfaces $X = X_\Phi$ of type $(\mathcal{O}_E, \Phi)$ where E is a non-biquadratic quartic CM field. It is known that such a CM abelian surface is the Jacobian J_C of a nonsingular CM curve C of genus 2. In Appendix A, we use work of Bouyer and Streng [BS15] to give models for genus 2 curves whose Jacobians have complex multiplication by a prescribed CM field E. In Appendix B, we give an algorithm to compute Shintani sets. All computer calculations in our examples were performed using `SageMath` [S$^+$09].

Example 2.1. Consider the cyclotomic field $E = \mathbb{Q}(\zeta_5)$, where $\zeta_5 := e^{2\pi i/5}$ is a primitive 5th root of unity. Then E is a CM extension of the real quadratic field $F = \mathbb{Q}(\sqrt{5})$ with narrow class number 1 and $E/\mathbb{Q}$ is cyclic. The Jacobian J_C of the genus 2 curve $C : y^2 = x^5 + 1$ is a CM abelian surface of type $(\mathcal{O}_E, \Phi)$.

Note that E has class number $h_E = 1$. Also, we have $d_E = 125$ and $d_F = 5$. The relative discriminant is $\mathfrak{D}_{E/F} = \sqrt{5}\mathcal{O}_F = \langle\sqrt{5}\rangle$. The ring of integers of F is

$$\mathcal{O}_F = \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{5}}{2}\right),$$

and the group of units is $\mathcal{O}_F^\times = \{\pm\varepsilon_5^n \mid n \in \mathbb{Z}\}$, where $\varepsilon_5 := (1+\sqrt{5})/2$ is the fundamental unit of F. Since $\varepsilon_5' < 0$, the subgroup of totally positive units is given by $\mathcal{O}_F^{\times,+} = \{\varepsilon_5^{2n} \mid n \in \mathbb{Z}\}$. Hence, we let $\varepsilon := \varepsilon_5^2 = (3+\sqrt{5})/2$ be the generator of $\mathcal{O}_F^{\times,+}$.

Since the roots of unity in E are the 10-th roots of unity $\mu_{10} = \{\pm\zeta_5^k \mid k = 0,1,2,3,4\}$, we have $\mathcal{O}_E^\times \cong \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}$ and $\mathcal{O}_F^\times \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. Hence, $\mu_{E/F} = [\mathcal{O}_E^\times : \mathcal{O}_F^\times] = 5$.

We computed the Shintani set

$$\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) = \left\{\frac{m}{5} + \frac{m}{5}\varepsilon \;\middle|\; m = 1,2,3,4\right\} = \left\{\left(\frac{5+\sqrt{5}}{10}\right)m \;\middle|\; m = 1,2,3,4\right\},$$

and the corresponding values of the quadratic ray class character $\chi_{E/F}$ modulo $\mathfrak{D}_{E/F}$,

$$\chi_{E/F}\left(\mathfrak{D}_{E/F}\left\langle\left(\tfrac{5+\sqrt{5}}{10}\right)m\right\rangle\right) = \chi_5(m) = \begin{cases} 1, & m = 1,4 \\ -1, & m = 2,3. \end{cases}$$

From the character values, we compute

$$\sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})\\ z=x+y\varepsilon}} \chi_{E/F}(\mathfrak{D}_{E/F}\langle z\rangle)B_2(x) = \sum_{m=1}^{4}\chi_5(m)B_2\left(\frac{m}{5}\right) = \frac{4}{25}.$$

Hence, $c = c(E/F) = \sqrt{5}/5$.

By substituting the preceding calculations in Theorem 1.1, we get

$$\exp[h_{\mathrm{Fal}}(J_C)] = \frac{1}{2\pi}\left(\frac{3+\sqrt{5}}{2}\right)^{\sqrt{5}/10} \prod_{m=1}^{4}\prod_{\pm}\Gamma_2((\tfrac{5\pm\sqrt{5}}{10})m,(1,\tfrac{3\pm\sqrt{5}}{2}))^{-\frac{5}{4}\chi_5(m)}.$$

Then expanding this expression yields the following result.

Corollary 2.2. *The Faltings height of the Jacobian J_C of the genus 2 curve $C: y^2 = x^5+1$ is given by*

$$\exp[h_{\mathrm{Fal}}(J_C)] = \frac{1}{2\pi}\left(\frac{3+\sqrt{5}}{2}\right)^{\sqrt{5}/10} \prod_{\pm} \frac{\Gamma_2(2\cdot(\frac{5\pm\sqrt{5}}{10}),(1,\frac{3\pm\sqrt{5}}{2}))^{5/4}}{\Gamma_2(1\cdot(\frac{5\pm\sqrt{5}}{10}),(1,\frac{3\pm\sqrt{5}}{2}))^{5/4}} \frac{\Gamma_2(3\cdot(\frac{5\pm\sqrt{5}}{10}),(1,\frac{3\pm\sqrt{5}}{2}))^{5/4}}{\Gamma_2(4\cdot(\frac{5\pm\sqrt{5}}{10}),(1,\frac{3\pm\sqrt{5}}{2}))^{5/4}}.$$

Numerically, the value of this Faltings height is $h_{\mathrm{Fal}}(J_C) \approx -3.2903$.

We now turn to our primary objective, which is to explicitly evaluate Faltings heights of Jacobians of genus 2 curves with complex multiplication by *non-abelian* quartic CM fields. According to Table 1, the non-abelian quartic CM fields whose corresponding Shintani sets have the least number of elements are

$$\mathbb{Q}\left(\sqrt{(-13+\sqrt{5})/2}\right) \quad \text{and} \quad \mathbb{Q}\left(\sqrt{-5-2\sqrt{2}}\right),$$

and therefore we work with these fields.

Example 2.3. Consider the field $E = \mathbb{Q}(\sqrt{\Delta})$ where $\Delta = (-13+\sqrt{5})/2 \ll 0$. Then E is a CM extension of the real quadratic field $F = \mathbb{Q}(\sqrt{5})$ with narrow class number 1 and $E/\mathbb{Q}$ is non-Galois with $\mathrm{Gal}(E^c/\mathbb{Q}) \cong D_4$. Define the genus 2 curve

$$C_a:\ y^2 = (-a+3)x^6 + (4a-8)x^5 + 10x^4 + (-a+20)x^3 + (4a+5)x^2 + (a+4)x + 1 \qquad (2.1)$$

where $a := \alpha^2 + 5$ and $\alpha = \pm\sqrt{\frac{-11+\sqrt{41}}{2}}$ is a root of $X^4 + 11X^2 + 20$. Then C_a is defined over the real quadratic field $\mathbb{Q}(\sqrt{41})$ and the Jacobian J_{C_a} is a CM abelian surface of type $(\mathcal{O}_E, \Phi)$ (see Appendix A, Example A.1).

Note that E has class number $h_E = 1$. Also, we have $d_E = 1025$ and $d_F = 5$. The relative discriminant is $\mathfrak{D}_{E/F} = \Delta\mathcal{O}_F = \langle\Delta\rangle$. Recall that $\varepsilon_5 := (1+\sqrt{5})/2$ is the fundamental unit of F and $\varepsilon := (3+\sqrt{5})/2$ is the generator of the group of totally positive units $\mathcal{O}_F^{\times,+}$.

Since the roots of unity in E are $\{\pm 1\}$, we have $\mathcal{O}_E^\times \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}$ and $\mathcal{O}_F^\times \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}$. Hence, $\mu_{E/F} = [\mathcal{O}_E^\times : \mathcal{O}_F^\times] = 1$.

We computed the Shintani set $\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$. This set can be visualized geometrically in $\mathbb{R}_+^2$ via the embedding $\alpha \longmapsto (\alpha, \alpha')$ as a finite subset of the Shintani cone

$$C(\varepsilon) := \left\{t_1(1,1) + t_2\left(\varepsilon, \varepsilon'\right) :\ t_1 > 0,\ t_2 \geq 0\right\} \subset \mathbb{R}_+^2,$$

as shown in Figure 1.[2]

[2]The shaded parallelogram in Figure 1 is the subset of the Shintani cone $C(\varepsilon)$ determined by the inequalities $0 < t_1 \leq 1$ and $0 \leq t_2 < 1$, which correspond to the inequalities appearing in the definition of $\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$.

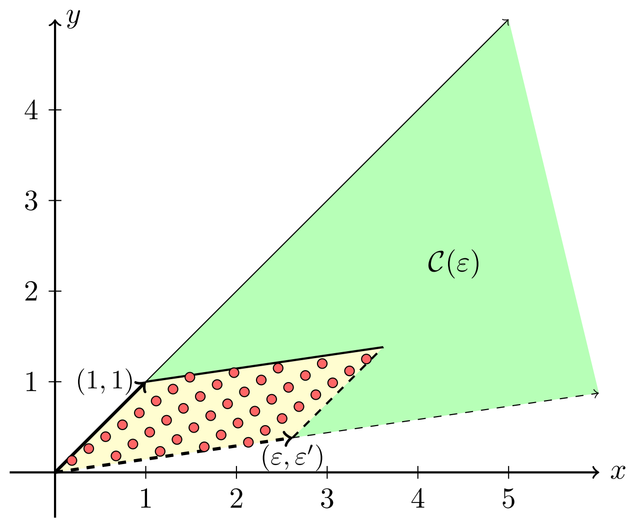

FIGURE 1. The embedding of $\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$ into $C(\varepsilon)$

In order to give a uniform description of the points in $\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$, it is convenient to express them in terms of a $\mathbb{Z}$-basis for $\mathfrak{D}_{E/F}^{-1}$. In particular, for the $\mathbb{Z}$-basis given by

$$\mathfrak{D}_{E/F}^{-1} = \mathbb{Z} \cdot 1 + \mathbb{Z}\left(\frac{13+\sqrt{5}}{82}\right),$$

we find that

$$\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) = \left\{ z_{m,n} := -m + (8m+n)\frac{13+\sqrt{5}}{82} \;\middle|\; 0 \le m \le 4,\ 1 \le n \le 8 \right\}.$$

We computed the corresponding values of the quadratic ray class character $\chi_{E/F}$ modulo $\mathfrak{D}_{E/F}$,

$$c_{m,n} := \chi_{E/F}\left(\mathfrak{D}_{E/F}\langle z_{m,n}\rangle\right) \in \{\pm 1\},$$

which are given in the following table.

Values of $c_{m,n}$								
m \ n	1	2	3	4	5	6	7	8
0	1	1	−1	1	1	−1	−1	1
1	1	1	−1	−1	−1	−1	−1	1
2	−1	1	−1	1	1	−1	1	−1
3	1	−1	−1	−1	−1	−1	1	1
4	1	−1	−1	1	1	−1	1	1

TABLE 2. The character values $c_{m,n} := \chi_{E/F}\left(\mathfrak{D}_{E/F}\langle z_{m,n}\rangle\right)$.

From the character values, we compute

$$\sum_{\substack{z \in \mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) \\ z = x + y\varepsilon}} \chi_{E/F}(\mathfrak{D}_{E/F}\langle z\rangle) B_2(x) = \sum_{\substack{0 \le m \le 4 \\ 1 \le n \le 8}} c_{m,n} B_2\left(\frac{-m+5n}{41}\right) = \frac{32}{41},$$

where we used

$$z_{m,n} = x + y\varepsilon = \frac{-m+5n}{41} + \frac{8m+n}{41}\varepsilon.$$

Hence, $c(E/F) = 8\sqrt{5}/41$.

By substituting the preceding calculations in Theorem 1.1, we get

$$\exp[h_{\mathrm{Fal}}(J_{C_a})] = \frac{1}{2\pi}\left(\frac{41}{5}\right)^{\frac{1}{4}}\left(\frac{3+\sqrt{5}}{2}\right)^{\frac{4\sqrt{5}}{41}} \prod_{\substack{0\le m\le 4\\ 1\le n\le 8}}\prod_{\pm}\Gamma_2(-m+(8m+n)\tfrac{13\pm\sqrt{5}}{82},(1,\tfrac{3\pm\sqrt{5}}{2}))^{-\frac{c_{m,n}}{4}}.$$

We have now proved the following result.

Corollary 2.4. *The Faltings height of the Jacobian J_{C_a} of the genus 2 curve C_a over $\mathbb{Q}(\sqrt{41})$ defined by (2.1) is given by*

$$\exp[h_{\mathrm{Fal}}(J_{C_a})] = \frac{1}{2\pi}\left(\frac{41}{5}\right)^{\frac{1}{4}}\left(\frac{3+\sqrt{5}}{2}\right)^{\frac{4\sqrt{5}}{41}} \prod_{\substack{0\le m\le 4\\ 1\le n\le 8}}\prod_{\pm}\Gamma_2(-m+(8m+n)\tfrac{13\pm\sqrt{5}}{82},(1,\tfrac{3\pm\sqrt{5}}{2}))^{-\frac{c_{m,n}}{4}},$$

where the numbers $c_{m,n} \in \{\pm 1\}$ are given in Table 2. Numerically, the value of this Faltings height is approximately $h_{\mathrm{Fal}}(J_{C_a}) \approx -3.102$.

Example 2.5. Consider the field $E = \mathbb{Q}(\sqrt{\Delta})$ where $\Delta = -5-2\sqrt{2} \ll 0$. Then E is a CM extension of the real quadratic field $F = \mathbb{Q}(\sqrt{2})$ with narrow class number 1 and $E/\mathbb{Q}$ is non-Galois with $\mathrm{Gal}(E^c/\mathbb{Q}) \cong D_4$. Define the genus 2 curve

$$C_a:\ y^2 = x^6 + (2a+4)x^5 + (3a+14)x^4 + (10a+8)x^3 + (-9a+32)x^2 + (16a-16)x - 4a + 1 \tag{2.2}$$

where $a := \alpha^2 + 2$ and $\alpha = \pm\sqrt{\frac{-5\pm\sqrt{17}}{2}}$ is a root of $X^4 + 5X^2 + 2$. Then C_a is defined over the real quadratic field $\mathbb{Q}(\sqrt{17})$ and the Jacobian J_{C_a} is a CM abelian surface of type $(\mathcal{O}_E, \Phi)$ (see Appendix A, Example A.2).

Note that E has class number $h_E = 1$. Also, we have $d_E = 1088$ and $d_F = 8$. The relative discriminant is $\mathfrak{D}_{E/F} = \Delta\mathcal{O}_F = \langle\Delta\rangle$. The fundamental unit of F is $\varepsilon_5 := 1+\sqrt{2}$ and $\varepsilon := 3+2\sqrt{2}$ is the generator of the group of totally positive units $\mathcal{O}_F^{\times,+}$.

Since the roots of unity in E are $\{\pm 1\}$, we have $\mathcal{O}_E^{\times} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ and $\mathcal{O}_F^{\times} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. Hence, $\mu_{E/F} = [\mathcal{O}_E^{\times} : \mathcal{O}_F^{\times}] = 1$.

We computed the Shintani set $\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$. This set can be visualized geometrically in $\mathbb{R}_+^2$ via the embedding $\alpha \longmapsto (\alpha, \alpha')$ as a finite subset of the Shintani cone

$$C(\varepsilon) := \left\{t_1(1,1) + t_2\left(\varepsilon, \varepsilon'\right):\ t_1 > 0,\ t_2 \ge 0\right\} \subset \mathbb{R}_+^2,$$

as shown in Figure 2.

In order to give a uniform description of the points in $\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$, it is convenient to express them in terms of a $\mathbb{Z}$-basis for $\mathfrak{D}_{E/F}^{-1}$. In particular, for the $\mathbb{Z}$-basis given by

$$\mathfrak{D}_{E/F}^{-1} = \mathbb{Z}\cdot 1 + \mathbb{Z}\left(\frac{6+\sqrt{2}}{17}\right),$$

we find that

$$\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) = \left\{ z_{m,n} := -m + (4m+n-1)\frac{6+\sqrt{2}}{17} \;\middle|\; 0 \le m \le 8,\ n \in S(m)\right\},$$

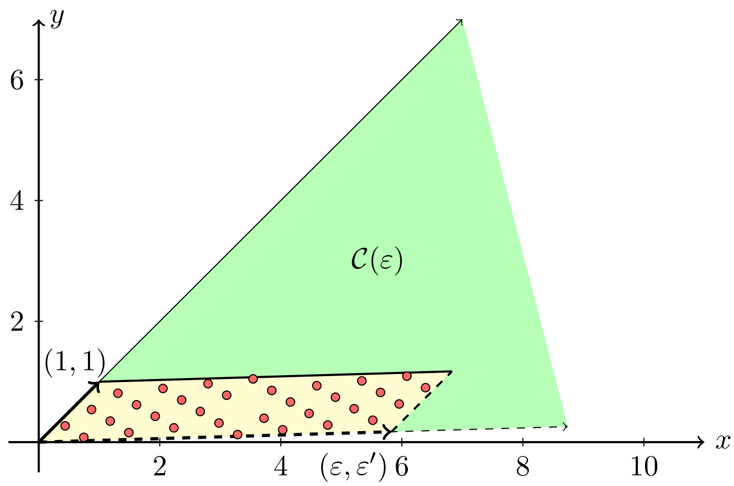

FIGURE 2. The embedding of $R(\varepsilon, \mathfrak{D}_{E/F}^{-1})$ into $C(\varepsilon)$.

where

$$S(m) := \begin{cases} \{2,3,4\} & \text{if } m = 0 \\ \{1,2,3,4\} & \text{if } m = 1,2,3 \\ \{1,3\} & \text{if } m = 4 \\ \{0,1,2,3\} & \text{if } m = 5,6,7 \\ \{0,1,2\} & \text{if } m = 8. \end{cases}$$

We computed the corresponding values of the quadratic ray class character $\chi_{E/F}$ modulo $\mathfrak{D}_{E/F}$,

$$c_{m,n} := \chi_{E/F}\left(\mathfrak{D}_{E/F}\langle z_{m,n}\rangle\right) \in \{\pm 1\},$$

which are given in the following table.

Values of $c_{m,n}$									
n \ m	0	1	2	3	4	5	6	7	8
0						1	-1	-1	1
1		1	1	-1	1	-1	-1	-1	-1
2	1	-1	1	1		1	1	-1	1
3	1	-1	-1	-1	1	-1	1	1	
4	-1	-1	-1	1					

TABLE 3. The character values $c_{m,n} := \chi_{E/F}(\mathfrak{D}_{E/F}\langle z_{m,n}\rangle)$.

From the character values, we compute

$$\sum_{\substack{z \in \mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) \\ z = x + y\varepsilon}} \chi_{E/F}(\mathfrak{D}_{E/F}\langle z\rangle) B_2(x) = \sum_{\substack{0 \le m \le 8 \\ n \in S(m)}} c_{m,n} B_2\left(\frac{2m + 9n - 9}{34}\right) = \frac{4}{17},$$

where we used

$$z_{m,n} = x + y\varepsilon = \frac{2m + 9n - 9}{34} + \frac{4m - n + 1}{34}\varepsilon.$$

Hence, $c(E/F) = 4\sqrt{2}/17$.

Substituting the preceding calculations in Theorem 1.1 yields

$$\exp[h_{\mathrm{Fal}}(J_{C_a})] = \frac{1}{2\pi}\left(\frac{17}{8}\right)^{\frac{1}{4}}(3+2\sqrt{2})^{\frac{2\sqrt{2}}{17}} \prod_{\substack{0\le m\le 4\\ n\in S(m)}}\prod_{\pm}\Gamma_2(-m+(4m+n-1)\tfrac{6\pm\sqrt{2}}{17},(1,3\pm 2\sqrt{2}))^{-\frac{c_{m,n}}{4}}.$$

We have now proved the following result.

Corollary 2.6. *The Faltings height of the Jacobian J_{C_a} of the genus 2 curve C_a over $\mathbb{Q}(\sqrt{17})$ defined by (2.2) is given by*

$$\exp[h_{\mathrm{Fal}}(J_{C_a})] = \frac{1}{2\pi}\left(\frac{17}{8}\right)^{\frac{1}{4}}(3+2\sqrt{2})^{\frac{2\sqrt{2}}{17}} \prod_{\substack{0\le m\le 4\\ n\in S(m)}}\prod_{\pm}\Gamma_2(-m+(4m+n-1)\tfrac{6\pm\sqrt{2}}{17},(1,3\pm 2\sqrt{2}))^{-\frac{c_{m,n}}{4}},$$

where the numbers $c_{m,n} \in \{\pm 1\}$ are given in Table 3. Numerically, the value of this Faltings height is approximately $h_{\mathrm{Fal}}(J_{C_a}) \approx -3.033$.

3. Ray class characters and L–functions

In this section, we review some facts we will need regarding Hecke Grössencharacters and narrow ray class characters, following [Neu99, Section VII.6]. Let K be a number field of degree $n = r_1+2r_2$, where r_1 is the number of real embeddings of K and r_2 is the number of pairs of complex conjugate embeddings of K. Let $\mathcal{O}_K$ be the ring of integers of K. Let $\sigma_t : K \hookrightarrow \mathbb{R}$, $t = 1, \ldots, r_1$ be the real embeddings, and let $\sigma_t : K \hookrightarrow \mathbb{C}$, $t = r_1 + 1, \ldots, r_1 + r_2$ be a fixed choice of complex embeddings such that the set of all complex embeddings is given by the σ_t and their conjugates $\overline{\sigma_t}$. We define the Minkoswki space $K_{\mathbb{R}} := \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong K \otimes_{\mathbb{Q}} \mathbb{R}$. Using the fixed choice of embeddings, we define a map $j : K \longrightarrow K_{\mathbb{R}}$ by

$$j(\alpha) := (\sigma_1(\alpha), \ldots, \sigma_{r_1}(\alpha), \sigma_{r_1+1}(\alpha), \ldots, \sigma_{r_1+r_2}(\alpha)).$$

Let $\mathfrak{m}$ be an integral ideal of K and $\mathcal{I}_K(\mathfrak{m})$ be the group of fractional ideals of K that are relatively prime to $\mathfrak{m}$. A *Grössencharacter modulo* $\mathfrak{m}$ is a character $\chi : \mathcal{I}_K(\mathfrak{m}) \longrightarrow \mathbb{S}^1$ for which there exists a pair of characters

$$\chi_{\mathrm{f}} : (\mathcal{O}_K/\mathfrak{m})^{\times} \longrightarrow \mathbb{S}^1, \quad \chi_{\infty} : K_{\mathbb{R}}^{\times} \longrightarrow \mathbb{S}^1,$$

such that for every $\alpha \in \mathcal{O}_K$ with $(\alpha, \mathfrak{m}) = 1$ we have

$$\chi(\langle\alpha\rangle) = \chi_{\mathrm{f}}(\alpha)\chi_{\infty}(\alpha).$$

Here when we write $\chi_{\infty}(\alpha)$, we identify α with its image $j(\alpha) \in K_{\mathbb{R}}$ to ease notation. The characters χ_{f} and χ_{∞} are called the finite and infinite parts of χ, respectively.

The *conductor* of a Grössencharakter χ modulo $\mathfrak{m}$ is the smallest integral ideal $\mathfrak{f}_{\chi}$ such that $\mathfrak{f}_{\chi}|\mathfrak{m}$ and χ is the restriction of a Grössencharakter χ' modulo $\mathfrak{f}_{\chi}$. A Grössencharakter χ modulo $\mathfrak{m}$ is *primitive* if it is not the restriction of a Grössencharakter χ' modulo $\mathfrak{m}'$ for any proper divisor $\mathfrak{m}'|\mathfrak{m}$ (in other words, the conductor is $\mathfrak{f}_{\chi} = \mathfrak{m}$).

A character $\lambda : K_{\mathbb{R}}^{\times} \longrightarrow \mathbb{S}^1$ can be described explicitly as follows. Let $\mathbf{x} = (x_t) \in K_{\mathbb{R}}$ and $\mathbf{e} = (e_t) \in \mathbb{C}^{r_1+r_2}$. We define $\mathbf{x}^{\mathbf{e}} := (x_t^{e_t})$, $|\mathbf{x}| := (|x_t|)$, and $N(\mathbf{x}) := \prod_t x_t$. An *admissible vector* is a vector $\mathbf{p} = (p_t) \in \mathbb{Z}^{r_1+r_2}$ such that $p_t \in \{0, 1\}$ for $t = 1, \ldots, r_1$. Given a character $\lambda : K_{\mathbb{R}}^{\times} \longrightarrow \mathbb{S}^1$, there exist uniquely determined vectors $\mathbf{p}, \mathbf{q}$ with $\mathbf{p} \in \mathbb{Z}^{r_1+r_2}$ admissible and $\mathbf{q} \in \mathbb{R}^{r_1+r_2}$ such that for every $\mathbf{x} \in K_{\mathbb{R}}^{\times}$, we have

$$\lambda(\mathbf{x}) = N(\mathbf{x}^{\mathbf{p}}|\mathbf{x}|^{-\mathbf{p}+i\mathbf{q}})$$

where

$$\mathbf{x}^{\mathbf{p}}|\mathbf{x}|^{-\mathbf{p}+i\mathbf{q}} := \left(x_t^{p_t}|x_t|^{-p_t+iq_t}\right) = \left(\operatorname{sign}(x_t)^{p_t}|x_t|^{iq_t}\right).$$

If a Grössencharacter χ modulo $\mathfrak{m}$ has infinite part $\chi_\infty : K_{\mathbb{R}}^\times \longrightarrow \mathbb{S}^1$ given by

$$\chi_\infty(\mathbf{x}) = N(\mathbf{x}^{\mathbf{p}}|\mathbf{x}|^{-\mathbf{p}+i\mathbf{q}}),$$

we say that χ is of type $(\mathbf{p},\mathbf{q})$.

We now define the narrow ray class characters modulo $\mathfrak{m}$. Let $\mathcal{P}_K^+(\mathfrak{m}) < \mathcal{I}_K(\mathfrak{m})$ be the subgroup of principal fractional ideals

$$\mathcal{P}_K^+(\mathfrak{m}) = \{\langle\alpha\rangle \in \mathcal{I}_K(\mathfrak{m}) \mid \alpha \in K,\, \alpha \gg 0,\, \alpha \overset{\times}{\equiv} 1 \pmod{\mathfrak{m}}\},$$

where the congruence $\alpha \overset{\times}{\equiv} 1 \pmod{\mathfrak{m}}$ means that if $\alpha = \beta/\gamma$ with $\beta, \gamma \in \mathcal{O}_K$ relatively prime to $\mathfrak{m}$, then $\beta \equiv \gamma \pmod{\mathfrak{m}}$. The *narrow ray class group modulo* $\mathfrak{m}$ is the quotient group

$$\mathcal{R}_K^+(\mathfrak{m}) := \mathcal{I}_K(\mathfrak{m})/\mathcal{P}_K^+(\mathfrak{m}).$$

A *narrow ray class character modulo* $\mathfrak{m}$ is a character $\widetilde{\chi} : \mathcal{R}_K^+(\mathfrak{m}) \longrightarrow \mathbb{S}^1$ of the narrow ray class group, or equivalently, a character $\widetilde{\chi} : \mathcal{I}_K(\mathfrak{m}) \longrightarrow \mathbb{S}^1$ such that $\widetilde{\chi}(\mathcal{P}_K^+(\mathfrak{m})) = \{1\}$. The *conductor* of $\widetilde{\chi}$ is the smallest integral ideal $\mathfrak{f}_\chi|\mathfrak{m}$ such that $\widetilde{\chi}$ factors through $\mathcal{R}_K^+(\mathfrak{f}_\chi)$.

Given a narrow ray class character $\widetilde{\chi}$ modulo $\mathfrak{m}$, there exists a Grössencharacter χ modulo $\mathfrak{m}$ of type $(\mathbf{p}, \mathbf{0})$ for some admissible vector $\mathbf{p} = (p_t) \in \mathbb{Z}^{r_1+r_2}$ with $p_t = 0$ for $t = r_1+1, \dots, r_1+r_2$, such that $\widetilde{\chi}(\mathfrak{a}) = \chi(\mathfrak{a})$ for every fractional ideal $\mathfrak{a} \in \mathcal{I}_K(\mathfrak{m})$ (see [Neu99, Chapter VII, Proposition 6.9]) In particular, for $\alpha \in \mathcal{O}_K$ relatively prime to $\mathfrak{m}$, we have

$$\widetilde{\chi}(\langle\alpha\rangle) = \chi_{\mathrm{f}}(\alpha) N\left(\left(\frac{\alpha}{|\alpha|}\right)^{\mathbf{p}}\right)$$

where $\chi_{\mathrm{f}} : (\mathcal{O}_K/\mathfrak{m})^\times \longrightarrow \mathbb{S}^1$ is the finite part of χ.

Assume now that K is a real quadratic field, so that $r_1 = 2$, $r_2 = 0$, $\sigma_1 = \mathrm{id}$, and σ_2 is the nontrivial embedding. The four possibilities for the admissible vector $\mathbf{p} \in \mathbb{Z}^2$ are $(0,0), (1,0), (0,1)$, and $(1,1)$. In particular, on principal integral ideals $\langle\alpha\rangle$ coprime to $\mathfrak{m}$, the narrow ray class characters modulo $\mathfrak{m}$ take one of the following forms:

$$\begin{aligned}
\widetilde{\chi}(\langle\alpha\rangle) &= \chi_{\mathrm{f}}(\alpha) & \mathbf{p} &= (0,0),\\
\widetilde{\chi}(\langle\alpha\rangle) &= \chi_{\mathrm{f}}(\alpha)\operatorname{sign}(\alpha) & \mathbf{p} &= (1,0),\\
\widetilde{\chi}(\langle\alpha\rangle) &= \chi_{\mathrm{f}}(\alpha)\operatorname{sign}(\alpha') & \mathbf{p} &= (0,1),\\
\widetilde{\chi}(\langle\alpha\rangle) &= \chi_{\mathrm{f}}(\alpha)\operatorname{sign}(N_{K/\mathbb{Q}}(\alpha)) & \mathbf{p} &= (1,1).
\end{aligned}$$

The L–function of a narrow ray class character χ modulo $\mathfrak{m}$ is defined by

$$L(\chi, s) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\chi(\mathfrak{a})}{N_{K/\mathbb{Q}}(\mathfrak{a})^s}, \quad \mathrm{Re}(s) > 1$$

where $N_{K/\mathbb{Q}}(\mathfrak{a})$ denotes the norm and the sum is over non-zero integral ideals (here we define $\chi(\mathfrak{a}) = 0$ when $\mathfrak{a}$ is not relatively prime to $\mathfrak{m}$). If the admissible vector corresponding to χ is $\mathbf{p} = (p_1, p_2)$, then the completed L–function is given by (see [Neu99, pp. 496-497])

$$\Lambda(\chi, s) := (d_K N_{K/\mathbb{Q}}(\mathfrak{m}))^{s/2}\pi^{-(2s+p_1+p_2)/2}\Gamma\left(\frac{s+p_1}{2}\right)\Gamma\left(\frac{s+p_2}{2}\right)L(\chi, s),$$

where d_K is the absolute value of the discriminant of K. If χ is primitive, the completed L–function satisfies the functional equation

$$\Lambda(\chi, s) = W(\chi)\Lambda(\overline{\chi}, 1-s), \tag{3.1}$$

where $W(\chi)$ is an explicit complex number of absolute value 1 (see [Neu99, Chapter VII, Theorem 8.5]).

Remark 3.1. A calculation with the functional equation shows that if χ has admissible vector $\mathbf{p} = (p_1, p_2)$, then the order of vanishing of $L(\chi, s)$ at $s = 0$ is $2 - (p_1 + p_2)$.

4. The combinatorics of Shintani sets

In this section, we undertake an extensive study of the combinatorial structure of certain finite sets of algebraic numbers which arise in Shintani's decomposition of ray class L–functions for real quadratic fields as finite linear combinations of Shintani zeta functions. We call these sets "Shintani sets". We will prove an explicit formula for the size of Shintani sets. Moreover, we will use this formula to prove certain orthogonality relations for ray class characters with respect to Shintani sets.

4.1. **Notation.** The following notation and assumptions will remain fixed throughout this section.

Let F be a real quadratic field with narrow class number 1. Then $F = \mathbb{Q}(\sqrt{p})$ for a prime p with $p = 2$ or $p \equiv 1 \pmod 4$. Let $\mathcal{O}_F$ be the ring of integers, $\mathcal{O}_F^\times$ be the group of units, and $\varepsilon_p > 1$ be the fundamental unit of F.

For a subset $S \subseteq F$, let S^+ denote the corresponding subset of totally positive elements of S. Since $\mathcal{O}_F^\times = \{\pm\varepsilon_p^n \mid n \in \mathbb{Z}\}$, we have $\mathcal{O}_F^{\times,+} = \{\varepsilon_p^n \mid n \in \mathbb{Z}\}$ if $\varepsilon_p \gg 0$ and $\mathcal{O}_F^{\times,+} = \{\varepsilon_p^{2n} \mid n \in \mathbb{Z}\}$ if $\varepsilon_p' < 0$, where ε_p' denotes the conjugate of ε_p. In particular, if we define $\varepsilon := \varepsilon_p$ if $\varepsilon_p \gg 0$ and $\varepsilon := \varepsilon_p^2$ if $\varepsilon_p' < 0$, then ε generates $\mathcal{O}_F^{\times,+}$.

Let $\mathfrak{f} \subset \mathcal{O}_F$ be an integral ideal of F. Then the *Shintani set associated to* $\mathfrak{f}$ is defined by

$$\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) := \left\{ z = x + y\varepsilon \in \mathfrak{f}^{-1} \;\middle|\; x, y \in \mathbb{Q},\ 0 < x \leq 1,\ 0 \leq y < 1 \right\}.$$

Similarly, the *restricted Shintani set associated to* $\mathfrak{f}$ is defined by

$$\mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) := \left\{ z = x + y\varepsilon \in \mathfrak{f}^{-1} \;\middle|\; x, y \in \mathbb{Q},\ 0 < x \leq 1,\ 0 \leq y < 1,\ \mathfrak{f}\langle z\rangle \text{ coprime to } \mathfrak{f} \right\}.$$

For brevity, we will simply refer to both of these sets as Shintani sets.

The set $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ is finite. To see this, let $\iota(x) := (x, x')$ denote the embedding of $x \in F$ into $\mathbb{R}^2$. Then $\iota(\mathfrak{f}^{-1})$ is a lattice in $\mathbb{R}^2$, and $\iota(\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})) \subset \iota(\mathfrak{f}^{-1})$ is the subset consisting of points (u, v) of the form

$$(u, v) = (x + y\varepsilon, x + y\varepsilon') \in \mathbb{R}^2$$

where $0 < u < 1 + \varepsilon$, $0 < v < 1 + \varepsilon'$. Since $\iota(\mathfrak{f}^{-1})$ is discrete, it follows that $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ is finite.

4.2. **Main results.** The main result of this section is the following theorem, which gives an explicit formula for the size of the Shintani sets.

Theorem 4.1. *Let $\mathfrak{f}$ be a non-zero integral ideal of F. Then the sizes of the associated Shintani sets are given by*

$$\#\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) = \frac{\operatorname{vol}(\mathbb{R}^2/\iota(\mathbb{Z} + \mathbb{Z}\varepsilon))}{\operatorname{vol}(\mathcal{O}_F)} \cdot N_{F/\mathbb{Q}}(\mathfrak{f}) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}} \cdot N_{F/\mathbb{Q}}(\mathfrak{f})$$

and

$$\#\mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) = \frac{\operatorname{vol}(\mathbb{R}^2/\iota(\mathbb{Z} + \mathbb{Z}\varepsilon))}{\operatorname{vol}(\mathcal{O}_F)} \cdot \varphi(\mathfrak{f}) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}} \cdot \varphi(\mathfrak{f}),$$

where

$$\varphi(\mathfrak{f}) := N_{F/\mathbb{Q}}(\mathfrak{f}) \cdot \prod_{\mathfrak{p}|\mathfrak{f}} \left(1 - \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{p})}\right)$$

is the generalized Euler φ-function for number fields.

We will also prove the following "orthogonality relations" for any narrow ray class character χ modulo $\mathfrak{f}$.

Theorem 4.2. *Let $\mathfrak{f}$ be an integral ideal of F and χ be a narrow ray class character modulo $\mathfrak{f}$. Then we have*

$$\sum_{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})} \chi(\mathfrak{f}\langle z\rangle) = \begin{cases} 0, & \chi \neq 1 \\ \dfrac{\varepsilon-\varepsilon'}{\sqrt{d_F}}\cdot\varphi(\mathfrak{f}), & \chi = 1. \end{cases}$$

4.3. **A group structure on $\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})$.** In order to prove Theorems 4.1 and 4.2, we will introduce a binary operation on the Shintani set $\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})$ which gives it the structure of a finite abelian group.

We will need the following modified fractional part functions. For $x\in\mathbb{R}$, define $\{x\}_{(0,1]}$ and $\{x\}_{[0,1)}$ by

$$\{x\}_{(0,1]} := \begin{cases} \{x\}, & \text{if } x\notin\mathbb{Z} \\ 1, & \text{if } x\in\mathbb{Z} \end{cases} \quad\text{and}\quad \{x\}_{[0,1)} := \{x\},$$

where $\{x\}$ is the usual fractional part of a real number x.

Now, since $\mathfrak{f}$ is a nonzero integral ideal of F, then $\mathfrak{f}^{-1}$ contains the ring of integers $\mathcal{O}_F$, so that we have the chain of abelian groups (under addition)

$$\mathbb{Z}+\mathbb{Z}\varepsilon \subseteq \mathcal{O}_F \subseteq \mathfrak{f}^{-1}.$$

In the following proposition, we relate the Shintani set $\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})$ to the quotient group

$$\mathfrak{f}^{-1}\Big/\mathbb{Z}+\mathbb{Z}\varepsilon.$$

Proposition 4.3. *For each $w\in\mathfrak{f}^{-1}$ there is a unique element $z_w\in\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})$ such that*

$$w - z_w \in \mathbb{Z}+\mathbb{Z}\varepsilon.$$

Hence, the Shintani set $\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})$ is a complete set of coset representatives for the quotient group

$$\mathfrak{f}^{-1}\Big/\mathbb{Z}+\mathbb{Z}\varepsilon.$$

Proof. Let $w\in\mathfrak{f}^{-1}$. Since $\{1,\varepsilon\}$ is a $\mathbb{Q}$-basis for F, then we can write $w = X+Y\varepsilon$ for some unique $X,Y\in\mathbb{Q}$. Now, define $z_w = x+y\varepsilon := \{X\}_{(0,1]}+\{Y\}_{[0,1)}\varepsilon$. Then

$$w = \begin{cases} z_w + \lfloor X\rfloor + \lfloor Y\rfloor\varepsilon & \text{if } X\notin\mathbb{Z} \\ z_w + \lfloor X\rfloor - 1 + \lfloor Y\rfloor\varepsilon & \text{if } X\in\mathbb{Z}. \end{cases}$$

Since $w\in\mathfrak{f}^{-1}$ and both $\lfloor X\rfloor+\lfloor Y\rfloor\varepsilon\in\mathfrak{f}^{-1}$ and $\lfloor X\rfloor-1+\lfloor Y\rfloor\varepsilon\in\mathfrak{f}^{-1}$, we see that $z_w\in\mathfrak{f}^{-1}$. By construction, $0<x\le 1$ and $0\le y<1$, so we conclude that $z_w = x+y\varepsilon\in\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})$.

To see that this element is unique, suppose that $\widetilde{z_w} = \widetilde{x}+\widetilde{y}\varepsilon$ is another such element. Then

$$z_w - \widetilde{z_w} = (x-\widetilde{x}) + (y-\widetilde{y})\varepsilon \in \mathbb{Z}+\mathbb{Z}\varepsilon.$$

By the uniqueness of the representation of an element of F as a $\mathbb{Q}$-linear combination of the basis elements $\{1,\varepsilon\}$, we see that $x-\widetilde{x}\in\mathbb{Z}$ and $y-\widetilde{y}\in\mathbb{Z}$. Then, since $-1<x-\widetilde{x}<1$ and $-1<y-\widetilde{y}<1$, we conclude that $x-\widetilde{x} = y-\widetilde{y} = 0$. Thus $z_w = \widetilde{z_w}$, and this proves uniqueness. □

Let

$$\oplus : \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) \times \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) \longrightarrow \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$$

be the binary operation defined as follows. For $z_1, z_2 \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$, by Proposition 4.3 we let $z_1 \oplus z_2$ be the unique element of $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ such that

$$z_1 + z_2 - z_1 \oplus z_2 \in \mathbb{Z} + \mathbb{Z}\varepsilon.$$

Hence $z_1 \oplus z_2$ is the unique element of $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ such that

$$(z_1 \oplus z_2) + \mathbb{Z} + \mathbb{Z}\varepsilon = (z_1 + \mathbb{Z} + \mathbb{Z}\varepsilon) + (z_2 + \mathbb{Z} + \mathbb{Z}\varepsilon).$$

Remark 4.4. Observe that if $z_1 = x_1 + y_1\varepsilon \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ and $z_2 = x_2 + y_2\varepsilon \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$, then

$$z_1 \oplus z_2 = \{x_1 + x_2\}_{(0,1]} + \{y_1 + y_2\}_{[0,1)}\varepsilon.$$

Proposition 4.5. *The Shintani set $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ is a finite abelian group with respect to the binary operation $\oplus$.*

Proof. First, observe that $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) \neq \varnothing$ since $1 \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$. Now, the binary operation $\oplus$ is clearly commutative, so we proceed to prove associativity, existence of a neutral element, and existence of inverses.

Associativity: Let $z_1, z_2, z_3 \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$. Then as an immediate consequence of the definition of $\oplus$, there are integers $m, n, \widetilde{m}, \widetilde{n} \in \mathbb{Z}$ such that

$$z_1 + z_2 + z_3 - (z_1 \oplus z_2) \oplus z_3 = m + n\varepsilon \in \mathbb{Z} + \mathbb{Z}\varepsilon$$

and

$$z_1 + z_2 + z_3 - z_1 \oplus (z_2 \oplus z_3) = \widetilde{m} + \widetilde{n}\varepsilon \in \mathbb{Z} + \mathbb{Z}\varepsilon.$$

Therefore, by the uniqueness part of Proposition 4.3, it follows that

$$(z_1 \oplus z_2) \oplus z_3 = z_1 \oplus (z_2 \oplus z_3).$$

Neutral element: We show that $1 \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ is the neutral element for the operation $\oplus$. Let $z \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$. Then since

$$z + 1 - z = 1 \in \mathbb{Z} + \mathbb{Z}\varepsilon,$$

we see by uniqueness that $z \oplus 1 = z$.

Existence of inverses: Let $z \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$. By Proposition 4.3, we define $\ominus z$ to be the unique element of $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ such that

$$1 - z - \ominus z \in \mathbb{Z} + \mathbb{Z}\varepsilon.$$

This immediately implies that

$$z + \ominus z - 1 \in \mathbb{Z} + \mathbb{Z}\varepsilon,$$

so again by uniqueness we have $z \oplus (\ominus z) = 1$. This completes the proof that $(\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}), \oplus)$ is a finite abelian group. ☐

Now, since F has narrow class number 1, we can write $\mathfrak{f} = \langle \alpha \rangle$ for some $\alpha \in \mathcal{O}_F$ with $\alpha \gg 0$, and hence

$$\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) = \left\{ z = x + y\varepsilon \in \frac{1}{\alpha}\mathcal{O}_F \;\middle|\; x, y \in \mathbb{Q},\ 0 < x \leq 1,\ 0 \leq y < 1 \right\}.$$

Note that if $z \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$, then $\alpha z \in \mathcal{O}_F$. We then define the *projection map*

$$\pi_\alpha : \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) \longrightarrow \mathcal{O}_F/\mathfrak{f}$$

by $\pi_\alpha(z) := \alpha z + \mathfrak{f}$.

Lemma 4.6. *The projection map*

$$\pi_\alpha : \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) \longrightarrow \mathcal{O}_F/\mathfrak{f}$$

is a surjective group homomorphism. Moreover, we have

$$\#\pi_\alpha^{-1}(w) = \#\ker(\pi_\alpha)$$

for all $w \in \mathcal{O}_F/\mathfrak{f}$.

Proof. We first show that π_α is a group homomorphism. Let $z_1, z_2 \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$. Then there are integers $m, n \in \mathbb{Z}$ such that

$$z_1 + z_2 = z_1 \oplus z_2 + m + n\varepsilon.$$

Hence $z_1 \oplus z_2 = z_1 + z_2 - (m + n\varepsilon)$, and we have

$$\begin{aligned} \pi_\alpha(z_1 \oplus z_2) &= \pi_\alpha(z_1 + z_2 - (m + n\varepsilon)) \\ &= \alpha(z_1 + z_2 - (m + n\varepsilon)) + \mathfrak{f} \\ &= \alpha z_1 + \alpha z_2 + \mathfrak{f} \\ &= \pi_\alpha(z_1) + \pi_\alpha(z_2). \end{aligned}$$

Next, we show that π_α is surjective. Let $w = \beta + \mathfrak{f} \in \mathcal{O}_F/\mathfrak{f}$, where $\beta \in \mathcal{O}_F$. Then since $\beta/\alpha \in \mathfrak{f}^{-1}$, by Proposition 4.3 there is a unique element $z_{\beta/\alpha} \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ such that $\beta/\alpha - z_{\beta/\alpha} \in \mathbb{Z} + \mathbb{Z}\varepsilon$. It follows that $z_{\beta/\alpha} = \beta/\alpha + m + n\varepsilon$ for some integers $m, n \in \mathbb{Z}$, and thus $\pi_\alpha(z_{\beta/\alpha}) = \beta + \mathfrak{f} = w$.

Finally, we note that by the first isomorphism theorem for groups, we have

$$\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})/\ker(\pi_\alpha) \cong \mathcal{O}_F/\mathfrak{f}.$$

Hence if $w \in \mathcal{O}_F/\mathfrak{f}$, then the size of the fiber $\pi_\alpha^{-1}(w)$ is given by $\#\pi_\alpha^{-1}(w) = \#\ker(\pi_\alpha)$. $\square$

4.4. **Pick's theorem and the size of** $\ker(\pi_\alpha)$**.** To prove Theorem 4.1, we will need an explicit formula for $\#\ker(\pi_\alpha)$. Here we prove such a formula by translating the problem of determining $\#\ker(\pi_\alpha)$ to a lattice point counting problem and employing a remarkable theorem of Pick (see e.g. [SS09, Section 3.5]) which gives a formula for the area of a lattice polygon in $\mathbb{R}^2$ in terms of the number of lattice points inside and on the boundary of the polygon.

Recall that $\mathfrak{f} = \alpha\mathcal{O}_F$. Hence for a point $z \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$, we have

$$z \in \ker(\pi_\alpha) \iff \alpha z \in \alpha\mathcal{O}_F \iff z \in \alpha^{-1}\alpha\mathcal{O}_F = \mathcal{O}_F.$$

Thus

$$\ker(\pi_\alpha) = \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) \cap \mathcal{O}_F = \{z = x + y\varepsilon \in \mathcal{O}_F \mid x, y \in \mathbb{Q},\ 0 < x \leq 1,\ 0 \leq y < 1\}.$$

Using the embedding $\iota : F \longrightarrow \mathbb{R}^2$ of F into the Minkowski space $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^2$ given by $\iota(x) := (x, x')$, we see that geometrically, the kernel $\ker(\pi_\alpha)$ is the set of lattice points of $\iota(\mathcal{O}_F)$ that lie in the half-open parallelogram

$$P_F := \{t_1(1,1) + t_2(\varepsilon, \varepsilon') \mid t_1, t_2 \in \mathbb{R},\ 0 < t_1 \leq 1,\ 0 \leq t_2 < 1\}.$$

In particular, this means that

$$\#\ker(\pi_\alpha) = \#(P_F \cap \iota(\mathcal{O}_F)).$$

Note this already shows that $\#\ker(\pi_\alpha)$ only depends on the base field F, and not on the ideal $\mathfrak{f} = \alpha\mathcal{O}_F$.

In the next proposition, we will make use of Pick's theorem to prove an explicit formula for $\#(P_F \cap \iota(\mathcal{O}_F))$.

Proposition 4.7. *We have*

$$\#\ker(\pi_\alpha) = \#(P_F \cap \iota(\mathcal{O}_F)) = \frac{\mathrm{vol}(\mathbb{R}^2/\iota(\mathbb{Z}+\mathbb{Z}\varepsilon))}{\mathrm{vol}(\mathcal{O}_F)} = \frac{\varepsilon-\varepsilon'}{\sqrt{d_F}}.$$

Proof. Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation mapping $\iota(\mathcal{O}_F)$ onto $\mathbb{Z}^2$. For example, if we fix the integral basis $\{1, \frac{d_F+\sqrt{d_F}}{2}\}$ for $\mathcal{O}_F$, such a linear transformation is given by

$$L(x,y) = -\frac{1}{\sqrt{d_F}}\begin{bmatrix} \frac{d_F-\sqrt{d_F}}{2} & -\frac{d_F+\sqrt{d_F}}{2} \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Since interior (resp. boundary) points of $\overline{P_F}$ are mapped to interior (resp. boundary) points of $\overline{L(P_F)}$, it follows that

$$\#(\overline{P_F} \cap \iota(\mathcal{O}_F)) = \#(\overline{L(P_F)} \cap \mathbb{Z}^2). \tag{4.1}$$

Now, recall that a polygon P in $\mathbb{R}^2$ whose vertices have integer coordinates is called a *lattice polygon*. If P is a lattice polygon in $\mathbb{R}^2$, then Pick's theorem asserts that

$$\mathrm{Area}(P) = I(P) + \frac{B(P)}{2} - 1,$$

where $I(P)$ denotes the number of lattice points in the interior of P and $B(P)$ denotes the number of lattice points on the boundary of P. With this notation, we have

$$\#(\overline{L(P_F)} \cap \mathbb{Z}^2) = I(\overline{L(P_F)}) + B(\overline{L(P_F)}).$$

Moreover, since $\overline{L(P_F)}$ is a lattice parallelogram, Pick's Theorem implies that

$$\mathrm{Area}(\overline{L(P_F)}) = I(\overline{L(P_F)}) + \frac{B(\overline{L(P_F)})}{2} - 1,$$

and thus

$$\#(\overline{L(P_F)} \cap \mathbb{Z}^2) = \mathrm{Area}(\overline{L(P_F)}) + \frac{B(\overline{L(P_F)})}{2} + 1 \tag{4.2}$$

By basic linear algebra, we know that

$$\begin{aligned} \mathrm{Area}(\overline{L(P_F)}) &= |\det(L)| \cdot \mathrm{Area}(\overline{P_F}) \\ &= \frac{1}{\sqrt{d_F}} \cdot \mathrm{Area}(\overline{P_F}) \\ &= \frac{1}{\sqrt{d_F}}\left|\det\begin{bmatrix} 1 & \varepsilon \\ 1 & \varepsilon' \end{bmatrix}\right| \\ &= \frac{\varepsilon-\varepsilon'}{\sqrt{d_F}}. \end{aligned} \tag{4.3}$$

Thus, it only remains to compute the value $B(\overline{L(P_F)})$, which we will now show is equal to 4. In fact, the 4 lattice points on the boundary are precisely the 4 vertices of the parallelogram $\overline{L(P_F)}$.

If we write $\varepsilon = a + b\left(\frac{d_F+\sqrt{d_F}}{2}\right)$ for some $a, b \in \mathbb{Z}$ with $b \geq 1$ (note that $b \geq 1$ since $b = (\varepsilon - \varepsilon')/\sqrt{d_F} > 0$), then from the explicit description of the linear transformation L given above, we see that the four vertices of $\overline{P_F}$ get mapped to the following vertices of $\overline{L(P_F)}$, as shown in Figure 3 and Figure 4.

$$\begin{aligned}(0,0) &\longmapsto (0,0)\\ (1,1) &\longmapsto (1,0)\\ (\varepsilon,\varepsilon') &\longmapsto (a,b)\\ (\varepsilon+1,\varepsilon'+1) &\longmapsto (a+1,b)\end{aligned}$$

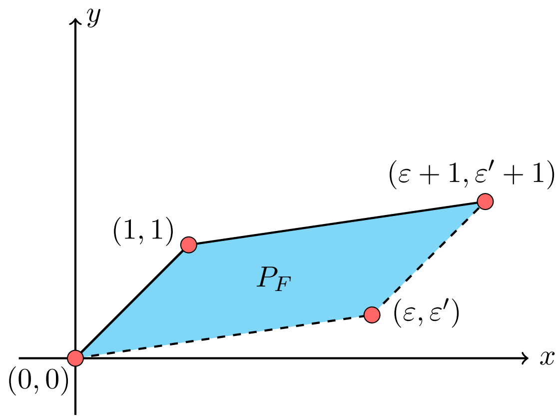

FIGURE 3. The parallelogram P_F and the lattice points in $\partial(\overline{P_F}) \cap \iota(\mathcal{O}_F)$.

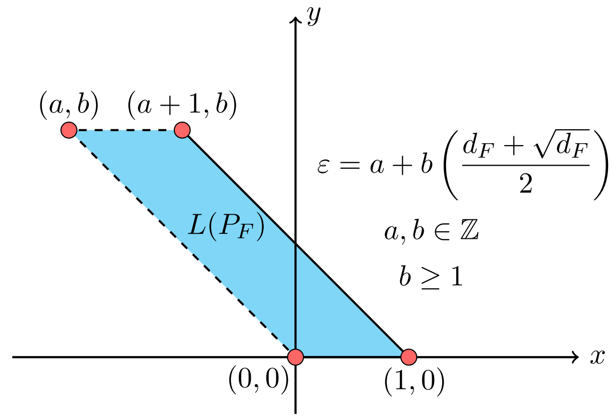

FIGURE 4. The lattice parallelogram $L(P_F)$ and the corresponding lattice points in $\partial(\overline{L(P_F)}) \cap \mathbb{Z}^2$.

It is clear that there are no lattice points on the line segments $\overline{(0,0)(1,0)}$ and $\overline{(a,b)(a+1,b)}$ other than the vertices. We now show that the same is true for the other two line segments. If there were a lattice point $(m,n) \in \mathbb{Z}^2$ lying on the line segment $\overline{(0,0)(a,b)}$ other than $(0,0)$ or (a,b), there would be a rational number $p/q \in \mathbb{Q}$ with $p,q \in \mathbb{Z}$, $\gcd(p,q) = 1$, and $0 < p/q < 1$, such that

$$(m,n) = \frac{p}{q}(a,b).$$

Therefore $qm = pa$ and $qn = pb$, and since $\gcd(p,q) = 1$, this implies that $q|a$ and $q|b$. Thus, writing $a = a_0 q$ and $b = b_0 q$ for some $a_0, b_0 \in \mathbb{Z}$, we see that

$$1 = N_{F/\mathbb{Q}}(\varepsilon) = N_{F/\mathbb{Q}}\left(a_0 q + b_0 q\left(\frac{d_F + \sqrt{d_F}}{2}\right)\right) = N_{F/\mathbb{Q}}(q) \cdot N_{F/\mathbb{Q}}\left(a_0 + b_0\left(\frac{d_F + \sqrt{d_F}}{2}\right)\right),$$

and this is a contradiction because $N_{F/\mathbb{Q}}(q) = q^2 > 1$. Hence the only lattice points on the line segment $\overline{(0,0)(a,b)}$ are $(0,0)$ and (a,b). Since the remaining line segment $\overline{(1,0)(a+1,b)}$ is just obtained from the previous one after translation by the vector $(1,0)$, we see that the only lattice points on $\overline{(1,0)(a+1,b)}$ are $(1,0)$ and $(a+1,b)$. This completes the proof that

$$B(\overline{L(P_F)}) = 4. \tag{4.4}$$

Now, by equations (4.1)–(4.4), we see that

$$\#(\overline{P_F} \cap \iota(\mathcal{O}_F)) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}} + 3.$$

Moreover, since the only vertex contained in the half-open parallelogram P_F is $(1,1)$, we have

$$\#(P_F \cap \iota(\mathcal{O}_F)) = \#(\overline{P_F} \cap \iota(\mathcal{O}_F)) - 3.$$

Hence the previous identity implies that

$$\#\ker(\pi_\alpha) = \#(P_F \cap \iota(\mathcal{O}_F)) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}}.$$

Finally, observe that

$$\mathrm{vol}(\mathbb{R}^2/\iota(\mathbb{Z} + \mathbb{Z}\varepsilon)) = \mathrm{Area}(P_F) = \varepsilon - \varepsilon'$$

and $\mathrm{vol}(\mathcal{O}_F) = \sqrt{d_F}$. This completes the proof of the proposition. ☐

4.5. **Proof of Theorem 4.1.** The Shintani sets can be written as the disjoint unions

$$\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) = \bigcup_{w \in \mathcal{O}_F/\mathfrak{f}}^{\cdot} \pi_\alpha^{-1}(w) \quad \text{and} \quad \mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) = \bigcup_{w \in (\mathcal{O}_F/\mathfrak{f})^\times}^{\cdot} \pi_\alpha^{-1}(w). \tag{4.5}$$

Then using Lemma 4.6, we see that

$$\#\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) = \sum_{w \in \mathcal{O}_F/\mathfrak{f}} \#\pi_\alpha^{-1}(w) = \#\ker(\pi_\alpha) \cdot N_{F/\mathbb{Q}}(\mathfrak{f})$$

and

$$\#\mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) = \sum_{w \in (\mathcal{O}_F/\mathfrak{f})^\times} \#\pi_\alpha^{-1}(w) = \#\ker(\pi_\alpha) \cdot \varphi(\mathfrak{f}).$$

Finally, it follows from Proposition 4.7 that

$$\#\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}} \cdot N_{F/\mathbb{Q}}(\mathfrak{f}) \quad \text{and} \quad \#\mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}} \cdot \varphi(\mathfrak{f}).$$

Note that the equality involving the volumes follows from the last line in the proof of Proposition 4.7. ☐

4.6. **Proof of Theorem 4.2.** Recall from Section 3 that χ takes one of the following forms on principal integral ideals $\langle \gamma \rangle$ coprime to $\mathfrak{f}$,

$$\begin{aligned}
\chi(\langle \gamma \rangle) &= \chi_{\mathrm{f}}(\gamma), \\
\chi(\langle \gamma \rangle) &= \chi_{\mathrm{f}}(\gamma)\,\mathrm{sign}(\gamma), \\
\chi(\langle \gamma \rangle) &= \chi_{\mathrm{f}}(\gamma)\,\mathrm{sign}(\gamma'), \\
\chi(\langle \gamma \rangle) &= \chi_{\mathrm{f}}(\gamma)\,\mathrm{sign}(N_{F/\mathbb{Q}}(\gamma)),
\end{aligned}$$

where $\chi_{\mathrm{f}} : (\mathcal{O}_F/\mathfrak{f})^\times \to \mathbb{S}^1$. Let $\mathfrak{f} = \langle \alpha \rangle$ for some $\alpha \in \mathcal{O}_F$ with $\alpha \gg 0$. Since every element of the Shintani set $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$ is totally positive, we see that for $z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$ we have $\alpha z \gg 0$, and it follows that

$$\mathrm{sign}(\alpha z) = \mathrm{sign}(\alpha' z') = \mathrm{sign}(N_{F/\mathbb{Q}}(\alpha z)) = 1.$$

In particular, for $z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$ we have

$$\chi(\mathfrak{f}\langle z \rangle) = \chi(\langle \alpha z \rangle) = \chi_{\mathrm{f}}(\alpha z).$$

Using the disjoint union (4.5), Lemma 4.6, and Proposition 4.7, we get

$$\begin{aligned}\sum_{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})}\chi(\mathfrak{f}\langle z\rangle) &= \sum_{w\in(\mathcal{O}_F/\mathfrak{f})^\times}\sum_{z\in\pi_\alpha^{-1}(w)}\chi_{\mathfrak{f}}(\alpha z)\\ &= \sum_{w\in(\mathcal{O}_F/\mathfrak{f})^\times}\chi_{\mathfrak{f}}(w)\cdot\#\pi_\alpha^{-1}(w)\\ &= \#\ker(\pi_\alpha)\sum_{w\in(\mathcal{O}_F/\mathfrak{f})^\times}\chi_{\mathfrak{f}}(w)\\ &= \frac{\varepsilon-\varepsilon'}{\sqrt{d_F}}\sum_{w\in(\mathcal{O}_F/\mathfrak{f})^\times}\chi_{\mathfrak{f}}(w).\end{aligned}$$

Finally, the orthogonality relations for characters of the finite group $(\mathcal{O}_F/\mathfrak{f})^\times$ yield

$$\sum_{w\in(\mathcal{O}_F/\mathfrak{f})^\times}\chi_{\mathfrak{f}}(w) = \begin{cases}0, & \chi\neq 1\\ \#\,(\mathcal{O}_F/\mathfrak{f})^\times = \varphi(\mathfrak{f}), & \chi = 1.\end{cases}$$

□

4.7. **Shintani cycles in $\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})$.** In this section we continue our combinatorial study of Shintani sets and their relation to ray class characters. In particular, using the vantage point provided by the group law defined on the Shintani set $\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})$, we revisit some results already observed by Shintani in [Shi77a]. This allows us to clarify and unify the theory, and moreover to set the stage for a potential generalization to higher dimensions (currently being investigated by the authors). We will need the following important facts.

Lemma 4.8. *For $w\in\mathfrak{f}^{-1}$ and $M,N\in\mathbb{Z}$, we have*

$$\gcd(\mathfrak{f}\langle w+M+N\varepsilon\rangle,\mathfrak{f})=1 \iff \gcd(\mathfrak{f}\langle w\rangle,\mathfrak{f})=1.$$

Proof. Since $\mathfrak{f}=\langle\alpha\rangle$ for some $\alpha\in\mathcal{O}_F$, we have

$$\begin{aligned}\gcd(\mathfrak{f}\langle w+M+N\varepsilon\rangle,\mathfrak{f})=1 &\iff \mathfrak{f}\langle w+M+N\varepsilon\rangle+\mathfrak{f}=\mathcal{O}_F\\ &\iff \langle\alpha(w+M+N\varepsilon)\rangle+\langle\alpha\rangle=\mathcal{O}_F\\ &\iff \langle\alpha w\rangle+\langle\alpha\rangle=\mathcal{O}_F\\ &\iff \mathfrak{f}\langle w\rangle+\mathfrak{f}=\mathcal{O}_F\\ &\iff \gcd(\mathfrak{f}\langle w\rangle,\mathfrak{f})=1.\end{aligned}$$

□

Proposition 4.9. *If $u\in\mathcal{O}_F^{\times,+}$ and $z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})$, there exists a unique $\delta(u,z)\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})$ such that*

$$\delta(u,z)-uz\in\mathbb{Z}+\mathbb{Z}\varepsilon.$$

Proof. By Proposition 4.3, we know there is a unique element $\delta:=\delta(u,z)\in\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})$ such that

$$\delta(u,z)-uz\in\mathbb{Z}+\mathbb{Z}\varepsilon. \tag{4.6}$$

Thus, to prove that $\delta\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})$, we need only show that $\gcd(\mathfrak{f}\langle\delta\rangle,\mathfrak{f})=1$. By (4.6), we have $\delta=uz+M+N\varepsilon$ for some integers $M,N\in\mathbb{Z}$. Then by Lemma 4.8, we have

$$\gcd(\mathfrak{f}\langle\delta\rangle,\mathfrak{f})=1 \iff \gcd(\mathfrak{f}\langle z\rangle,\mathfrak{f})=1.$$

However, since $z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})$, it follows that $\gcd(\mathfrak{f}\langle z\rangle,\mathfrak{f})=1$.

□

Using Proposition 4.9, we define a map

$$\begin{aligned} * : \mathcal{O}_F^{\times,+} \times \mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) &\longrightarrow \mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) \\ (u, z) &\longmapsto u * z := \delta(u, z). \end{aligned}$$

Proposition 4.10. *The map $*$ defines a group action of $\mathcal{O}_F^{\times,+}$ on $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$.*

Proof. First, we prove that $1 * z = z$ for all $z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$. Since $z - z = 0 \in \mathbb{Z} + \mathbb{Z}\varepsilon$, by uniqueness, we must have $1 * z := \delta(1, z) = z$.

Next, we prove that $u * (v * z) = (uv) * z$ for all $u, v \in \mathcal{O}_F^{\times,+}$ and $z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$. This is equivalent to $\delta(u, \delta(v, z)) = \delta(uv, z)$. We know that

$$\delta(v, z),\ \delta(uv, z),\ \delta(u, \delta(v, z)) \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$$

are the unique elements such that

$$\delta(v, z) - vz,\ \delta(uv, z) - (uv)z,\ \delta(u, \delta(v, z)) - u\delta(v, z) \in \mathbb{Z} + \mathbb{Z}\varepsilon,$$

respectively. Hence we can write

$$\delta(v, z) - vz = m_1 + n_1\varepsilon \tag{4.7}$$

and

$$\delta(u, \delta(v, z)) - u\delta(v, z) = m_2 + n_2\varepsilon \tag{4.8}$$

for some integers $m_1, n_1, m_2, n_2 \in \mathbb{Z}$. Solving for $\delta(v, z)$ in (4.7) and substituting the resulting equation into (4.8), we get

$$\delta(u, \delta(v, z)) - (uv)z = u(m_1 + n_1\varepsilon) + m_2 + n_2\varepsilon. \tag{4.9}$$

We claim that the right hand side of (4.9) is in $\mathbb{Z} + \mathbb{Z}\varepsilon$, and hence by uniqueness, $\delta(u, \delta(v, z)) = \delta(uv, z)$. Since $u = \varepsilon^n$ for some $n \in \mathbb{Z}$, it suffices to prove that $m_1\varepsilon^n + n_1\varepsilon^{n+1} \in \mathbb{Z} + \mathbb{Z}\varepsilon$. To do this, we will show that $\varepsilon^N \in \mathbb{Z} + \mathbb{Z}\varepsilon$ for every $N \in \mathbb{Z}$. Now, since ε is an algebraic integer of degree 2, it satisfies

$$\varepsilon^2 = a\varepsilon + b$$

for some integers $a, b \in \mathbb{Z}$, and hence $\varepsilon^2 \in \mathbb{Z} + \mathbb{Z}\varepsilon$. Then using an inductive argument, we can conclude that $\varepsilon^N \in \mathbb{Z} + \mathbb{Z}\varepsilon$ for any $N \geq 0$. On the other hand, for negative powers, we use the fact that $\varepsilon^{-1} = \varepsilon'$. Writing ε in terms of the integral basis $\{1, (d_F + \sqrt{d_F})/2\}$, we see that

$$\varepsilon = r + s\left(\frac{d_F + \sqrt{d_F}}{2}\right)$$

for some integers $r, s \in \mathbb{Z}$. Hence

$$\varepsilon^{-1} = \varepsilon' = r + s\left(\frac{d_F - \sqrt{d_F}}{2}\right) = (2r + sd_F) - \varepsilon \in \mathbb{Z} + \mathbb{Z}\varepsilon.$$

Moreover, since ε' is also an algebraic integer of degree 2, we see that ε^{-1} satisfies

$$\varepsilon^{-2} = \tilde{a}\varepsilon + \tilde{b}$$

for some integers $\tilde{a}, \tilde{b} \in \mathbb{Z}$. Therefore, as before, we conclude by an inductive argument that $\varepsilon^N \in \mathbb{Z} + \mathbb{Z}\varepsilon$ for any $N \leq -1$. Thus the claim is proved, which completes the proof of the proposition. $\square$

An $\mathcal{O}_F^{\times,+}$-orbit in $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$ is called a *Shintani cycle* and denoted by C.

Given $z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$, we define $\widehat{z} := \ominus z$, where $\ominus z \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ is the inverse of z under the group law defined in Proposition 4.5.

Lemma 4.11. *The map $z \longmapsto \widehat{z}$ is an involution on $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$.*

Proof. Clearly, we have $\widehat{\widehat{z}} = \ominus(\ominus z) = z$. Hence, it remains to prove that $\widehat{z} \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$, i.e., that $\gcd(\mathfrak{f}\langle\widehat{z}\rangle, \mathfrak{f}) = 1$. Now, by definition of $\ominus z$, we have

$$\ominus z - (-z) \in \mathbb{Z} + \mathbb{Z}\varepsilon,$$

and therefore there exist integers $m, n \in \mathbb{Z}$ such that $\widehat{z} = \ominus z = m + n\varepsilon - z$. It follows from Lemma 4.8 that

$$\gcd(\mathfrak{f}\langle\widehat{z}\rangle, \mathfrak{f}) = \gcd(\mathfrak{f}\langle m + n\varepsilon - z\rangle, \mathfrak{f}) = \gcd(\mathfrak{f}\langle z\rangle, \mathfrak{f}) = 1.$$

Thus, we conclude that $\widehat{z} \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$. ☐

Given a cycle C in $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$, define the set

$$\widehat{C} := \{\widehat{z} : \ z \in C\}.$$

Lemma 4.12. *The set $\widehat{C}$ is a cycle in $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$.*

Proof. Let C_{z_0} denote the cycle containing an element $z_0 \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$. Then it suffices to prove that $\widehat{C_{z_0}} = C_{\widehat{z_0}}$. First, we prove that $\widehat{C_{z_0}} \subseteq C_{\widehat{z_0}}$. Let $w \in \widehat{C_{z_0}}$. Then $w = \widehat{z}$ for some $z \in C_{z_0}$. Since $z \in C_{z_0}$, there exists $u \in \mathcal{O}_F^{\times,+}$ such that $u * z_0 = z$, or equivalently, $z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$ is the unique element such that

$$z - uz_0 \in \mathbb{Z} + \mathbb{Z}\varepsilon. \tag{4.10}$$

We claim that $u * \widehat{z_0} = \widehat{z}$, or equivalently, that $\widehat{z} - u\widehat{z_0} \in \mathbb{Z} + \mathbb{Z}\varepsilon$, which implies that $w = \widehat{z} \in C_{\widehat{z_0}}$. Write

$$\widehat{z_0} = m_1 + n_1\varepsilon - z_0,$$
$$\widehat{z} = m_2 + n_2\varepsilon - z$$

for some integers $m_1, m_2, n_1, n_2 \in \mathbb{Z}$. Substituting these equations yields

$$\widehat{z} - u\widehat{z_0} = -(z - uz_0) + m_2 + n_2\varepsilon - u(m_1 + n_1\varepsilon).$$

By (4.10), we have $-(z - uz_0) \in \mathbb{Z} + \mathbb{Z}\varepsilon$, and by the argument in the last paragraph of the proof of Proposition 4.10, we have $u(m_1 + n_1\varepsilon) \in \mathbb{Z} + \mathbb{Z}\varepsilon$. Hence $\widehat{z} - u\widehat{z_0} \in \mathbb{Z} + \mathbb{Z}\varepsilon$, which proves the claim. This completes the proof that $\widehat{C_{z_0}} \subseteq C_{\widehat{z_0}}$. A symmetric argument can be used to prove that $C_{\widehat{z_0}} \subseteq \widehat{C_{z_0}}$. ☐

The cycle $\widehat{C}$ is called the *opposite cycle*. It follows from Lemma 4.11 that the map $C \longmapsto \widehat{C}$ is an involution on the set of cycles in $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$.

Lemma 4.13. *Let χ be a narrow ray class character modulo $\mathfrak{f}$. Then for $z \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ and integers $M, N \in \mathbb{Z}$ such that $z + M + N\varepsilon \gg 0$, we have*

$$\chi(\mathfrak{f}\langle z + M + N\varepsilon\rangle) = \chi(\mathfrak{f}\langle z\rangle).$$

Proof. Suppose that $\gcd(\mathfrak{f}\langle z\rangle, \mathfrak{f}) \neq 1$. Then by Lemma 4.8, we have trivially that

$$\chi(\mathfrak{f}\langle z + M + N\varepsilon\rangle) = \chi(\mathfrak{f}\langle z\rangle) = 0.$$

Next, suppose that $\gcd(\mathfrak{f}\langle z\rangle, \mathfrak{f}) = 1$. Then by Lemma 4.8, we have $\gcd(\mathfrak{f}\langle z + M + N\varepsilon\rangle, \mathfrak{f}) = 1$. Now, recall that if $\gamma \in \mathcal{O}_F$ with $\gamma \gg 0$ and relatively prime to $\mathfrak{f}$, then

$$\chi(\langle\gamma\rangle) = \chi_{\mathrm{f}}(\gamma)\chi_\infty(\gamma) = \chi_{\mathrm{f}}(\gamma).$$

Since F has narrow class number 1, $\mathfrak{f} = \langle\alpha\rangle$ for some $\alpha \in \mathcal{O}_F$ with $\alpha \gg 0$. Because $\alpha \gg 0$, $z \gg 0$, and $z + M + N\varepsilon \gg 0$, it follows that

$$\chi(\mathfrak{f}\langle z + M + N\varepsilon\rangle) = \chi(\langle\alpha(z + M + N\varepsilon)\rangle) = \chi_{\mathrm{f}}(\alpha(z + M + N\varepsilon)) = \chi_{\mathrm{f}}(\alpha z) = \chi(\mathfrak{f}\langle z\rangle),$$

where we used the fact that $\alpha(z + M + N\varepsilon) \equiv \alpha z \pmod{\mathfrak{f}}$ and that $\chi_{\mathrm{f}} : (\mathcal{O}_F/\mathfrak{f})^\times \longrightarrow \mathbb{S}^1$. ☐

Lemma 4.14. *Let χ be a narrow ray class character modulo $\mathfrak{f}$ and C be a cycle in $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$. Then the value $\chi(\mathfrak{f}\langle z\rangle)$ is independent of the choice of $z \in C$.*

Proof. Let $z_1, z_2 \in C$. Then there exists $u \in \mathcal{O}_F^{\times,+}$ such that $u * z_1 = z_2$. This implies that there exist integers $M, N \in \mathbb{Z}$ such that $z_2 - uz_1 = M + N\varepsilon$. Since $uz_1 \gg 0$, we have $z_2 - (M + N\varepsilon) \gg 0$. It follows from Lemma 4.13 that

$$\chi(\mathfrak{f}\langle z_1\rangle) = \chi(\mathfrak{f}\langle uz_1\rangle) = \chi(\mathfrak{f}\langle z_2 - (M + N\varepsilon)\rangle) = \chi(\mathfrak{f}\langle z_2\rangle).$$

□

By Lemma 4.14, given $z \in C$ we may define $\chi(C) := \chi(\mathfrak{f}\langle z\rangle)$.

Proposition 4.15. *Let χ be a narrow ray class character modulo $\mathfrak{f}$ with admissible vector $\mathbf{p}$. Then*

$$\chi(\widehat{C}) = \begin{cases} \chi(C), & \text{if } \mathbf{p} = (0,0) \text{ or } (1,1) \\ -\chi(C), & \text{if } \mathbf{p} = (1,0) \text{ or } (0,1). \end{cases}$$

Proof. Let $\langle\gamma\rangle$ be a principal integral ideal coprime to $\mathfrak{f}$. Then the infinite part of χ is given by (see Section 3)

$$\chi_\infty(\gamma) = \begin{cases} 1, & \text{if } \mathbf{p} = (0,0) \\ \text{sign}(\gamma), & \text{if } \mathbf{p} = (1,0) \\ \text{sign}(\gamma'), & \text{if } \mathbf{p} = (0,1) \\ \text{sign}(N_{F/\mathbb{Q}}(\gamma)), & \text{if } \mathbf{p} = (1,1). \end{cases}$$

It follows that

$$\frac{\chi_\infty(\gamma)}{\chi_\infty(-\gamma)} = \begin{cases} 1, & \text{if } \mathbf{p} = (0,0) \text{ or } (1,1) \\ -1, & \text{if } \mathbf{p} = (1,0) \text{ or } (0,1). \end{cases}$$

Now, because

$$\chi_{\mathrm{f}}(-\gamma) = \frac{\chi(\langle -\gamma\rangle)}{\chi_\infty(-\gamma)} = \frac{\chi(\langle\gamma\rangle)}{\chi_\infty(-\gamma)} = \frac{\chi_\infty(\gamma)}{\chi_\infty(-\gamma)}\chi_{\mathrm{f}}(\gamma),$$

we have

$$\chi_{\mathrm{f}}(-\gamma) = \begin{cases} \chi_{\mathrm{f}}(\gamma), & \text{if } \mathbf{p} = (0,0) \text{ or } (1,1) \\ -\chi_{\mathrm{f}}(\gamma), & \text{if } \mathbf{p} = (1,0) \text{ or } (0,1). \end{cases} \tag{4.11}$$

Let $\mathfrak{f} = \langle\alpha\rangle$ for some $\alpha \in \mathcal{O}_F$ with $\alpha \gg 0$. Given $z \in C$, write $\widehat{z} = m + n\varepsilon - z$ for some integers $m, n \in \mathbb{Z}$. Then

$$\begin{aligned} \chi(\widehat{C}) = \chi(\mathfrak{f}\langle\widehat{z}\rangle) &= \chi(\langle\alpha(m + n\varepsilon - z)\rangle) \\ &= \chi_{\mathrm{f}}(\alpha(m + n\varepsilon - z))\chi_\infty(\alpha(m + n\varepsilon - z)) \\ &= \chi_{\mathrm{f}}(\alpha(m + n\varepsilon - z)) \\ &= \chi_{\mathrm{f}}(-\alpha z), \end{aligned} \tag{4.12}$$

where for the fourth equality we used $\alpha(m + n\varepsilon - z) \gg 0$, and for the last equality we used $\alpha(m + n\varepsilon - z) \equiv -\alpha z \pmod{\mathfrak{f}}$ and $\chi_{\mathrm{f}} : (\mathcal{O}_F/\mathfrak{f})^\times \to \mathbb{S}^1$. A similar calculation shows that

$$\chi(C) = \chi_{\mathrm{f}}(\alpha z). \tag{4.13}$$

The proposition now follows from (4.11), (4.12) and (4.13). □

Corollary 4.16. *If there exists a narrow ray class character χ modulo $\mathfrak{f}$ with admissible vector $\mathbf{p} = (1,0)$ or $(0,1)$, then the involution $C \longmapsto \widehat{C}$ on the set of cycles in $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$ is fixed point free.*

Proof. Suppose that $C \longmapsto \widehat{C}$ has a fixed point C_0. Then $\widehat{C_0} = C_0$, so that $\chi(\widehat{C_0}) = \chi(C_0)$. On the other hand, by Proposition 4.15 we have $\chi(\widehat{C_0}) = -\chi(C_0)$. It follows that $\chi(C_0) = -\chi(C_0)$, or $\chi(C_0) = 0$. But this is impossible, since $\chi(C_0)$ takes values in the unit circle $\mathbb{S}^1$ (recall that $\chi(C_0) := \chi(\mathfrak{f}\langle z\rangle)$ for $z \in C_0$ and $\gcd(\mathfrak{f}\langle z\rangle, \mathfrak{f}) = 1$ for $z \in C_0$). Thus $C \longmapsto \widehat{C}$ is fixed point free. □

If a narrow ray class character χ modulo $\mathfrak{f}$ has admissible vector $(1,0)$ or $(0,1)$, then one can use the symmetry relations satisfied by χ with respect to Shintani cycles to give an alternative proof of the orthogonality relations in Theorem 4.2.

Recall that $B_2(x) = x^2 - x - 1/6$ is the second Bernoulli polynomial.

Proposition 4.17. *If χ is a narrow ray class character modulo $\mathfrak{f}$ with admissible vector $\mathbf{p} = (1,0)$ or $(0,1)$, then*

$$\sum_{z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})} \chi(\mathfrak{f}\langle z\rangle) = 0$$

and

$$\sum_{\substack{z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) \\ z = x + y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle) B_2(x) = 0.$$

Proof. By Corollary 4.16, the involution $C \longmapsto \widehat{C}$ is fixed point free. Hence, there is an even number of cycles $C_1, \ldots, C_{2n}$ in $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$, and a choice of cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_n}$ giving a disjoint union

$$\mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) = C_{i_1} \cup \widehat{C_{i_1}} \cup C_{i_2} \cup \widehat{C_{i_2}} \cup \cdots \cup C_{i_n} \cup \widehat{C_{i_n}}. \tag{4.14}$$

It follows from Proposition 4.15 and (4.14) that

$$\sum_{z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})} \chi(\mathfrak{f}\langle z\rangle) = \sum_{j=1}^{n} \#C_{i_j} \left(\chi(C_{i_j}) + \chi(\widehat{C_{i_j}}) \right) = \sum_{j=1}^{n} \#C_{i_j} \left(\chi(C_{i_j}) - \chi(C_{i_j}) \right) = 0.$$

Next, given $z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$, write $z = x_z + y_z\varepsilon$ and $\widehat{z} = x_{\widehat{z}} + y_{\widehat{z}}\varepsilon$. By definition of the involution $z \longmapsto \widehat{z}$, we have $x_{\widehat{z}} = x_z$ or $x_{\widehat{z}} = 1 - x_z$. It follows from Proposition 4.15, (4.14), and the relation

$$B_2(x) = B_2(1-x)$$

that

$$\begin{aligned}
\sum_{\substack{z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) \\ z = x_z + y_z\varepsilon}} \chi(\mathfrak{f}\langle z\rangle) B_2(x_z) &= \sum_{j=1}^{n} \sum_{\substack{z \in C_{i_j} \\ z = x_z + y_z\varepsilon}} \left(\chi(C_{i_j}) B_2(x_z) + \chi(\widehat{C_{i_j}}) B_2(x_{\widehat{z}}) \right) \\
&= \sum_{j=1}^{n} \sum_{\substack{z \in C_{i_j} \\ z = x_z + y_z\varepsilon}} B_2(x_z) \left(\chi(C_{i_j}) + \chi(\widehat{C_{i_j}}) \right) \\
&= \sum_{j=1}^{n} \sum_{\substack{z \in C_{i_j} \\ z = x_z + y_z\varepsilon}} B_2(x_z) \left(\chi(C_{i_j}) - \chi(C_{i_j}) \right) \\
&= 0.
\end{aligned}$$

□

Remark 4.18. The involution $C \longmapsto \widehat{C}$ may not be fixed point free. For example, let $F = \mathbb{Q}(\sqrt{5})$, so that $\varepsilon = \frac{3+\sqrt{5}}{2}$, and consider the integral ideal $\mathfrak{f} = \langle\sqrt{5}\rangle$. Then the Shintani set decomposes into the union of two cycles

$$\mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) = C_1 \cup C_2,$$

where

$$C_1 = \left\{\frac{1+\varepsilon}{5}, \frac{4+4\varepsilon}{5}\right\} \quad \text{and} \quad C_2 = \left\{\frac{2+2\varepsilon}{5}, \frac{3+3\varepsilon}{5}\right\}.$$

A straightforward calculation shows that $\widehat{C_1} = C_1$ and $\widehat{C_2} = C_2$. In particular, by Corollary 4.16, there are no narrow ray class characters χ modulo $\mathfrak{f}$ with admissible vector $\mathbf{p} = (1,0)$ or $(0,1)$.

5. Shintani zeta functions

In this section, we state a formula of Shintani which relates the derivative of the Shintani double zeta function at $s = 0$ to logarithms of the Barnes double Gamma function. We first recall the definition of the Barnes double Gamma function given in the introduction (see e.g. [Shi77a, Section 1]). Let z be a complex number and $\omega = (\omega_1, \omega_2)$ be a pair of complex numbers. Assume that z, ω_1 and ω_2 have positive real part. For a complex number $w \in \mathbb{C} \setminus (\infty, 0]$, let $w^s = \exp(s \log w)$, where $\log w = \log|w| + i \arg w$ with $|\arg w| < \pi$. Define the *double Hurwitz zeta function*

$$\zeta_H(s, z, \omega) := \sum_{m,n=0}^{\infty} \frac{1}{(z + m\omega_1 + n\omega_2)^s}, \quad \mathrm{Re}(s) > 2.$$

The function $\zeta_H(s, z, \omega)$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at $s = 1, 2$. In analogy with Lerch's identity (1.2), the *normalized Barnes double Gamma function* is defined by

$$\log\left(\Gamma_2^*(z, \omega)\right) := \frac{d}{ds}\zeta_H(s, z, \omega)\bigg|_{s=0}.$$

The function $\Gamma_2^*(z, \omega)$ has a simple pole at $z = 0$. Let $\rho_2(\omega)^{-1}$ denote the residue of $\Gamma_2^*(z, \omega)$ at $z = 0$. Then the *Barnes double gamma function* is defined by

$$\Gamma_2(z, \omega) := \rho_2(\omega)\Gamma_2^*(z, \omega).$$

Let $\mathbf{a} = (a_1, a_2)$ be a pair of positive real numbers and $\mathbf{x} = (x_1, x_2) \neq (0, 0)$ be a pair of non-negative real numbers. Then the *Shintani double zeta function* is defined by

$$\zeta(s, \mathbf{a}, \mathbf{x}) := \sum_{m,n=0}^{\infty} \frac{1}{\{(x_1 + m + (x_2 + n)a_1)(x_1 + m + (x_2 + n)a_2)\}^s}, \quad \mathrm{Re}(s) > 1.$$

The function $\zeta(s, \mathbf{a}, \mathbf{x})$ has a meromorphic continuation to $\mathbb{C}$. Shintani calculated the first two terms in the Taylor expansion of $\zeta(s, \mathbf{a}, \mathbf{x})$ at $s = 0$. In particular, he proved that (see [Shi77a, Proposition 3]),

$$\begin{aligned} \frac{d}{ds}\zeta(s, \mathbf{a}, \mathbf{x})\bigg|_{s=0} &= \log\left\{\Gamma_2^*(x_1 + x_2 a_1, (1, a_1))\Gamma_2^*(x_1 + x_2 a_2, (1, a_2))\right\} \\ &\quad + \frac{a_1 - a_2}{4a_1 a_2}\log\left(\frac{a_2}{a_1}\right)B_2(x_1) \end{aligned} \tag{5.1}$$

where $B_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomials.

Remark 5.1. For real numbers $u, v > 0$, Shintani [Shi80, Proposition 1] proved the identities

$$\Gamma_2(u,(1,v)) = (2\pi)^{u/2}\exp\left(\left\{\frac{(u-u^2)}{2v}-\frac{u}{2}\right\}\log v+\frac{(u^2-u)\gamma}{2v}\right)$$
$$\times\,\Gamma(u)\prod_{n=1}^{\infty}\frac{\Gamma(u+nv)}{\Gamma(1+nv)}\exp\left\{\frac{u-u^2}{2nv}+(1-u)\log(nv)\right\}$$

and

$$\rho_2((1,u)) = (2\pi)^{3/4}\exp\left\{-\frac{\gamma}{12v}-\frac{v}{12}+v\zeta'(-1)+\left(\frac{v}{12}-\frac{1}{4}+\frac{1}{12v}\right)\log v\right\}$$
$$\times\prod_{n=1}^{\infty}\frac{(2\pi)^{1/2}}{\Gamma(1+nv)}\exp\left\{\frac{1}{12nv}+\left(\frac{1}{2}+nv\right)\log(nv)-nv\right\},$$

where γ is Euler's constant and $\zeta(s)$ is the Riemann zeta function. In particular, the Barnes double Gamma values appearing on the right hand side of (5.1) can be expressed in terms of these infinite products.

Remark 5.2. We used the first identity in the previous remark in order to approximate the values of the Barnes double Gamma function appearing in the examples of Section 2.

6. Special values of ray class L–functions for real quadratic fields

In this section, we use work of Shintani [Shi77a] and the results of Section 4 to evaluate the derivative at $s = 0$ of the L–function of a narrow ray class character. We let notation and assumptions be as in Section 4. In particular, F is a real quadratic field of narrow class number 1 and $\mathfrak{f} \subset \mathcal{O}_F$ is an integral ideal.

6.1. **Shintani's decomposition of $L(\chi, s)$.** For the convenience of the reader, we begin by giving a detailed, self-contained treatment of Shintani's [Shi77a] decomposition of the L–function of a narrow ray class character in terms of Shintani zeta functions (see Proposition 6.3). This is the two-dimensional analog of the identity (1.3).

The key to Shintani's decomposition is the following parameterization of the set $\mathfrak{f}^{-1,+}$.

Proposition 6.1. *Let F be a real quadratic field of narrow class number 1 and $\mathfrak{f} \subset \mathcal{O}_F$ be an integral ideal. Then the map*

$$\phi : \mathcal{O}_F^{\times,+} \times \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1}) \times \mathbb{Z}^2_{\geq 0} \longrightarrow \mathfrak{f}^{-1,+}$$

defined by

$$\phi((u, z, (m, n))) = u(z + m + n\varepsilon)$$

is a bijection.

Observe that the group $F^\times$ (resp. $\mathcal{O}_F^{\times,+}$) acts on $\mathbb{R}^2$ (resp. $\mathbb{R}^2_+$) via the embedding $\iota(x) := (x, x')$ by componentwise multiplication. The proof of Proposition 6.1 then rests on the construction of a particular fundamental domain for the action of $\mathcal{O}_F^{\times,+}$ on $\mathbb{R}^2_+$.

Proposition 6.2. *The Shintani cone*

$$\mathcal{C}(\varepsilon) := \left\{x(1,1) + y\left(\varepsilon, \varepsilon'\right) : \ x > 0,\ y \geq 0\right\} \subset \mathbb{R}^2_+$$

is a fundamental domain for the action of $\mathcal{O}_F^{\times,+}$ on $\mathbb{R}^2_+$. In particular,

$$\mathbb{R}^2_+ = \dot{\bigcup_{n\in\mathbb{Z}}} \varepsilon^n \mathcal{C}(\varepsilon).$$

Proof. Define the norm one hypersurface

$$\mathcal{S} := \{(t_1, t_2) \in \mathbb{R}_+^2 \mid t_1 t_2 = 1\} \subset \mathbb{R}_+^2.$$

Let $(t_1, t_2) \in \mathcal{S}$ and define $s := \log_\varepsilon t_1$. Since $t_2 = 1/t_1$, we have $(t_1, t_2) = (\varepsilon^s, \frac{1}{\varepsilon^s}) = (\varepsilon^s, (\varepsilon')^s)$. It follows that

$$\mathcal{S} = \{(\varepsilon^s, (\varepsilon')^s) \mid s \in \mathbb{R}\} = \dot{\bigcup_{n\in\mathbb{Z}}} \{(\varepsilon^{n+t}, (\varepsilon')^{n+t}) \mid 0 \leq t < 1\} = \dot{\bigcup_{n\in\mathbb{Z}}} \varepsilon^n \mathcal{F}, \tag{6.1}$$

where

$$\mathcal{F} := \{(\varepsilon^t, (\varepsilon')^t) \mid 0 \leq t < 1\} = \mathcal{S} \cap \mathcal{C}(\varepsilon).$$

Now, given $(v_1, v_2) \in \mathbb{R}_+^2$, let

$$P = \frac{1}{\sqrt{v_1 v_2}}(v_1, v_2)$$

be the projection of (v_1, v_2) onto $\mathcal{S}$. Then by the decomposition (6.1), there exists a unique $n \in \mathbb{Z}$ such that

$$P \in \varepsilon^n \mathcal{F} \subset \varepsilon^n \mathcal{C}(\varepsilon).$$

It follows that the ray $\overrightarrow{OP} \subset \varepsilon^n \mathcal{C}(\varepsilon)$. Since $(v_1, v_2) \in \overrightarrow{OP}$, we have shown

$$\mathbb{R}_+^2 = \dot{\bigcup_{n\in\mathbb{Z}}} \varepsilon^n \mathcal{C}(\varepsilon),$$

as required. ∎

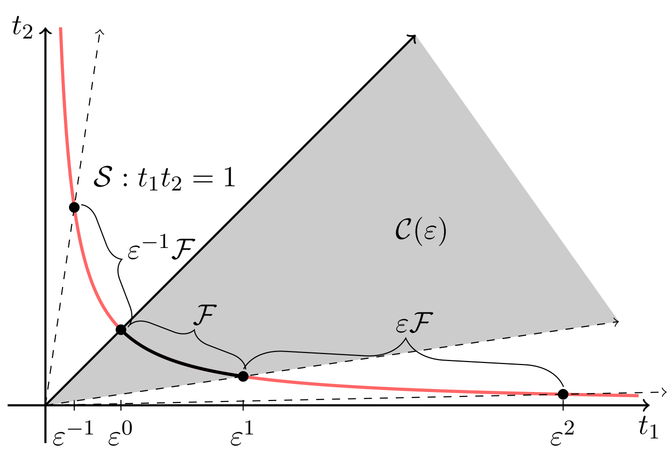

FIGURE 5. Covering $\mathcal{S}$ by translates of $\mathcal{F}$.

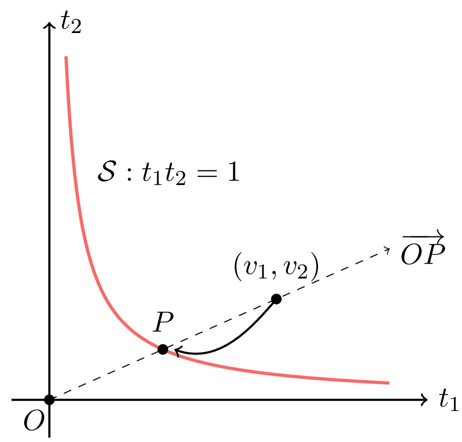

FIGURE 6. The projection P of (v_1, v_2) onto $\mathcal{S}$.

Proof of Proposition 6.1. We can now prove Proposition 6.1.

Injectivity of ϕ: Suppose that

$$\phi((u_1, z_1, (m_1, n_1))) = \phi((u_2, z_2, (m_2, n_2))).$$

Then

$$u_1(z_1 + m_1 + n_1\varepsilon) = u_2(z_2 + m_2 + n_2\varepsilon).$$

Write $z_1, z_2 \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ as

$$z_1 = x_1 + y_1\varepsilon, \quad z_2 = x_2 + y_2\varepsilon$$

for some rational numbers $x_1, x_2, y_1, y_2 \in \mathbb{Q}$ satisfying the inequalities

$$0 < x_1, x_2 \leq 1, \quad 0 \leq y_1, y_2 < 1. \tag{6.2}$$

Then

$$z_1 + m_1 + n_1\varepsilon = (x_1 + m_1) + (y_1 + n_1)\varepsilon, \quad z_2 + m_2 + n_2\varepsilon = (x_2 + m_2) + (y_2 + n_2)\varepsilon$$

with $x_1 + m_1,\ x_2 + m_2 > 0$ and $y_1 + n_1,\ y_2 + n_2 \geq 0$. In particular, the images of $z_1 + m_1 + n_1\varepsilon$ and $z_2 + m_2 + n_2\varepsilon$ under the embedding $\iota : F \hookrightarrow \mathbb{R}^2$ are in $\mathcal{C}(\varepsilon)$. Since $\mathcal{C}(\varepsilon)$ is a fundamental domain for the action of $\mathcal{O}_F^{\times,+}$ on $\mathbb{R}^2_+$, we have $u_1 = u_2$. Next, observe that

$$z_1 + m_1 + n_1\varepsilon = z_2 + m_2 + n_2\varepsilon \iff x_1 + m_1 - x_2 - m_2 = (y_2 + n_2 - y_1 - n_1)\varepsilon.$$

Since $\varepsilon \notin \mathbb{Q}$, we have

$$x_1 + m_1 - x_2 - m_2 = 0, \quad y_2 + n_2 - y_1 - n_1 = 0. \tag{6.3}$$

This implies that $x_1 - x_2 = m_2 - m_1 \in \mathbb{Z}$ and $y_1 - y_2 = n_2 - n_1 \in \mathbb{Z}$. Apply (6.2) to get $x_1 = x_2$ and $y_1 = y_2$, then apply (6.3) to get $m_1 = m_2$ and $n_1 = n_2$. We conclude that

$$(u_1, z_1, (m_1, n_1)) = (u_2, z_2, (m_2, n_2)),$$

and hence ϕ is injective.

Surjectivity of ϕ: Let $\gamma \in \mathfrak{f}^{-1,+}$. Then by Proposition 4.3, there is a unique $z \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ such that

$$\gamma - z \in \mathbb{Z} + \mathbb{Z}\varepsilon.$$

Thus, there are integers $m, n \in \mathbb{Z}$ such that $\gamma = z + m + n\varepsilon$. We conclude that

$$\phi((1, z, (m, n))) = 1 \cdot (z + m + n\varepsilon) = \gamma,$$

and hence ϕ is surjective. □

We can now prove Shintani's decomposition.

Proposition 6.3. *Let F be a real quadratic field of narrow class number 1 and let χ be a narrow ray class character modulo $\mathfrak{f}$ where $\mathfrak{f} \subset \mathcal{O}_F$ is an integral ideal. Then*

$$L(\chi, s) = N_{F/\mathbb{Q}}(\mathfrak{f})^{-s} \sum_{\substack{z \in \mathcal{R}(\varepsilon, \mathfrak{f}^{-1}) \\ z = x + y\varepsilon}} \chi(\mathfrak{f}\langle z \rangle)\zeta(s, (\varepsilon, \varepsilon'), (x, y)), \tag{6.4}$$

where

$$\zeta(s, (\varepsilon, \varepsilon'), (x, y)) = \sum_{m,n=0}^{\infty} \frac{1}{\{(x + m + (y + n)\varepsilon)(x + m + (y + n)\varepsilon')\}^s}$$

is the Shintani zeta function $\zeta(s, \mathbf{a}, \mathbf{x})$ for the choices $\mathbf{a} = (\varepsilon, \varepsilon') \in \mathbb{R}^2_+$ and $\mathbf{x} = (x, y) \in \mathbb{R}^2_{\geq 0}$ with $(x, y) \neq (0, 0)$.

Proof. Since F has narrow class number 1, every integral ideal $\mathfrak{a} \subset \mathcal{O}_F$ is equivalent in the strict sense to $\mathfrak{f}$, and hence can be written as $\mathfrak{a} = \langle \gamma \rangle \mathfrak{f}$ for a unique $\gamma \in \mathfrak{f}^{-1,+}/\mathcal{O}_F^{\times,+}$. Therefore

$$L(\chi, s) = \sum_{\mathfrak{a} \subset \mathcal{O}_F} \frac{\chi(\mathfrak{a})}{N_{F/\mathbb{Q}}(\mathfrak{a})^s} = N_{F/\mathbb{Q}}(\mathfrak{f})^{-s} \sum_{\gamma \in \mathfrak{f}^{-1,+}/\mathcal{O}_F^{\times,+}} \frac{\chi(\mathfrak{f}\langle \gamma \rangle)}{N_{F/\mathbb{Q}}(\gamma)^s}.$$

Now, by Proposition 6.1 and Lemma 4.13, we have

$$\begin{aligned}\sum_{\gamma\in\mathfrak{f}^{-1,+}/\mathcal{O}_F^{\times,+}} \frac{\chi(\mathfrak{f}\langle\gamma\rangle)}{N_{F/\mathbb{Q}}(\gamma)^s} &= \sum_{z\in\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})} \sum_{m,n=0}^{\infty} \frac{\chi(\mathfrak{f}\langle z+m+n\varepsilon\rangle)}{N_{F/\mathbb{Q}}(z+m+n\varepsilon)^s} \\ &= \sum_{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})} \sum_{m,n=0}^{\infty} \frac{\chi(\mathfrak{f}\langle z\rangle)}{N_{F/\mathbb{Q}}(z+m+n\varepsilon)^s} \\ &= \sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})\\ z=x+y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle) \sum_{m,n=0}^{\infty} \frac{1}{\{(x+m+(y+n)\varepsilon)(x+m+(y+n)\varepsilon')\}^s} \\ &= \sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})\\ z=x+y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle)\zeta(s,(\varepsilon,\varepsilon'),(x,y)),\end{aligned}$$

where we replaced $\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})$ by $\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})$ since $\chi(\mathfrak{f}\langle z\rangle)=0$ for $z\in\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})\setminus\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})$. □

6.2. **Special values of ray class L–functions.** We are now in a position to evaluate the special value $L'(\chi,0)$.

Proposition 6.4. *Let F be a real quadratic field of narrow class number 1 and let χ be a narrow ray class character modulo $\mathfrak{f}$ where $\mathfrak{f}\subset\mathcal{O}_F$ is an integral ideal. Then*

$$\begin{aligned}L'(\chi,0) = -\log\left(N_{F/\mathbb{Q}}(\mathfrak{f})\right)L(\chi,0) &+ \sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})\\ z=x+y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle)\log\left\{\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))\right\} \\ &+ \frac{\varepsilon-\varepsilon'}{2}\log(\varepsilon') \sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})\\ z=x+y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle)B_2(x).\end{aligned} \tag{6.5}$$

Proof. Differentiating (6.4) yields

$$L'(\chi,s) = -\log\left(N_{F/\mathbb{Q}}(\mathfrak{f})\right)L(\chi,s) + N_{F/\mathbb{Q}}(\mathfrak{f})^{-s} \sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})\\ z=x+y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle)\frac{d}{ds}\zeta(s,(\varepsilon,\varepsilon'),(x,y)). \tag{6.6}$$

Then using (6.6) and (5.1), we get

$$\begin{aligned}L'(\chi,0) = -\log\left(N_{F/\mathbb{Q}}(\mathfrak{f})\right)L(\chi,0) &+ \sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})\\ z=x+y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle)\log\left\{\Gamma_2^*(z,(1,\varepsilon))\Gamma_2^*(z',(1,\varepsilon'))\right\} \\ &+ \frac{\varepsilon-\varepsilon'}{2}\log(\varepsilon') \sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})\\ z=x+y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle)B_2(x).\end{aligned} \tag{6.7}$$

By Theorem 4.2, we have the orthogonality relations

$$\sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})\\ z=x+y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle) = 0.$$

Then because $\Gamma_2^*(z,\omega) = \Gamma_2(z,\omega)/\rho_2(\omega)$ and $\rho_2(\omega)$ is independent of z, the identity (6.7) simplifies to

$$L'(\chi,0) = -\log\left(N_{F/\mathbb{Q}}(\mathfrak{f})\right)L(\chi,0) + \sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})\\ z=x+y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle)\log\left\{\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))\right\}$$
$$+ \frac{\varepsilon-\varepsilon'}{2}\log(\varepsilon') \sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})\\ z=x+y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle)B_2(x).$$

□

Remark 6.5. Suppose that χ is a narrow ray class character modulo $\mathfrak{f}$ with admissible vector $\mathbf{p} = (1,0)$ or $(0,1)$. If we assume in addition that χ is primitive, then as observed in Remark 3.1, the functional equation (3.1) implies that $L(\chi,0) = 0$. It now follows from Proposition 4.17 that the identity (6.5) simplifies to

$$L'(\chi,0) = \sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{f}^{-1})\\ z=x+y\varepsilon}} \chi(\mathfrak{f}\langle z\rangle)\log\left\{\Gamma_2(z,(1,\varepsilon))\Gamma_2(z',(1,\varepsilon'))\right\}.$$

7. The character of a relative quadratic extension

In this section, we summarize some facts we will need regarding the ray class character $\chi_{L/K}$ associated to a quadratic extension L/K of number fields (see e.g. [Sie65, pp. 144–146]). In particular, the explicit description of $\chi_{L/K}$ given here is used in the algorithm implemented in `SageMath` to compute the character values appearing in the examples of Section 2 (see also Appendix B).

We first recall the definition of the quadratic residue symbol associated to a number field K. Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal and $\alpha \in \mathcal{O}_K$ be relatively prime to $\mathfrak{p}$. If $\mathfrak{p}$ is relatively prime to 2, the quadratic residue symbol is defined by

$$\left(\frac{\alpha}{\mathfrak{p}}\right) := \begin{cases} 1, & \text{if } \alpha \equiv \xi^2 \pmod{\mathfrak{p}} \text{ has a solution } \xi \in \mathcal{O}_K \\ -1, & \text{if } \alpha \equiv \xi^2 \pmod{\mathfrak{p}} \text{ has no solution } \xi \in \mathcal{O}_K. \end{cases}$$

If $\mathfrak{p}$ divides 2 and $a := \mathrm{ord}_{\mathfrak{p}}(2)$, the quadratic residue symbol is defined by

$$\left(\frac{\alpha}{\mathfrak{p}}\right) := \begin{cases} 1, & \text{if } \alpha \equiv \xi^2 \pmod{\mathfrak{p}^{2a+1}} \text{ has a solution } \xi \in \mathcal{O}_K \\ -1, & \text{if } \alpha \equiv \xi^2 \pmod{\mathfrak{p}^{2a+1}} \text{ has no solution } \xi \in \mathcal{O}_K \text{, but} \\ & \quad \alpha \equiv \xi^2 \pmod{\mathfrak{p}^{2a}} \text{ has a solution } \xi \in \mathcal{O}_K. \end{cases}$$

Finally, if $\mathfrak{p}$ is not relatively prime to α, define $\left(\frac{\alpha}{\mathfrak{p}}\right) = 0$.

Next, we extend the definition of the quadratic residue symbol to fractional ideals of K. If $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})}$ is an integral ideal of K, define

$$\left(\frac{\alpha}{\mathfrak{a}}\right) := \prod_{\mathfrak{p}} \left(\frac{\alpha}{\mathfrak{p}}\right)^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})}.$$

If $\mathfrak{a}$ is a fractional ideal of K and we write $\mathfrak{a} = \mathfrak{b}\mathfrak{c}^{-1}$ for relatively prime integral ideals $\mathfrak{b},\mathfrak{c}$ of K with $\mathfrak{c}$ relatively prime to α, define

$$\left(\frac{\alpha}{\mathfrak{a}}\right) := \left(\frac{\alpha}{\mathfrak{b}}\right)\left(\frac{\alpha}{\mathfrak{c}}\right)^{-1}.$$

Let L be a quadratic extension of K and let $\Delta \in \mathcal{O}_K$ be an algebraic integer such that $L = K(\sqrt{\Delta})$. Then there exists an integral ideal $\mathfrak{i} \subset \mathcal{O}_K$ such that $\mathfrak{i}^2\langle\Delta\rangle = \mathfrak{D}_{L/K}$ (see e.g. [Gol72, p.

3]). In particular, if $\mathfrak{p}$ is a prime ideal of K which is relatively prime to $\mathfrak{D}_{L/K}$, then $\mathfrak{p}$ is relatively prime to Δ. For a fractional ideal $\mathfrak{a} \in \mathcal{I}_K(\mathfrak{D}_{L/K})$, define

$$\chi_{L/K}(\mathfrak{a}) := \left(\frac{\Delta}{\mathfrak{a}}\right).$$

It is clear from the definition that $\chi_{L/K}$ is a homomorphism. Using quadratic reciprocity (see e.g. Hecke [Hec81]), one can prove the following result.

Proposition 7.1. *The character* $\chi_{L/K} : \mathcal{I}_K(\mathfrak{D}_{L/K}) \longrightarrow \{\pm 1\}$ *is a primitive narrow ray class character modulo* $\mathfrak{D}_{E/F}$.

Similarly, using quadratic reciprocity, one can prove the following result which gives a description of the finite and infinite parts of the ray class character $\chi_{L/K}$ when K is real quadratic.

Proposition 7.2. *If K is a real quadratic field, $L = K(\sqrt{\Delta})$, and $\alpha \in \mathcal{O}_K$ is relatively prime to $\mathfrak{D}_{L/K}$, then*

$$\chi_{L/K}(\langle\alpha\rangle) = \chi_{\mathrm{f}}(\alpha)\chi_\infty(\alpha),$$

where the finite part of $\chi_{L/K}$ is $\chi_{\mathrm{f}}(\alpha) := \chi_{L/K}(\langle\alpha\rangle)/\chi_\infty(\alpha)$ and the infinite part of $\chi_{L/K}$ is

$$\chi_\infty(\alpha) := \begin{cases} 1, & \Delta \gg 0 \\ \mathrm{sign}(\alpha), & \Delta > 0,\ \Delta' < 0 \\ \mathrm{sign}(\alpha'), & \Delta < 0,\ \Delta' > 0 \\ \mathrm{sign}(N_{K/\mathbb{Q}}(\alpha)), & \Delta \ll 0. \end{cases}$$

8. Proof of Theorem 1.1

As discussed in the introduction, Tonghai Yang [Yan10a, Yan10b, Yan13] proved the Colmez conjecture for a large class of non-biquadratic quartic CM fields, including the first *non-abelian* cases. We start by briefly explaining how the Colmez conjecture for quartic CM fields can be deduced from the averaged Colmez conjecture.

Let E be a CM field of degree $2n$, and let $\Phi(E)$ be the set of CM types for E. In [BSM16, Proposition 5.1], it is shown that if the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\Phi(E)$ is transitive, then the recently proved averaged Colmez conjecture [AGHM15, YZ15] implies the Colmez conjecture for E, and moreover, the Colmez conjecture takes the form

$$h_{\mathrm{Fal}}(X) = -\frac{1}{2}\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} - \frac{1}{4}\log\left(\frac{|d_E|}{d_F}\right) - \frac{n}{2}\log(2\pi), \tag{8.1}$$

where $L(\chi_{E/F}, s)$ is the L–function of the Hecke character $\chi_{E/F}$ associated to the quadratic extension E/F.

Now, suppose that E is a quartic CM field. Then the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\Phi(E)$ is transitive if and only if E is non-biquadratic; that is, if and only if E is cyclic or non-abelian (see e.g. [Shi98, Example 8.4 (2)]). Therefore, if E is a non-biquadratic quartic CM field and $X = X_\Phi$ is a CM abelian surface of type $(\mathcal{O}_E, \Phi)$, then the Colmez conjecture is true and takes the form

$$h_{\mathrm{Fal}}(X) = -\frac{1}{2}\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} - \frac{1}{4}\log\left(\frac{d_E}{d_F}\right) - \log(2\pi). \tag{8.2}$$

Note that for quartic CM fields, we can replace $|d_E|$ with d_E in (8.1) since by the conductor-discriminant formula, the sign of d_E equals $(-1)^{r_2}$ where $r_2 = 2$ is the number of pairs of complex conjugate embeddings of E.

For the remaining part of the proof, we assume that F is a real quadratic field with narrow class number 1 and E is a CM extension of F with $E/\mathbb{Q}$ non-biquadratic.

Since E/F is a CM extension, by Propositions 7.1 and 7.2 the Hecke character $\chi_{E/F}$ is a primitive narrow ray class character modulo $\mathfrak{D}_{E/F}$ with admissible vector $\mathbf{p} = (1, 1)$. In particular, by Remark 3.1 we have $L(\chi_{E/F}, 0) \neq 0$. This special value can be determined using the analytic class number formula.

Lemma 8.1. *We have*

$$L(\chi_{E/F}, 0) = 2h_E/\mu_{E/F}, \tag{8.3}$$

where $\mu_{E/F} := [\mathcal{O}_E^\times : \mathcal{O}_F^\times]$.

Proof. For a number field K of degree $n = r_1 + 2r_2$ and signature (r_1, r_2), the Taylor expansion of the Dedekind zeta function $\zeta_K(s)$ at $s = 0$ has the form

$$\zeta_K(s) = -\frac{h_K R_K}{w_K} s^{r_1+r_2-1} + O(s^{r_1+r_2}),$$

where h_K is the class number, R_K is the regulator, and w_K is the number of roots of unity contained in K. Now, since E and F have signatures $(r_1, r_2) = (0, 2)$ and $(2, 0)$ respectively, and since $h_F = 1$ and $w_F = 2$, then the Taylor expansions of the respective Dedekind zeta functions at $s = 0$ are given by

$$\zeta_E(s) = -\frac{h_E R_E}{w_E} s + O(s^2) \quad \text{and} \quad \zeta_F(s) = -\frac{R_F}{2} s + O(s^2).$$

Moreover, since $L(\chi_{E/F}, s)$ is holomorphic at $s = 0$, then comparing the leading terms on both sides of the identity $\zeta_E(s) = \zeta_F(s)L(\chi_{E/F}, s)$ shows that

$$-\frac{h_E R_E}{w_E} = -\frac{R_F}{2} L(\chi_{E/F}, 0),$$

or equivalently, that

$$L(\chi_{E/F}, 0) = \frac{2h_E R_E}{w_E R_F}. \tag{8.4}$$

By [Was97, Proposition 4.16], we have the formula

$$\frac{R_E}{R_F} = \frac{2}{[\mathcal{O}_E^\times : W_E \mathcal{O}_F^\times]}, \tag{8.5}$$

where W_E is the group of roots of unity contained in E. From the tower of subgroups $\mathcal{O}_F^\times \subseteq W_E \mathcal{O}_F^\times \subseteq \mathcal{O}_E^\times$, we get the relation

$$[\mathcal{O}_E^\times : \mathcal{O}_F^\times] = [\mathcal{O}_E^\times : W_E \mathcal{O}_F^\times] \cdot [W_E \mathcal{O}_F^\times : \mathcal{O}_F^\times]. \tag{8.6}$$

Writing $W_E = \{\pm u_1, \ldots, \pm u_r\}$ for $r = w_E/2$ and observing that $W_E \cap \mathcal{O}_F^\times = \{\pm 1\}$, one can see that the different left cosets of $\mathcal{O}_F^\times$ in $W_E \mathcal{O}_F^\times$ are precisely the cosets $u_i \mathcal{O}_F^\times$ for $i = 1, \ldots, w_E/2$. Therefore we have $[W_E \mathcal{O}_F^\times : \mathcal{O}_F^\times] = w_E/2$. Combining this with (8.6), we obtain

$$[\mathcal{O}_E^\times : W_E \mathcal{O}_F^\times] = \frac{2[\mathcal{O}_E^\times : \mathcal{O}_F^\times]}{w_E}. \tag{8.7}$$

Finally, combining the identities (8.4), (8.5), and (8.7) gives the formula (8.3). $\square$

Continuing with the proof of Theorem 1.1, we use (6.5) and (8.3) to get

$$\begin{aligned}\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} &= -\log\left(N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F})\right) \\ &\quad + \frac{\mu_{E/F}}{2h_E}\sum_{z\in\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})}\chi_{E/F}(\mathfrak{D}_{E/F}\langle z\rangle)\log\left(\prod_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})}\Gamma_2(z^\sigma,(1,\varepsilon^\sigma))\right) \\ &\quad + \frac{\varepsilon-\varepsilon'}{4}\log(\varepsilon')\frac{\mu_{E/F}}{h_E}\sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})\\ z=x+y\varepsilon}}\chi_{E/F}(\mathfrak{D}_{E/F}\langle z\rangle)B_2(x).\end{aligned} \tag{8.8}$$

The identity (8.8) can be written as

$$\frac{L'(\chi_{E/F},0)}{L(\chi_{E/F},0)} = \log\left(\frac{(\varepsilon')^{C_1}}{N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F})}\prod_{z\in\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})}\prod_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})}\Gamma_2(z^\sigma,(1,\varepsilon^\sigma))^{C_2}\right), \tag{8.9}$$

where

$$C_1 := \frac{\varepsilon-\varepsilon'}{4}\frac{\mu_{E/F}}{h_E}\sum_{\substack{z\in\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})\\ z=x+y\varepsilon}}\chi_{E/F}(\mathfrak{D}_{E/F}\langle z\rangle)B_2(x)$$

and

$$C_2 := \frac{\mu_{E/F}\chi_{E/F}(\mathfrak{D}_{E/F}\langle z\rangle)}{2h_E}.$$

Then combining (8.2) and (8.9) yields

$$h_{\mathrm{Fal}}(X) = \log\left(\frac{1}{2\pi}\left(\frac{d_F}{d_E}\right)^{\frac{1}{4}}\left(\frac{N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F})}{(\varepsilon)^{C_1}}\right)^{\frac{1}{2}}\prod_{z\in\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})}\prod_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})}\Gamma_2(z^\sigma,(1,\varepsilon^\sigma))^{-\frac{C_2}{2}}\right). \tag{8.10}$$

We have $\varepsilon\varepsilon' = 1$, so that $1/(\varepsilon')^{C_1} = \varepsilon^{C_1}$. Also, $N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F}) = d_E/d_F^2$, and thus

$$\left(\frac{d_F}{d_E}\right)^{1/4} N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F})^{1/2} = \left(\frac{d_E}{d_F^3}\right)^{1/4}.$$

Finally, from these calculations and (8.10), we get

$$\exp[h_{\mathrm{Fal}}(X)] = \frac{1}{2\pi}\left(\frac{d_E}{d_F^3}\right)^{1/4}\varepsilon^{C_1/2}\prod_{z\in\mathcal{R}(\varepsilon,\mathfrak{D}_{E/F}^{-1})}\prod_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})}\Gamma_2(z^\sigma,(1,\varepsilon^\sigma))^{-C_2/2}.$$

This completes the proof of Theorem 1.1. □

Appendix A. Models for CM curves of genus 2

In this appendix, we discuss the interesting (and difficult) problem of finding explicit models for CM curves of genus 2. In particular, we explain how to use the work of Bouyer and Streng [BS15] to find models for the CM curves of genus 2 whose Faltings heights we evaluated in the examples of Section 2.

A.1. **Background.** Recall that an abelian variety A is called *simple* if the only abelian subvarieties of A are $\{0\}$ and A. Moreover, if E is a CM field and $\Phi \in \Phi(E)$ is a CM type for E, then Φ is said to be *primitive* if every CM abelian variety of that type is simple (see e.g. [Shi98, Section 8.2]).

Suppose now that E is a quartic CM field, and let E^c denote the Galois closure of $E/\mathbb{Q}$. Then there are three possibilities for $\mathrm{Gal}(E^c/\mathbb{Q})$. When $E/\mathbb{Q}$ is Galois, E can be biquadratic with Galois group $\mathrm{Gal}(E/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, or cyclic with Galois group $\mathrm{Gal}(E/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$. When $E/\mathbb{Q}$ is non-Galois, then $\mathrm{Gal}(E^c/\mathbb{Q}) \cong D_4$.

Let $X = X_\Phi$ be a CM abelian surface of type $(\mathcal{O}_E, \Phi)$. Shimura showed that Φ is not primitive if and only if E is biquadratic (see [Shi98, Example 8.4 (2), p. 64]). Moreover, it is known that a principally polarized abelian surface is either the Jacobian of a smooth curve of genus 2 or the product of two elliptic curves (see e.g. [BL04, Corollary 11.8.2 (a)]). Therefore, the CM abelian surface $X = X_\Phi$ is the Jacobian of a CM curve of genus 2 if E is non-biquadratic, and it is the product of two CM elliptic curves if E is biquadratic.

When evaluating Faltings heights of CM abelian surfaces, we want to exclude the "degenerate case" of a product of two CM elliptic curves, so we assume that E is a non-biquadratic quartic CM field. In particular, X must be the Jacobian J_C of a nonsingular CM curve C of genus 2 defined over $\overline{\mathbb{Q}}$. The curve C is hyperelliptic, so is birational to an affine curve $C : y^2 = f(x)$ where $f(x) \in \overline{\mathbb{Q}}[x]$ has degree 5 or 6 with no roots of multiplicity greater than 1. There is an algorithm to find a model for such a curve C, which can be described roughly as follows (see e.g. [vWam99, Algorithm 1]).

Given a CM abelian surface X, one finds a corresponding CM point τ in the Siegel upper half-plane $\mathfrak{h}_2$ and uses this to compute the Rosenhain normal form of C. The coefficients of the Rosenhain normal form will usually not be easily recognizable algebraic numbers, and thus more sophisticated analysis must be performed. In particular, one computes the Igusa invariants of the Rosenhain normal form, then uses an algorithm of Mestre [Mes91] to find a number field k and corresponding equation $C : y^2 = f(x)$ with $f(x) \in k[x]$. This involves finding a k-rational point in $\mathbb{P}^2_k$ for a certain conic whose coefficients are constructed from the Igusa invariants. Note that the coefficients of this equation may be very large, hence one may need to perform further reduction to make the coefficients smaller.

Van Wamelen [vWam99] gave a list of 19 CM curves of genus 2 which are defined over $\mathbb{Q}$. These curves have complex multiplication by a cyclic quartic CM field E. In fact, a result of Shimura (see e.g. Proposition 5.17 and the second paragraph of p. 132 in [Shi94]) implies that a CM curve of genus 2 with complex multiplication by a non-Galois quartic CM field E cannot be defined over $\mathbb{Q}$. Very recently, Bouyer and Streng [BS15] gave a list of CM curves of genus 2 which have complex multiplication by a non-Galois quartic CM field E and which are defined over the real quadratic subfield $\widetilde{F}$ of the reflex field $\widetilde{E}$ of E. In Section 2, we use Theorem 1.1 to evaluate the Faltings heights of Jacobians of certain curves in these lists.

A.2. **Models for CM curves of genus 2.** In preparation for our examples, we use the work of Bouyer and Streng [BS15] to find models for genus 2 curves whose Jacobians have complex multiplication by non-abelian quartic CM fields. For convenience, we briefly describe how such models can be found using the tables in [BS15] (a more detailed explanation can be found in [BS15, Section 5.3]).

Let E be a quartic CM field with real quadratic subfield F. Then E is given up to isomorphism by a unique triple $[d_F, A, B]$ with

$$E \cong \mathbb{Q}[X]/\langle X^4 + AX^2 + B\rangle, \quad A, B \in \mathbb{Z}_+.$$

Similarly, let $\widetilde{E}$ be the reflex field of E (strictly speaking, this is the reflex field associated to a particular choice of CM type for E; see [BS15, section 5.1]). Then $\widetilde{E}$ is a quartic CM field with real quadratic subfield $\widetilde{F}$ which is given up to isomorphism by a unique triple $[d_{\widetilde{F}}, \widetilde{A}, \widetilde{B}]$ with

$$\widetilde{E} \cong \mathbb{Q}[X]/\langle X^4 + \widetilde{A}X^2 + \widetilde{B}\rangle, \quad \widetilde{A}, \widetilde{B} \in \mathbb{Z}_+.$$

Table 2b in [BS15, pp. 528–534] contains a list of models for genus 2 curves C whose Jacobians J_C have complex multiplication by a specified non-abelian quartic CM field E (see [BS15, Theorems 1.1 and 5.1]). The CM field E is specified by the triple $[d_F, A, B]$ in the first column of the table. The corresponding reflex field $\widetilde{E}$ is specified by the triple $[d_{\widetilde{F}}, \widetilde{A}, \widetilde{B}]$ in the second column of the table. The third column of the table gives an integer $a \in \mathcal{O}_{\widetilde{F}}$ which is defined as a quadratic polynomial in a root α of $X^4 + \widetilde{A}X^2 + \widetilde{B}$. The last column of the table gives a genus 2 curve $C : y^2 = f(x)$ whose Jacobian J_C is a CM abelian surfaces of type $(\mathcal{O}_E, \Phi)$, and which has coefficients of the form $ma + n \in \mathcal{O}_{\widetilde{F}}$ for integers $m, n \in \mathbb{Z}$. In particular, one gets a curve C for each of the four roots α of $X^4 + \widetilde{A}X^2 + \widetilde{B}$.

Example A.1 (The field $E = \mathbb{Q}(\sqrt{\Delta_1})$ where $\Delta_1 = (-13 + \sqrt{5})/2$)**.** The real quadratic subfield of E is $F = \mathbb{Q}(\sqrt{5})$. Moreover, since $\sqrt{\Delta_1}$ is a root of $X^4 + 13X^2 + 41$, we have

$$E \cong \mathbb{Q}[X]/\langle X^4 + 13X^2 + 41\rangle.$$

Thus, E is specified by the triple $[d_F, A, B] = [5, 13, 41]$ in the first column of Table 2b. The reflex field $\widetilde{E}$ is specified by the triple $[d_{\widetilde{F}}, \widetilde{A}, \widetilde{B}] = [41, 11, 20]$ in the second column of Table 2b. The integer $a \in \mathcal{O}_{\widetilde{F}}$ in the third column of Table 2b is given by $a = \alpha^2 + 5$ where α is a root of $X^4 + 11X^2 + 20$. Then the last column of Table 2b gives the genus 2 curve

$$C_a : \; y^2 = (-a + 3)x^6 + (4a - 8)x^5 + 10x^4 + (-a + 20)x^3 + (4a + 5)x^2 + (a + 4)x + 1$$

defined over $\widetilde{F} = \mathbb{Q}(\sqrt{41})$ whose Jacobian J_{C_a} is a CM abelian surface of type $(\mathcal{O}_E, \Phi)$.

There is a curve C_a corresponding to each of the four roots α of $X^4 + 11X^2 + 20$. These roots are given by

$$\alpha = \pm\sqrt{\frac{-11 \pm \sqrt{41}}{2}}.$$

For example, if we take $\alpha = \sqrt{\frac{-11+\sqrt{41}}{2}}$, then $a = (-1 + \sqrt{41})/2$ and the curve C_a is defined by the equation

$$y^2 = \left(\frac{7 - \sqrt{41}}{2}\right)x^6 + \left(-10 + 2\sqrt{41}\right)x^5 + 10x^4 + \left(\frac{41 - \sqrt{41}}{2}\right)x^3 + \left(3 + 2\sqrt{41}\right)x^2 + \left(\frac{7 + \sqrt{41}}{2}\right)x + 1.$$

Example A.2 (The field $E = \mathbb{Q}(\sqrt{\Delta_2})$ where $\Delta_2 = -5 - 2\sqrt{2}$)**.** The real quadratic subfield of E is $F = \mathbb{Q}(\sqrt{2})$. Moreover, since $\sqrt{\Delta_2}$ is a root of $X^4 + 10X^2 + 17$, we have

$$E \cong \mathbb{Q}[X]/\langle X^4 + 10X^2 + 17\rangle.$$

Thus, E is specified by the triple $[d_F, A, B] = [8, 10, 17]$ in the first column of Table 2b. The reflex field $\widetilde{E}$ is specified by the triple $[d_{\widetilde{F}}, \widetilde{A}, \widetilde{B}] = [17, 5, 2]$ in the second column of Table 2b. The integer $a \in \mathcal{O}_{\widetilde{F}}$ in the third column of Table 2b is given by $a = \alpha^2 + 2$ where α is a root of $X^4 + 5X^2 + 2$. Then the last column of Table 2b gives the genus 2 curve

$$C_a : \; y^2 = x^6 + (2a + 4)x^5 + (3a + 14)x^4 + (10a + 8)x^3 + (-9a + 32)x^2 + (16a - 16)x - 4a + 1$$

defined over $\widetilde{F} = \mathbb{Q}(\sqrt{17})$ whose Jacobian J_{C_a} is a CM abelian surface of type $(\mathcal{O}_E, \Phi)$.

There is a curve C_a corresponding to each of the four roots α of X^4+5X^2+2. These roots are given by

$$\alpha = \pm\sqrt{\frac{-5\pm\sqrt{17}}{2}}.$$

For example, if we take $\alpha = \sqrt{\frac{-5+\sqrt{17}}{2}}$, then $a=(-5+\sqrt{17})/2$ and the curve C_a is defined by the equation

$$y^2 = x^6 + (3+\sqrt{17})x^5 + \left(\frac{25+3\sqrt{17}}{2}\right)x^4 + (3+5\sqrt{17})x^3 + \left(\frac{73-9\sqrt{17}}{2}\right)x^2 + (-24+8\sqrt{17})x + 10 - 2\sqrt{17}.$$

Appendix B. Computing Shintani sets

In this appendix, we describe an algorithm which can be implemented in `SageMath` to compute Shintani sets. In particular, this is the algorithm we used to compute the Shintani sets appearing in the examples of Section 2. The basic idea is to reduce the problem of computing a Shintani set to the geometric problem of finding integral points inside a polygonal region in $\mathbb{R}^2$.

We start by recalling some definitions from Section 4. Let F be a real quadratic field and $\mathfrak{f}\subset\mathcal{O}_F$ be an integral ideal of F. Let $\varepsilon>1$ be the generator of the group of totally positive units $\mathcal{O}_F^{\times,+}$. Then the *Shintani set associated to* $\mathfrak{f}$ is

$$\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1}) := \left\{ z = x + y\varepsilon \in \mathfrak{f}^{-1} \;\middle|\; x,y\in\mathbb{Q},\ 0<x\le 1,\ 0\le y<1 \right\}.$$

Similarly, the *restricted Shintani set associated to* $\mathfrak{f}$ is defined by

$$\mathcal{R}(\varepsilon,\mathfrak{f}^{-1}) := \left\{ z = x + y\varepsilon \in \mathfrak{f}^{-1} \;\middle|\; x,y\in\mathbb{Q},\ 0<x\le 1,\ 0\le y<1,\ \mathfrak{f}\langle z\rangle \text{ coprime to } \mathfrak{f} \right\}.$$

In Section 4, we showed that both of these sets are finite. In fact, we proved in Theorem 4.1 that

$$\#\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1}) = \frac{\varepsilon-\varepsilon'}{\sqrt{d_F}}\cdot N_{F/\mathbb{Q}}(\mathfrak{f}) \quad\text{and}\quad \#\mathcal{R}(\varepsilon,\mathfrak{f}^{-1}) = \frac{\varepsilon-\varepsilon'}{\sqrt{d_F}}\cdot\varphi(\mathfrak{f}), \tag{B.1}$$

where

$$\varphi(\mathfrak{f}) := N_{F/\mathbb{Q}}(\mathfrak{f})\cdot\prod_{\mathfrak{p}\mid\mathfrak{f}}\left(1-\frac{1}{N_{F/\mathbb{Q}}(\mathfrak{p})}\right)$$

is the generalized Euler φ-function for number fields.

Now, observe that since $\{1,\varepsilon\}$ is a $\mathbb{Q}$-basis for F, then every element of F, and in particular every element of $\mathfrak{f}^{-1}$, can be written uniquely as $x+y\varepsilon$ for some $x,y\in\mathbb{Q}$. Hence, the Shintani set $\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})$ is just the subset of F consisting of points $z=x+y\varepsilon\in\mathfrak{f}^{-1}$ with (x,y) a rational point in the half open square $(0,1]\times[0,1)$.

Our strategy to compute $\widetilde{\mathcal{R}}(\varepsilon,\mathfrak{f}^{-1})$ rests on the fact that $\mathfrak{f}^{-1}$ is a $\mathbb{Z}$-submodule of F of rank 2. Therefore, we first compute a $\mathbb{Z}$-basis $\{\alpha,\beta\}$ for $\mathfrak{f}^{-1}$ and express every element $z\in\mathfrak{f}^{-1}=\mathbb{Z}\alpha+\mathbb{Z}\beta$ as $z=m\alpha+n\beta$ for some unique integers $m,n\in\mathbb{Z}$. Write the coordinate vector of z with respect to the $\mathbb{Z}$-basis $\{\alpha,\beta\}$ as

$$[z]_{\{\alpha,\beta\}} = \begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{Z}^2.$$

Next, we find the coordinate vector $[z]_{\{1,\varepsilon\}}\in\mathbb{Q}^2$ and determine whether it lies in the square $(0,1]\times[0,1)$. Thus, let $P\in\mathrm{GL}_2(\mathbb{Q})$ be the change of basis matrix $P=[P_1P_2]$ with columns given by

$$P_1 = [\alpha]_{\{1,\varepsilon\}} = \begin{bmatrix} a \\ c \end{bmatrix} \in \mathbb{Q}^2 \qquad\text{and}\qquad P_2 = [\beta]_{\{1,\varepsilon\}} = \begin{bmatrix} b \\ d \end{bmatrix} \in \mathbb{Q}^2.$$

Then we have the relation $[z]_{\{1,\varepsilon\}} = P[z]_{\{\alpha,\beta\}}$, which is given explicitly by

$$[z]_{\{1,\varepsilon\}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} am+bn \\ cm+dn \end{bmatrix}.$$

It follows that the points in the Shintani set $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$ are the points $z = m\alpha + n\beta$ with $m, n \in \mathbb{Z}$ satisfying the inequalities

$$\begin{cases} 0 < am + bn \leq 1 \\ 0 \leq cm + dn < 1. \end{cases}$$

Since each solution $(m, n) \in \mathbb{Z}^2$ to this system of inequalities gives a different point $z \in \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$, and the latter set is finite, this system has a finite number of solutions.

We summarize the preceding discussion in the following algorithm.

Algorithm B.1 Shintani sets for real quadratic fields

INPUT: A pair $(F, \mathfrak{f})$ consisting of a real quadratic field F and a nonzero integral ideal $\mathfrak{f} \subset \mathcal{O}_F$.
OUTPUT: The Shintani set $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$ associated to the integral ideal $\mathfrak{f}$.

1: Compute the unique generator $\varepsilon > 1$ of the group of totally positive units $\mathcal{O}_F^{\times,+}$.
2: Compute a $\mathbb{Z}$-basis $\{\alpha, \beta\}$ for the fractional ideal $\mathfrak{f}^{-1}$.
3: Since both $\{1, \varepsilon\}$ and $\{\alpha, \beta\}$ are $\mathbb{Q}$-bases for F as a $\mathbb{Q}$-vector space, compute the change of basis matrix $P = [P_1 P_2] \in \mathrm{GL}_2(\mathbb{Q})$ with columns given by the coordinate vectors

$$P_1 = [\alpha]_{\{1,\varepsilon\}} = \begin{bmatrix} a \\ c \end{bmatrix} \in \mathbb{Q}^2 \qquad \text{and} \qquad P_2 = [\beta]_{\{1,\varepsilon\}} = \begin{bmatrix} b \\ d \end{bmatrix} \in \mathbb{Q}^2.$$

4: Find all pairs $(m, n) \in \mathbb{Z}^2$ that satisfy the system of inequalities

$$\begin{cases} 0 < am + bn \leq 1 \\ 0 \leq cm + dn < 1. \end{cases}$$

5: For each pair $(m, n) \in \mathbb{Z}^2$ found in Step 4, compute the algebraic number $z_{m,n} = m\alpha + n\beta \in \mathfrak{f}^{-1}$. The set of all such numbers $z_{m,n}$ is the Shintani set $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{f}^{-1})$.
6: Discard every number $z_{m,n}$ such that $\mathfrak{f}\langle z_{m,n}\rangle$ is not coprime to $\mathfrak{f}$. The remaining set of numbers is the Shintani set $\mathcal{R}(\varepsilon, \mathfrak{f}^{-1})$.

To illustrate Algorithm B.1, we show how it can be used to compute the Shintani sets appearing in the examples from Section 2.

Example B.1. We start by showing how to compute the Shintani set from Example 2.1. In that example, we worked with the CM extension E/F given by $F = \mathbb{Q}(\sqrt{5})$ and $E = \mathbb{Q}(\zeta_5)$, where $\zeta_5 = e^{2\pi i/5}$. Thus, we apply the algorithm to the input pair $(F, \mathfrak{D}_{E/F})$, where the relative discriminant ideal is given by $\mathfrak{D}_{E/F} = \sqrt{5}\mathcal{O}_F$.

In Step 1 we compute $\varepsilon = (3 + \sqrt{5})/2$. Note that the formulas (B.1) imply that

$$\#\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}} \cdot N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F}) = 1 \cdot 5 = 5$$

and

$$\#\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}} \cdot \varphi(\mathfrak{D}_{E/F}) = 1 \cdot 4 = 4,$$

as can be easily checked since $\varphi(\mathfrak{D}_{E/F}) = N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F}) - 1$ because $\mathfrak{D}_{E/F}$ is a prime ideal (see also the first entry in Table 1 in the introduction of the paper).

In Step 2 we compute the $\mathbb{Z}$-basis $\{\alpha, \beta\}$ for $\mathfrak{D}_{E/F}^{-1}$ given by $\alpha = 1$ and $\beta = (5 + \sqrt{5})/10$.

In Step 3 we compute the change of basis matrix P to be

$$P = \begin{bmatrix} 1 & 1/5 \\ 0 & 1/5 \end{bmatrix}.$$

In Step 4 we find all pairs $(m, n) \in \mathbb{Z}^2$ that satisfy the inequalities

$$\begin{cases} 0 < m + \dfrac{n}{5} \leq 1 \\ 0 \leq \dfrac{n}{5} < 1. \end{cases} \tag{B.2}$$

The inequalities (B.2) determine the shaded region in the mn-plane shown in Figure B.1. The pairs $(m, n) \in \mathbb{Z}^2$ satisfying (B.2) are the 5 integral points (m, n) lying in the interior of the region, i.e., the pairs $(1, 0), (0, 1), (0, 2), (0, 3)$ and $(0, 4)$.

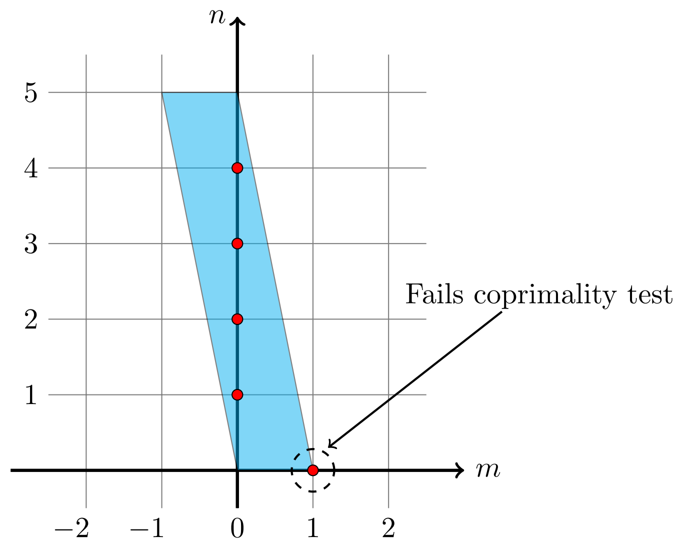

FIGURE B.1. The region determined by the inequalities (B.2) and the integral points (m, n) satisfying them.

In Step 5 we use the pairs (m, n) to compute the numbers $z_{m,n} = m\alpha + n\beta$ appearing in the Shintani set $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$, and we get

$$\begin{aligned} \widetilde{\mathcal{R}}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) &= \{z_{1,0}\} \cup \{z_{0,n} \mid n = 1, 2, 3, 4\} \\ &= \{1\} \cup \left\{ n \left(\frac{5 + \sqrt{5}}{10} \right) \,\middle|\, n = 1, 2, 3, 4 \right\}. \end{aligned}$$

In Step 6 we discard the numbers $z_{m,n}$ such that $\mathfrak{D}_{E/F}\langle z_{m,n} \rangle$ is not coprime to $\mathfrak{D}_{E/F}$. We find that the only such number is $z_{1,0} = 1$, which corresponds to the circled pair $(1, 0)$ shown in Figure B.1. The remaining set of 4 numbers is the Shintani set $\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$ appearing in Example 2.1.

Example B.2. We now compute the Shintani set from Example 2.3. In that example, we worked with the CM extension E/F given by $F = \mathbb{Q}(\sqrt{5})$ and $E = F(\sqrt{\Delta})$, where $\Delta = (-13 + \sqrt{5})/2$. Thus, we apply the algorithm to the input pair $(F, \mathfrak{D}_{E/F})$, where the relative discriminant ideal is given by $\mathfrak{D}_{E/F} = \Delta \mathcal{O}_F$.

In Step 1 we compute $\varepsilon = (3+\sqrt{5})/2$. Note that the formulas (B.1) imply that

$$\#\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}} \cdot N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F}) = 1 \cdot 41 = 41$$

and

$$\#\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}} \cdot \varphi(\mathfrak{D}_{E/F}) = 1 \cdot 40 = 40,$$

as can be easily checked since $\varphi(\mathfrak{D}_{E/F}) = N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F}) - 1$ because $\mathfrak{D}_{E/F}$ is a prime ideal (see also the second entry in Table 1 in the introduction of the paper).

In Step 2 we compute the $\mathbb{Z}$-basis $\{\alpha, \beta\}$ for $\mathfrak{D}_{E/F}^{-1}$ given by $\alpha = 1$ and $\beta = (13+\sqrt{5})/82$.

In Step 3 we compute the change of basis matrix P to be

$$P = \begin{bmatrix} 1 & 5/41 \\ 0 & 1/41 \end{bmatrix}.$$

In Step 4 we find all pairs $(m,n) \in \mathbb{Z}^2$ that satisfy the inequalities

$$\begin{cases} 0 < m + \dfrac{5}{41} n \leq 1 \\ 0 \leq \dfrac{n}{41} < 1. \end{cases} \tag{B.3}$$

The inequalities (B.3) determine the shaded region in the mn-plane shown in Figure B.2. The pairs $(m,n) \in \mathbb{Z}^2$ satisfying (B.3) are the 41 integral points (m,n) lying in the interior of the region, which are given in the following table.

The pairs (m,n) satisfying the inequalities (B.3)						
m	1	0	−1	−2	−3	−4
Values of n	0	1	9	17	25	33
		2	10	18	26	34
		3	11	19	27	35
		4	12	20	28	36
		5	13	21	29	37
		6	14	22	30	38
		7	15	23	31	39
		8	16	24	32	40

In Step 5 we use the pairs (m,n) to compute the numbers $z_{m,n} = m\alpha + n\beta$ appearing in the Shintani set $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$. By making the linear change of variables $(m,n) \mapsto (-m, 8m+n)$ and letting $\widetilde{z}_{m,n} := z_{-m,8m+n}$, we can write the Shintani set as

$$\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) = \left\{ \widetilde{z}_{m,n} := -m + (8m+n)\frac{13+\sqrt{5}}{82} \;\middle|\; 0 \leq m \leq 4,\ 1 \leq n \leq 8 \right\}.$$

In Step 6 we discard the numbers $\widetilde{z}_{m,n}$ such that $\mathfrak{D}_{E/F}\langle \widetilde{z}_{m,n} \rangle$ is not coprime to $\mathfrak{D}_{E/F}$. We find that the only such number is $\widetilde{z}_{-1,0} = z_{1,0} = 1$, which corresponds to the circled pair $(1,0)$ shown in Figure B.2. The remaining set of 40 numbers is the Shintani set $\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$ appearing in Example 2.3.

Example B.3. To illustrate Algorithm B.1, we show how it can be used to compute the Shintani set from Example 2.5. In that example, we worked with the CM extension E/F given by $F = \mathbb{Q}(\sqrt{2})$ and $E = F(\sqrt{\Delta})$, where $\Delta = -5 - 2\sqrt{2}$. Thus, we apply the algorithm to the input pair $(F, \mathfrak{D}_{E/F})$, where the relative discriminant ideal is given by $\mathfrak{D}_{E/F} = \Delta\mathcal{O}_F$.

In Step 1 we compute $\varepsilon = 3 + 2\sqrt{2}$. Note that the formulas (B.1) imply that

$$\#\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}} \cdot N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F}) = 2 \cdot 17 = 34$$

and

$$\#\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) = \frac{\varepsilon - \varepsilon'}{\sqrt{d_F}} \cdot \varphi(\mathfrak{D}_{E/F}) = 2 \cdot 16 = 32,$$

as can be easily checked since $\varphi(\mathfrak{D}_{E/F}) = N_{F/\mathbb{Q}}(\mathfrak{D}_{E/F}) - 1$ because $\mathfrak{D}_{E/F}$ is a prime ideal (see also the third entry in Table 1 in the introduction of the paper).

In Step 2 we compute the $\mathbb{Z}$-basis $\{\alpha, \beta\}$ for $\mathfrak{D}_{E/F}^{-1}$ given by $\alpha = 1$ and $\beta = (6 + \sqrt{2})/17$.

In Step 3 we compute the change of basis matrix P to be

$$P = \begin{bmatrix} 1 & 9/34 \\ 0 & 1/34 \end{bmatrix}.$$

In Step 4 we find all pairs $(m, n) \in \mathbb{Z}^2$ that satisfy the inequalities

$$\begin{cases} 0 < m + \dfrac{9}{34} n \leq 1 \\ 0 \leq \dfrac{n}{34} < 1. \end{cases} \tag{B.4}$$

The inequalities (B.4) determine the shaded region in the mn-plane shown in Figure B.3. The pairs $(m, n) \in \mathbb{Z}^2$ satisfying (B.4) are the 34 integral points (m, n) lying in the interior of the region, which are given in the following table.

The pairs (m,n) satisfying the inequalities (B.4)										
m	1	0	-1	-2	-3	-4	-5	-6	-7	-8
Values of n	0	1	4	8	12	16	19	23	27	31
		2	5	9	13	17	20	24	28	32
		3	6	10	14	18	21	25	29	33
			7	11	15		22	26	30	

In Step 5 we use the pairs (m, n) to compute the numbers $z_{m,n} = m\alpha + n\beta$ appearing in the Shintani set $\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$. By making the linear change of variables $(m, n) \mapsto (-m, 4m + n - 1)$ and letting $\widetilde{z}_{m,n} := z_{-m,4m+n-1}$, we can write the Shintani set as

$$\widetilde{\mathcal{R}}(\varepsilon, \mathfrak{D}_{E/F}^{-1}) = \left\{ \widetilde{z}_{m,n} := -m + (4m + n - 1)\frac{6 + \sqrt{2}}{17} \;\middle|\; 0 \leq m \leq 8,\ n \in \widetilde{S}(m) \right\},$$

where

$$\widetilde{S}(m) := \begin{cases} \{0\} & \text{if } m = -1 \\ \{2,3,4\} & \text{if } m = 0 \\ \{1,2,3,4\} & \text{if } m = 1,2,3 \\ \{1,2,3\} & \text{if } m = 4 \\ \{0,1,2,3\} & \text{if } m = 5,6,7 \\ \{0,1,2\} & \text{if } m = 8. \end{cases}$$

In Step 6 we discard the numbers $\widetilde{z}_{m,n}$ such that $\mathfrak{D}_{E/F}\langle \widetilde{z}_{m,n} \rangle$ is not coprime to $\mathfrak{D}_{E/F}$. We find that these numbers are $\widetilde{z}_{-1,0} = z_{1,0}$ and $\widetilde{z}_{4,2} = z_{-4,17}$, which correspond to the circled pairs $(1, 0)$ and $(-4, 17)$ shown in Figure B.3. The remaining set of 32 numbers is the Shintani set $\mathcal{R}(\varepsilon, \mathfrak{D}_{E/F}^{-1})$ appearing in Example 2.5.

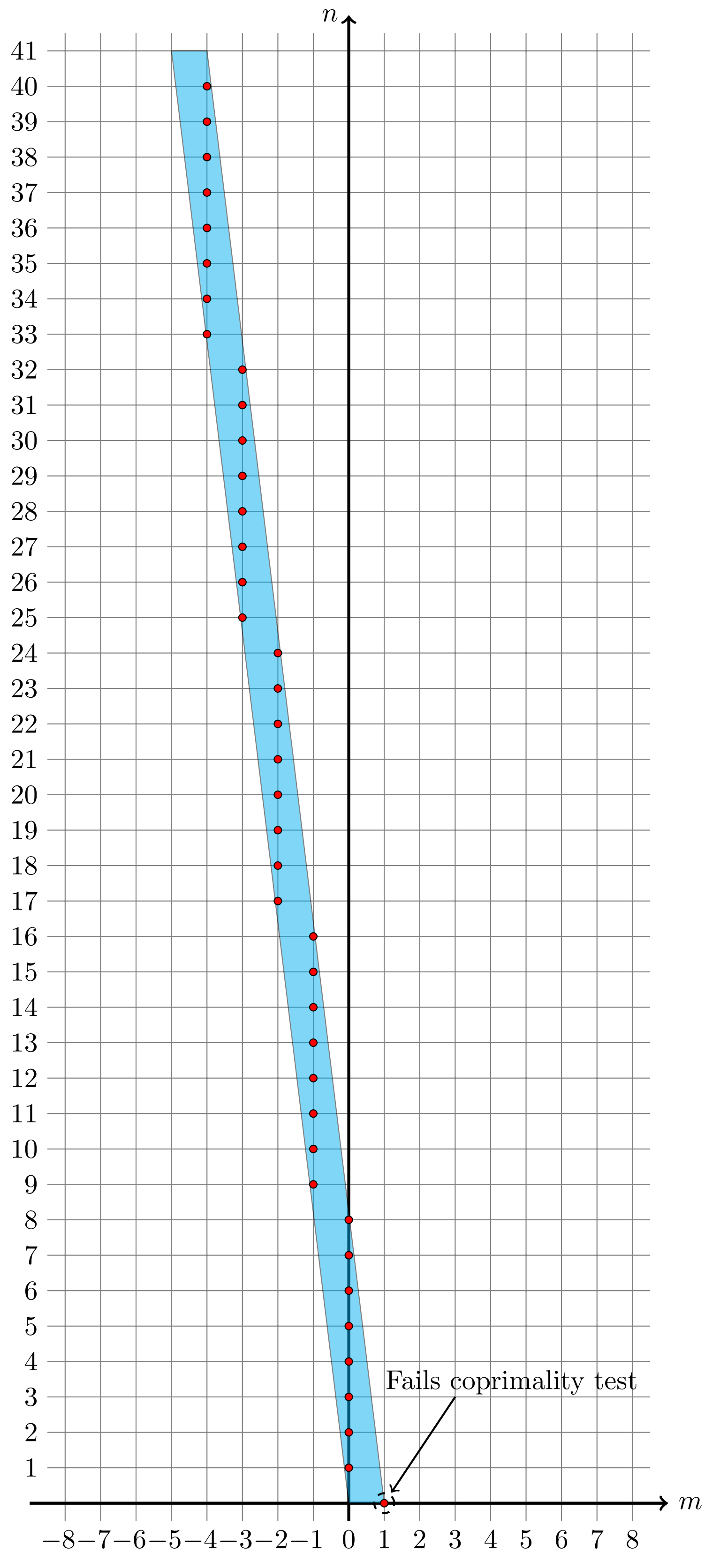

FIGURE B.2. The region determined by the inequalities (B.3) and the integral points (m, n) satisfying them.

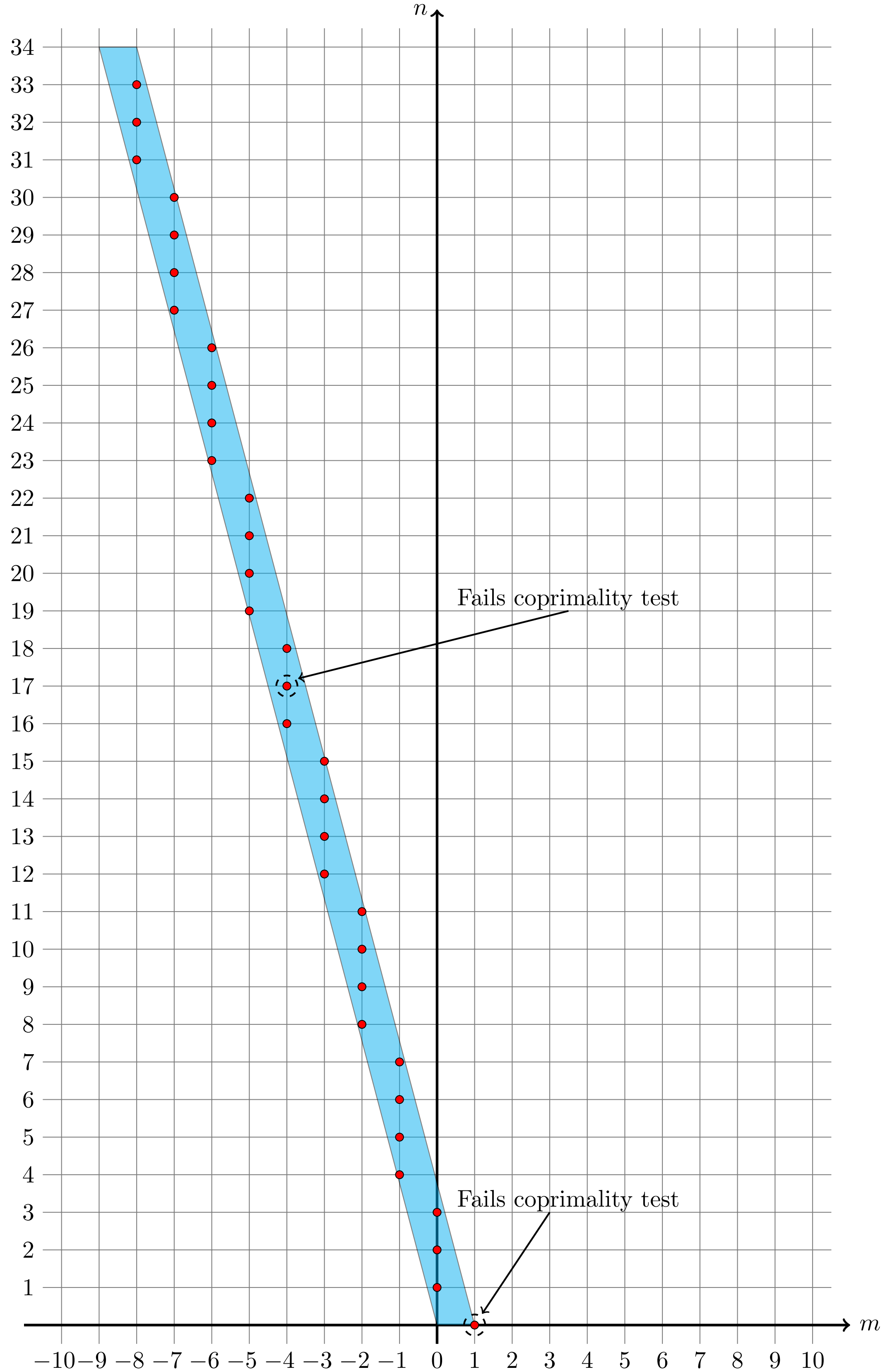

FIGURE B.3. The region determined by the inequalities (B.4) and the integral points (m, n) satisfying them.

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