

# EFFECTIVE BOUNDS FOR THE ANDREWS SPT-FUNCTION

MADLINE LOCUS DAWSEY\* AND RIAD MASRI

ABSTRACT. In this paper, we establish an asymptotic formula with an effective bound on the error term for the Andrews smallest parts function  $\text{spt}(n)$ . We use this formula to prove recent conjectures of Chen concerning inequalities which involve the partition function  $p(n)$  and  $\text{spt}(n)$ . Further, we strengthen one of the conjectures, and prove that for every  $\epsilon > 0$  there is an effectively computable constant  $N(\epsilon) > 0$  such that for all  $n \geq N(\epsilon)$ , we have

$$\frac{\sqrt{6}}{\pi} \sqrt{n} p(n) < \text{spt}(n) < \left( \frac{\sqrt{6}}{\pi} + \epsilon \right) \sqrt{n} p(n).$$

Due to the conditional convergence of the Rademacher-type formula for  $\text{spt}(n)$ , we must employ methods which are completely different from those used by Lehmer to give effective error bounds for  $p(n)$ . Instead, our approach relies on the fact that  $p(n)$  and  $\text{spt}(n)$  can be expressed as traces of singular moduli.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

The smallest parts function  $\text{spt}(n)$  of Andrews is defined for any integer  $n \geq 1$  as the number of smallest parts among the integer partitions of size  $n$ . For example, the partitions of  $n = 4$  are (with the smallest parts underlined):

$$\begin{aligned} &\underline{4}, \\ &3 + \underline{1}, \\ &\underline{2} + \underline{2}, \\ &2 + \underline{1} + \underline{1}, \\ &\underline{1} + \underline{1} + \underline{1} + \underline{1}, \end{aligned}$$

and so  $\text{spt}(4) = 10$ . The  $\text{spt}$ -function has many remarkable properties. For example, Andrews [3] proved the following analogues of the well-known Ramanujan congruences for the partition function  $p(n)$ :

$$\begin{aligned} \text{spt}(5n + 4) &\equiv 0 \pmod{5}, \\ \text{spt}(7n + 5) &\equiv 0 \pmod{7}, \\ \text{spt}(13n + 6) &\equiv 0 \pmod{13}. \end{aligned}$$

One can compute  $\text{spt}(n)$  by making use of the generating function

$$\sum_{n=1}^{\infty} \text{spt}(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^n)^2 (q^{n+1}; q)_{\infty}},$$

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\*This author was previously known as Madeline Locus.

where

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

We use this generating function to compute the values of  $\text{spt}(n)$  required for this paper.

In this paper, we will prove the following asymptotic formula for  $\text{spt}(n)$  with an effective bound on the error term.

**Theorem 1.1.** *Let  $\mu(n) := \pi\sqrt{24n-1}/6$ . Then for all  $n \geq 1$ , we have*

$$\text{spt}(n) = \frac{\sqrt{3}}{\pi\sqrt{24n-1}} e^{\mu(n)} + E_s(n),$$

where

$$|E_s(n)| < (5.78 \times 10^{43}) 2^{q(n)} (24n-1)^2 e^{\mu(n)/2}$$

with

$$q(n) := \frac{\log(24n-1)}{|\log(\log(24n-1)) - 1.1714|}.$$

Our interest in proving the effective bounds of Theorem 1.1 was motivated in part by the following recent conjectures of Chen [10] concerning inequalities which involve  $p(n)$  and  $\text{spt}(n)$ .

**Conjecture (Chen).**

(1) *For  $n \geq 5$ , we have*

$$\frac{\sqrt{6}}{\pi} \sqrt{n} p(n) < \text{spt}(n) < \sqrt{n} p(n).$$

(2) *For  $(a, b) \neq (2, 2)$  or  $(3, 3)$ , we have*

$$\text{spt}(a) \text{spt}(b) > \text{spt}(a+b).$$

(3) *For  $n \geq 36$ , we have*

$$\text{spt}(n)^2 > \text{spt}(n-1) \text{spt}(n+1).$$

(4) *For  $n > m > 1$ , we have*

$$\text{spt}(n)^2 > \text{spt}(n-m) \text{spt}(n+m).$$

(5) *For  $n \geq 13$ , we have*

$$\frac{\text{spt}(n-1)}{\text{spt}(n)} \left(1 + \frac{1}{n}\right) > \frac{\text{spt}(n)}{\text{spt}(n+1)}.$$

(6) *For  $n \geq 73$ , we have*

$$\frac{\text{spt}(n-1)}{\text{spt}(n)} \left(1 + \frac{\pi}{\sqrt{24n^{3/2}}}\right) > \frac{\text{spt}(n)}{\text{spt}(n+1)}.$$

**Remark.** Conjectures (1) and (2) are slight modifications of Chen's original claims.

By combining Theorem 1.1 with classical work of Lehmer [20] which gives effective bounds for  $p(n)$ , we will prove the following result.

**Theorem 1.2.** *All of Chen's conjectures are true.*

We will also use Theorem 1.1 to prove the following more precise version of Theorem 1.2 regarding Conjecture (1).

**Theorem 1.3** (Refined Theorem 1.2 (1)). *For each  $\epsilon > 0$ , there is an effectively computable constant  $N(\epsilon) > 0$  such that for all  $n \geq N(\epsilon)$ , we have*

$$\frac{\sqrt{6}}{\pi} \sqrt{np(n)} < \text{spt}(n) < \left( \frac{\sqrt{6}}{\pi} + \epsilon \right) \sqrt{np(n)}.$$

**Remark.** The constant  $N(\epsilon)$  of Theorem 1.3 can be computed in practice. For example, by letting  $\epsilon = 1 - \sqrt{6}/\pi$  in Theorem 1.3, we get Theorem 1.2 (1) for  $n \geq N(1 - \sqrt{6}/\pi)$ . Then, by computing  $N(1 - \sqrt{6}/\pi) = 12788$ , we use a computer to verify Theorem 1.2 (1) in the exceptional range  $5 \leq n < 12788$ .

**Remark.** In analogy with the Hardy-Ramanujan asymptotic for  $p(n)$ , Bringmann [6] used the circle method to establish the asymptotic

$$\text{spt}(n) \sim \frac{1}{\pi\sqrt{8n}} e^{\pi\sqrt{\frac{2n}{3}}}$$

as  $n \rightarrow \infty$ . Bringmann's asymptotic for  $\text{spt}(n)$  implies that

$$(1) \quad \text{spt}(n) \sim \frac{\sqrt{6}}{\pi} \sqrt{np(n)}$$

as  $n \rightarrow \infty$ . Theorem 1.3 refines the asymptotic (1).

We now describe our approach to Theorem 1.1. In particular, we explain some of the difficulties involved in proving effective bounds for  $\text{spt}(n)$ .

In [25], Rademacher established an exact formula for  $p(n)$  as the absolutely convergent infinite sum

$$(2) \quad p(n) = \frac{2\pi}{(24n-1)^{3/4}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} I_{3/2} \left( \frac{\pi\sqrt{24n-1}}{6c} \right),$$

where  $I_\nu$  is the  $I$ -Bessel function,  $A_c(n)$  is the Kloosterman sum

$$A_c(n) := \sum_{\substack{d \pmod{c} \\ (d,c)=1}} e^{\pi i s(d,c)} e^{-\frac{2\pi i dn}{c}},$$

and  $s(d, c)$  is the classical Dedekind sum

$$s(d, c) := \sum_{r=1}^{c-1} \frac{r}{c} \left( \frac{dr}{c} - \left\lfloor \frac{dr}{c} \right\rfloor - \frac{1}{2} \right).$$

Recently, Ahlgren and Andersen [1] gave a Rademacher-type exact formula for  $\text{spt}(n)$  as the conditionally convergent infinite sum

$$(3) \quad \text{spt}(n) = \frac{\pi}{6} (24n-1)^{\frac{1}{4}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} (I_{1/2} - I_{3/2}) \left( \frac{\pi\sqrt{24n-1}}{6c} \right).$$

In order to give an effective bound on the error term for  $p(n)$ , Lehmer [20] truncated the absolutely convergent sum (2) and applied bounds for the Kloosterman sum  $A_c(n)$ . On the other hand, since the formula (3) is only conditionally convergent, bounding  $\text{spt}(n)$  is a much more delicate matter. In fact, to resolve the difficult problem of proving that (3) converges, Ahlgren and Andersen used advanced methods from the spectral theory of automorphic forms. To give an effective bound on the error term for  $\text{spt}(n)$ , we will use different types of formulas for  $p(n)$  and  $\text{spt}(n)$  which express these functions as traces of singular moduli.

To state these formulas, consider the weight  $-2$  weakly holomorphic modular form for  $\Gamma_0(6)$  defined by

$$g(z) := \frac{1}{2} \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2}, \quad z = x + iy \in \mathbb{H}.$$

By applying the Maass weight-raising operator to  $g(z)$ , one gets the following weight zero weak Maass form for  $\Gamma_0(6)$ :

$$P(z) := - \left( \frac{1}{2\pi i} \frac{d}{dz} + \frac{1}{2\pi y} \right) g(z).$$

Bruinier and Ono [8] proved the following formula for  $p(n)$ .

**Theorem** (Bruinier–Ono). *For all  $n \geq 1$ , we have*

$$(4) \quad p(n) = \frac{1}{24n-1} \sum_{[Q]} P(\tau_Q),$$

where the sum is over the  $\Gamma_0(6)$  equivalence classes of discriminant  $-24n+1$  positive definite, integral binary quadratic forms  $Q = [a, b, c]$  such that  $6|a$  and  $b \equiv 1 \pmod{12}$ , and  $\tau_Q$  is the Heegner point given by the root  $Q(\tau_Q, 1) = 0$  in the complex upper half-plane  $\mathbb{H}$ .

Similarly, consider the weight zero weakly holomorphic modular form for  $\Gamma_0(6)$  defined by

$$(5) \quad f(z) := \frac{1}{24} \frac{E_4(z) - 4E_4(2z) - 9E_4(3z) + 36E_4(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2}.$$

Ahlgren and Andersen [1] proved the following analogue of (4) for  $\text{spt}(n)$ .

**Theorem** (Ahlgren–Andersen). *For all  $n \geq 1$ , we have*

$$(6) \quad \text{spt}(n) = \frac{1}{12} \sum_{[Q]} (f(\tau_Q) - P(\tau_Q)).$$

Identities which express Fourier coefficients of harmonic weak Maass forms as traces of singular moduli have been used in many contexts to give strong asymptotic formulas. For example, Bringmann and Ono [7] expressed  $p(n)$  as a twisted trace of singular moduli by arithmetically reformulating Rademacher's exact formula (2) for  $p(n)$ . Folsom and the second author [16] combined the Bringmann-Ono formula with spectral methods and subconvexity bounds for quadratic twists of modular  $L$ -functions to give an asymptotic formula for  $p(n)$  with a power-saving error term. In particular, by calculating the main term in this asymptotic formula in terms of the truncated main term in Rademacher's exact formula for  $p(n)$ , these authors improved the exponent in Lehmer's bound [20]. This exponent was further improved by Ahlgren and Andersen [2].

In the works [4, 22, 23], spectral methods and subconvexity bounds were again used to give asymptotic formulas with power-saving error terms for twisted traces of singular moduli. These results were applied to study a variety of arithmetic problems, including the distribution of  $p(n)$  and  $\text{spt}(n)$ , and the distribution of partition ranks. Note that although the constants in the error terms of these results are effective, it would be very difficult to actually give explicit numerical values for these constants because of the methods involved in the proofs. There is an alternative approach, which we now describe.

Using (4), the formula (6) can be written as

$$(7) \quad \text{spt}(n) = \frac{1}{12}S(n) - \frac{24n-1}{12}p(n),$$

where  $S(n)$  is the trace of singular moduli for  $f(z)$  given by

$$S(n) := \sum_{[Q]} f(\tau_Q).$$

Then by applying Lehmer's effective bounds for  $p(n)$  in (7), we will reduce the proof of Theorem 1.1 to the following asymptotic formula for  $S(n)$  with an effective bound on the error term.

**Theorem 1.4.** *For all  $n \geq 1$ , we have*

$$S(n) = 2\pi(24n-1)^{1/4}I_{1/2}(\pi\sqrt{24n-1}/6) + E(n),$$

where

$$|E(n)| < (3.45 \times 10^{44})2^{q(n)+1}(24n-1)^2e^{\mu(n)/2}$$

with

$$q(n) := \frac{\log(24n-1)}{|\log(\log(24n-1)) - 1.1714|}.$$

In [14], Dewar and Murty used the formula (4) to derive the Hardy-Ramanujan asymptotic formula for  $p(n)$  without using the circle method. We will use a related approach to prove Theorem 1.4. In order to give effective bounds, additional care must be taken. For instance, we must give effective bounds for the Fourier coefficients of  $f(z)$ , which requires a careful study of the asymptotic properties of Bessel functions in transitional ranges.

**Organization.** The paper is organized as follows. In Section 2, we review some facts regarding quadratic forms and Heegner points. In Section 3, we prove Theorem 1.4. In Section 4, we study the asymptotic properties of  $p(n)$  and  $\text{spt}(n)$ , and prove Theorem 1.1. In Section 5, we prove Theorem 1.3. Finally, in Section 6, we prove the remaining conjectures.

**Acknowledgments.** The authors would like to thank Ken Ono for many valuable suggestions regarding this work. They would also like to thank Adrian Barquero-Sanchez, Sheng-Chi Liu, Wei-Lun Tsai, and Matt Young for very helpful discussions, Michael Griffin and Lea Beneish for computing the values of  $\text{spt}(n)$  which were required to prove the conjectures, and Karl Mahlburg for comments which helped to improve the exposition of the paper.

## 2. QUADRATIC FORMS AND HEEGNER POINTS

Let  $N \geq 1$  be a positive integer and  $D < 0$  be a negative discriminant coprime to  $N$ . Let  $\mathcal{Q}_{D,N}$  be the set of positive definite, integral binary quadratic forms

$$Q(X, Y) = [a_Q, b_Q, c_Q](X, Y) = a_Q X^2 + b_Q XY + c_Q Y^2$$

of discriminant  $b_Q^2 - 4a_Q c_Q = D < 0$  with  $a_Q \equiv 0 \pmod{N}$ . There is a (right) action of  $\Gamma_0(N)$  on  $\mathcal{Q}_{D,N}$  defined by

$$Q = [a_Q, b_Q, c_Q] \mapsto Q \circ \sigma = [a_Q^\sigma, b_Q^\sigma, c_Q^\sigma],$$

where for  $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)$  we have

$$\begin{aligned} a_Q^\sigma &= a_Q \alpha^2 + b_Q \alpha \gamma + c_Q \gamma^2, \\ b_Q^\sigma &= 2a_Q \alpha \beta + b_Q (\alpha \delta + \beta \gamma) + 2c_Q \gamma \delta, \\ c_Q^\sigma &= a_Q \beta^2 + b_Q \beta \delta + c_Q \delta^2. \end{aligned}$$

Given a solution  $r \pmod{2N}$  of  $r^2 \equiv D \pmod{4N}$ , we define the subset of forms

$$\mathcal{Q}_{D,N,r} := \{Q = [a_Q, b_Q, c_Q] \in \mathcal{Q}_{D,N} : b_Q \equiv r \pmod{2N}\}.$$

Then the group  $\Gamma_0(N)$  also acts on  $\mathcal{Q}_{D,N,r}$ . The number of  $\Gamma_0(N)$  equivalence classes in  $\mathcal{Q}_{D,N,r}$  is given by the Hurwitz-Kronecker class number  $H(D)$ .

The preceding facts remain true if we restrict to the subset  $\mathcal{Q}_{D,N}^{\text{prim}}$  of primitive forms in  $\mathcal{Q}_{D,N}$ ; i.e., those forms with

$$(a_Q, b_Q, c_Q) = 1.$$

In this case, the number of  $\Gamma_0(N)$  equivalence classes in  $\mathcal{Q}_{D,N,r}^{\text{prim}}$  is given by the class number  $h(D)$ .

To each form  $Q \in \mathcal{Q}_{D,N}$ , we associate a Heegner point  $\tau_Q$  which is the root of  $Q(X, 1)$  given by

$$\tau_Q = \frac{-b_Q + \sqrt{D}}{2a_Q} \in \mathbb{H}.$$

The Heegner points  $\tau_Q$  are compatible with the action of  $\Gamma_0(N)$  in the sense that if  $\sigma \in \Gamma_0(N)$ , then

$$(8) \quad \sigma(\tau_Q) = \tau_{Q \circ \sigma^{-1}}.$$

## 3. PROOF OF THEOREM 1.4

In this section we prove Theorem 1.4, which will be used crucially in the proof of Theorem 1.1.

Let  $D_n := -24n + 1$  for  $n \in \mathbb{Z}^+$  and define the trace of  $f(z)$  by

$$S(n) := \sum_{[Q] \in \mathcal{Q}_{D_n, 6, 1} / \Gamma_0(6)} f(\tau_Q).$$

First, we decompose  $S(n)$  as a linear combination of traces involving primitive forms. Let  $\Delta < 0$  be any discriminant with  $\Delta \equiv 1 \pmod{24}$  and define the class polynomials

$$H_n(X) := \prod_{[Q] \in \mathcal{Q}_{D_n, 6, 1} / \Gamma_0(6)} (X - f(\tau_Q))$$

and

$$\widehat{H}_\Delta(X) := \prod_{[Q] \in \mathcal{Q}_{\Delta, 6, 1}^{\text{prim}} / \Gamma_0(6)} (X - f(\tau_Q)).$$

Let  $\{W_m\}_{m|6}$  be the group of Atkin-Lehner involutions for  $\Gamma_0(6)$ . Since  $f|_0W_m = f$  for  $m = 1, 6$  and  $f|_0W_m = -f$  for  $m = 2, 3$ , then arguing exactly as in the proof of [9, Lemma 7] we get the identity

$$(9) \quad H_n(X) = \prod_{\substack{u > 0 \\ u^2 | D_n}} \varepsilon(u)^{h(D_n/u^2)} \widehat{H}_{D_n/u^2}(\varepsilon(u)X),$$

where  $\varepsilon(u) = 1$  if  $u \equiv \pm 1 \pmod{12}$  and  $\varepsilon(u) = -1$  otherwise. Comparing terms on both sides of (9) yields the class number relation

$$H(D_n) = \sum_{\substack{u > 0 \\ u^2 | D_n}} h(D_n/u^2)$$

and the decomposition

$$(10) \quad S(n) = \sum_{\substack{u > 0 \\ u^2 | D_n}} \varepsilon(u)^{h(D_n/u^2)+1} S_u(n),$$

where

$$S_u(n) := \sum_{[Q] \in \mathcal{Q}_{D_n/u^2, 6, 1}^{\text{prim}} / \Gamma_0(6)} f(\tau_Q).$$

Next, following [14] we express  $S_u(n)$  as a trace involving primitive forms of level 1. The group  $\Gamma_0(6)$  has index 12 in  $SL_2(\mathbb{Z})$ . We choose the following 12 right coset representatives:

$$\begin{aligned} \gamma_\infty &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \gamma_{1/3, r} &:= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad r = 0, 1; \\ \gamma_{1/2, s} &:= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad s = 0, 1, 2; \\ \gamma_{0, t} &:= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t = 0, 1, 2, 3, 4, 5. \end{aligned}$$

We denote this set of coset representatives by  $\mathcal{C}_6$ . The matrices  $\gamma \in \mathcal{C}_6$  are scaling matrices for the four cusps  $\{\infty, 1/3, 1/2, 0\}$  of the modular curve  $X_0(6)$ , which have widths 1, 2, 3, and 6, respectively. In particular, we have  $\gamma_\infty(\infty) = \infty$ ,  $\gamma_{1/3, r}(\infty) = 1/3$ ,  $\gamma_{1/2, s}(\infty) = 1/2$ , and  $\gamma_{0, t}(\infty) = 0$ .

Now, let  $\mathcal{Q}_\Delta$  denote a set of primitive, reduced forms representing the equivalence classes in  $\mathcal{Q}_{\Delta,1}^{\text{prim}}/SL_2(\mathbb{Z})$ . For each  $Q \in \mathcal{Q}_\Delta$ , there is a unique choice of coset representative  $\gamma_Q \in \mathcal{C}_6$  such that

$$[Q \circ \gamma_Q^{-1}] \in \mathcal{Q}_{\Delta,6,1}^{\text{prim}}/\Gamma_0(6).$$

This induces a bijection

$$(11) \quad \begin{aligned} \mathcal{Q}_\Delta &\longrightarrow \mathcal{Q}_{\Delta,6,1}^{\text{prim}}/\Gamma_0(6) \\ Q &\longmapsto [Q \circ \gamma_Q^{-1}]; \end{aligned}$$

see the proposition on page 505 in [17], or more concretely, [14, Lemma 3], where an explicit list of the matrices  $\gamma_Q \in \mathcal{C}_6$  is given.

Using the bijection (11) and the compatibility relation (8) for Heegner points, the trace  $S_u(n)$  can be expressed as

$$(12) \quad S_u(n) = \sum_{[Q] \in \mathcal{Q}_{D_n/u^2,6,1}^{\text{prim}}/\Gamma_0(6)} f(\tau_Q) = \sum_{Q \in \mathcal{Q}_{D_n/u^2}} f(\gamma_Q(\tau_Q)).$$

Therefore, to study the asymptotic distribution of  $S_u(n)$ , we need the Fourier expansion of  $f(z)$  with respect to the scaling matrices  $\gamma_\infty$ ,  $\gamma_{1/3,r}$ ,  $\gamma_{1/2,s}$ , and  $\gamma_{0,t}$ .

Observe that the Fourier expansion of  $f(z)$  at the cusp  $\infty$  is given explicitly by (see [1, equation (4.6)])

$$f|_0\gamma_\infty(z) = e(-z) - e(-\bar{z}) + a(0) + \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \{a(m)e(mz) + a(-m)e(-m\bar{z})\}, \quad e(z) := e^{2\pi iz}$$

where

$$\begin{aligned} a(0) &= 4\pi^2 \sum_{\ell|6} \mu(\ell) \sum_{\substack{0 < c \equiv 0 \pmod{6/\ell} \\ (c,\ell)=1}} \frac{S(-\bar{\ell}, 0; c)}{(\sqrt{\ell}c)^2}, \\ a(m) &= 2\pi \sum_{\ell|6} \mu(\ell) \sum_{\substack{0 < c \equiv 0 \pmod{6/\ell} \\ (c,\ell)=1}} \frac{S(-\bar{\ell}, m; c)}{\sqrt{\ell}c} I_1\left(\frac{4\pi\sqrt{m}}{\sqrt{\ell}c}\right), \quad m > 0, \\ a(m) &= 2\pi \sum_{\ell|6} \mu(\ell) \sum_{\substack{0 < c \equiv 0 \pmod{6/\ell} \\ (c,\ell)=1}} \frac{S(-\bar{\ell}, m; c)}{\sqrt{\ell}c} J_1\left(\frac{4\pi\sqrt{|m|}}{\sqrt{\ell}c}\right), \quad m < 0. \end{aligned}$$

Here  $\mu(\ell)$  is the Möbius function,  $S(a, b; c)$  is the Kloosterman sum

$$S(a, b; c) := \sum_{\substack{d \pmod{c} \\ (c,d)=1}} e\left(\frac{a\bar{d} + bd}{c}\right),$$

and  $I_1, J_1$  are the usual Bessel functions of order 1 (note that  $\bar{d}$  is the multiplicative inverse of  $d \pmod{c}$ ).

Since  $f(z)$  is an eigenfunction for the Atkin-Lehner involutions of level 6, the Fourier expansion of  $f(z)$  with respect to the scaling matrices  $\gamma_{1/3,r}, \gamma_{1/2,s}, \gamma_{0,t}$  can be determined



from the Fourier expansion at  $\infty$ . In particular, if  $\zeta_6 := e(1/6)$  is a primitive sixth root of unity, we have (see for example [14, (3.2)])

$$\begin{aligned} f|_0\gamma_{1/3,r}(z) &= (-1)^r [e(-z/2) - e(-\bar{z}/2)] + a(0) + \sum_{m=1}^{\infty} a_{m,r}(z), \\ f|_0\gamma_{1/2,s}(z) &= \zeta_6^{3-2s} [e(-z/3) - e(-\bar{z}/3)] - a(0) + \sum_{m=1}^{\infty} b_{m,s}(z), \\ f|_0\gamma_{0,t}(z) &= \zeta_6^{-t} [e(-z/6) - e(-\bar{z}/6)] + a(0) + \sum_{m=1}^{\infty} c_{m,t}(z), \end{aligned}$$

where

$$\begin{aligned} a_{m,r}(z) &:= \frac{(-1)^r}{\sqrt{m}} [a(m)e(mz/2) + a(-m)e(-m\bar{z}/2)], \\ b_{m,s}(z) &:= \frac{1}{\sqrt{m}} [\zeta_6^{3+2ms} a(m)e(mz/3) + \zeta_6^{3-2ms} a(-m)e(-m\bar{z}/3)], \\ c_{m,t}(z) &:= \frac{1}{\sqrt{m}} [\zeta_6^{mt} a(m)e(mz/6) + \zeta_6^{-mt} a(-m)e(-m\bar{z}/6)]. \end{aligned}$$

More generally, given a form  $Q \in \mathcal{Q}_\Delta$  and corresponding coset representative  $\gamma_Q \in \mathcal{C}_6$ , we let  $h_Q \in \{1, 2, 3, 6\}$  be the width of the cusp  $\gamma_Q(\infty)$  and  $\zeta_Q$  be the sixth root of unity such that

$$(13) \quad f|_0\gamma_Q(z) = \zeta_Q [e(-z/h_Q) - e(-\bar{z}/h_Q)] \pm a(0) + \sum_{m=1}^{\infty} C_{m,Q}(z),$$

where

$$C_{m,Q}(z) := \frac{1}{\sqrt{m}} [\zeta_Q^{-1} a(m)e(mz/h_Q) + \zeta_Q a(-m)e(-m\bar{z}/h_Q)].$$

In the following lemma we give effective bounds for the Fourier coefficients  $a(m)$ .

**Lemma 3.1.** *The following bounds hold.*

(1) For  $m = 0$  we have

$$|a(0)| \leq \frac{8}{3}\pi^4.$$

(2) For  $m < 0$  we have

$$|a(m)| \leq C_1|m|,$$

where

$$C_1 := 16\sqrt{2}(4\pi)^{3/2} \cdot \frac{4\sqrt{6}}{e^2 \log(2) \log(3)} + 16\pi^2 \zeta^2(3/2).$$

(3) For  $m > 0$  we have

$$|a(m)| \leq C_2 m \exp(4\pi\sqrt{m}),$$

where

$$C_2 := \frac{32}{3}\sqrt{2}(4\pi)^{3/2} \cdot \frac{4\sqrt{6}}{e^2 \log(2) \log(3)} + 16\pi^2 e \zeta^2(3/2).$$

*Proof.* We first estimate  $|a(0)|$ . Since  $(\ell, c) = 1$ , we can evaluate the Ramanujan sum as

$$S(-\bar{\ell}, 0; c) = \sum_{\substack{d \pmod{c} \\ (c, d) = 1}} e\left(\frac{-\bar{\ell}d}{c}\right) = \sum_{\substack{d \pmod{c} \\ (c, d) = 1}} e\left(\frac{\bar{\ell}d}{c}\right) = \mu(c),$$

where the last equality follows from [19, equation (3.4)]. Therefore,

$$|S(-\bar{\ell}, 0; c)| = |\mu(c)| \leq 1,$$

and we get

$$\begin{aligned} |a(0)| &\leq 4\pi^2 \sum_{\ell|6} |\mu(\ell)| \sum_{\substack{0 < c \equiv 0 \pmod{6/\ell} \\ (c, \ell) = 1}} \frac{|S(-\bar{\ell}, 0; c)|}{|\sqrt{\ell}c|^2} \\ &\leq 4\pi^2 \sum_{\ell|6} \sum_{\substack{0 < c \equiv 0 \pmod{6/\ell} \\ (c, \ell) = 1}} \frac{1}{c^2} \\ &\leq 16\pi^2 \sum_{c=1}^{\infty} \frac{1}{c^2} \\ &= 16\pi^2 \zeta(2) = \frac{8}{3}\pi^4. \end{aligned}$$

We next estimate  $|a(m)|$  for  $m < 0$ . Using the series

$$J_1(x) = \frac{x}{2} \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k! (k+1)!}, \quad x > 0$$

we find that

$$(14) \quad |J_1(x)| \leq \frac{x}{2} \quad \text{for } 0 < x < 1.$$

On the other hand, by [24, page 3] we have the uniform asymptotic formula

$$J_1(x) = \sqrt{\frac{2}{\pi x}} \left\{ \cos(x - 3\pi/4) \left[ 1 + R_2^{(J)}(x, 1) \right] - \sin(x - 3\pi/4) \left[ \frac{3}{8x} - R_3^{(J)}(x, 1) \right] \right\},$$

where the error terms satisfy the bounds (see [24, equation (1.30)])

$$\left| R_2^{(J)}(x, 1) \right| < 1, \quad \left| R_3^{(J)}(x, 1) \right| < 1 \quad \text{for } x \geq 1.$$

Hence

$$(15) \quad |J_1(x)| \leq 6\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}} \quad \text{for } x \geq 1.$$

Let  $M = 4\pi\sqrt{|m|}/\sqrt{\ell}$ . Then using the Weil bound

$$(16) \quad |S(a, b; c)| \leq \tau(c)(a, b, c)^{1/2} c^{1/2}$$

and the estimates (14) and (15), we get

$$|a(m)| \leq 2\pi \sum_{\ell|6} |\mu(\ell)| \sum_{\substack{0 < c \equiv 0 \pmod{6/\ell} \\ (c, \ell) = 1}} \frac{|S(-\bar{\ell}, m; c)|}{\sqrt{\ell}c} |J_1(M/c)|$$

$$\leq 6\sqrt{2}|m|^{-1/4}S_1 + 4\pi^2|m|^{1/2}S_2,$$

where

$$S_1 := \sum_{\ell|6} \ell^{-1/4} \sum_{\substack{0 < c < M \\ (c,\ell)=1}} \tau(c) (-\bar{\ell}, m, c)^{1/2}$$

and

$$S_2 := \sum_{\ell|6} \ell^{-1} \sum_{\substack{c \geq M \\ (c,\ell)=1}} \frac{\tau(c) (-\bar{\ell}, m, c)^{1/2}}{c^{3/2}}.$$

For all  $\epsilon > 0$  we have the following effective bound for the divisor function,

$$(17) \quad \tau(n) \leq C(\epsilon)n^\epsilon$$

where

$$C(\epsilon) := \prod_{p < 2^{1/\epsilon}} \max_{\alpha \geq 0} \frac{\alpha + 1}{p^{\alpha\epsilon}}.$$

For future reference, we note that

$$C(1/2) = \frac{4\sqrt{6}}{e^2 \log(2) \log(3)}.$$

Then using (17) with  $\epsilon = 1/2$  we get

$$|S_1| \leq C(1/2)|m|^{1/2} \sum_{\ell|6} \ell^{-1/4} \sum_{\substack{0 < c < M \\ (c,\ell)=1}} c^{1/2} \leq \frac{8}{3}(4\pi)^{3/2}C(1/2)|m|^{5/4}.$$

Also, we have

$$|S_2| \leq 4|m|^{1/2} \sum_{c=1}^{\infty} \frac{\tau(c)}{c^{3/2}} = 4|m|^{1/2}\zeta^2(3/2).$$

Then after combining estimates, we get

$$|a(m)| \leq C_1|m|, \quad m < 0$$

where

$$C_1 := 16\sqrt{2}(4\pi)^{3/2} \cdot \frac{4\sqrt{6}}{e^2 \log(2) \log(3)} + 16\pi^2\zeta^2(3/2).$$

Finally, we estimate  $|a(m)|$  for  $m > 0$ . Using the inequality (see [21, (6.25)])

$$|I_1(x)| < \frac{1}{2}(1 + \exp(-2x)) \frac{x}{2} \exp(x) \quad \text{for } x > 0$$

we find that

$$(18) \quad |I_1(x)| \leq \frac{e}{2}x \quad \text{for } 0 < x < 1.$$

On the other hand, by [24, equation (A.4)] we have

$$I_1(x) = \mp \frac{i}{\pi} K_1(xe^{\mp\pi i}) \pm \frac{i}{\pi} e^{\pm\pi i} K_1(x), \quad x > 0$$

where  $K_1$  is the usual Bessel function. Then

$$|I_1(x)| \leq \frac{1}{\pi} |K_1(xe^{-\pi i})| + \frac{1}{\pi} |K_1(xe^{\pi i})| + \frac{2}{\pi} |K_1(x)|.$$

Now, by [24, page 26] we have the uniform asymptotic formulas

$$K_1(xe^{\mp\pi i}) = \pm i \sqrt{\frac{\pi}{2x}} \exp(x) \left[ 1 + R_1^{(K)}(xe^{\mp\pi i}, 1) \right]$$

and

$$K_1(x) = \sqrt{\frac{\pi}{2x}} \exp(-x) \left[ 1 + R_1^{(K)}(x, 1) \right],$$

where the error terms satisfy the bounds (see [24, equations (1.26) and (B.1)])

$$\left| R_1^{(K)}(xe^{\mp\pi i}, 1) \right| \leq \frac{3}{8} \sup_{r \geq 1} |\Lambda_1(2xe^{\mp\pi i}r)| \leq 1 \quad \text{for } x \geq 1$$

and (see [24, equation (1.30)])

$$\left| R_1^{(K)}(x, 1) \right| < 1 \quad \text{for } x \geq 1.$$

Here for  $p > 0$  and  $\omega \neq 0$ , we have

$$\Lambda_p(\omega) := \omega^p e^\omega \Gamma(1-p, \omega)$$

where  $\Gamma(1-p, \omega)$  is the incomplete Gamma function. Hence we get

$$(19) \quad |I_1(x)| \leq 2\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}} (\exp(x) + \exp(-x)) \leq 4\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}} \exp(x) \quad \text{for } x \geq 1.$$

Again, let  $M = 4\pi\sqrt{m}/\sqrt{\ell}$ . Then using the estimates (16), (18) and (19), we get

$$\begin{aligned} |a(m)| &\leq 2\pi \sum_{\ell|6} |\mu(\ell)| \sum_{\substack{0 < c \equiv 0 \pmod{6/\ell} \\ (c, \ell) = 1}} \frac{|S(-\bar{\ell}, m; c)|}{\sqrt{\ell}c} |I_1(M/c)| \\ &\leq 4\sqrt{2}m^{-1/4}S_3 + 4\pi^2em^{1/2}S_4, \end{aligned}$$

where

$$S_3 := \sum_{\ell|6} \ell^{-1/4} \sum_{\substack{0 < c < M \\ (c, \ell) = 1}} \tau(c) (-\bar{\ell}, m, c)^{1/2} \exp(4\pi\sqrt{m}/\sqrt{\ell}c)$$

and

$$S_4 := \sum_{\ell|6} \ell^{-1} \sum_{\substack{c \geq M \\ (c, \ell) = 1}} \frac{\tau(c) (-\bar{\ell}, m, c)^{1/2}}{c^{3/2}}.$$

As before, using (17) with  $\epsilon = 1/2$  we get

$$\begin{aligned} |S_3| &\leq C(1/2)m^{1/2}\exp(4\pi\sqrt{m}) \sum_{\ell|6} \ell^{-1/4} \sum_{\substack{0 < c < M \\ (c, \ell) = 1}} c^{1/2} \\ &\leq \frac{8}{3}(4\pi)^{3/2}C(1/2)m^{5/4}\exp(4\pi\sqrt{m}), \end{aligned}$$

and

$$|S_4| = |S_2| \leq 4m^{1/2} \sum_{c=1}^{\infty} \frac{\tau(c)}{c^{3/2}} = 4m^{1/2} \zeta^2(3/2).$$

Then combining estimates yields

$$|a(m)| \leq C_2 m \exp(4\pi\sqrt{m}), \quad m > 0$$

where

$$C_2 := \frac{32}{3} \sqrt{2} (4\pi)^{3/2} \cdot \frac{4\sqrt{6}}{e^2 \log(2) \log(3)} + 16\pi^2 e \zeta^2(3/2).$$

□

We are now in position to prove Theorem 1.4, which we restate for the convenience of the reader.

**Theorem 3.2.** *For all  $n \geq 1$ , we have*

$$S(n) = 2\pi(24n - 1)^{1/4} I_{1/2}(\pi\sqrt{24n - 1}/6) + E(n),$$

where

$$|E(n)| < (3.45 \times 10^{44}) 2^{q(n)+1} (24n - 1)^2 \exp(\pi\sqrt{24n - 1}/12)$$

with

$$q(n) := \frac{\log(24n - 1)}{|\log(\log(24n - 1)) - 1.1714|}.$$

*Proof.* By (10), (12) and (13), we have

$$\begin{aligned} S(n) &= \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u)^{h(D_n/u^2)+1} S_u(n) \\ &= \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u)^{h(D_n/u^2)+1} \sum_{Q \in \mathcal{Q}_{D_n/u^2}} f|_0 \gamma_Q(\tau_Q) \\ &= \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u)^{h(D_n/u^2)+1} \sum_{Q \in \mathcal{Q}_{D_n/u^2}} \zeta_Q [e(-\tau_Q/h_Q) - e(-\overline{\tau}_Q/h_Q)] + E_1(n), \end{aligned}$$

where

$$E_1(n) := \pm a(0) \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u)^{h(D_n/u^2)+1} h(D_n/u^2) + \sum_{m=1}^{\infty} \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u)^{h(D_n/u^2)+1} \sum_{Q \in \mathcal{Q}_{D_n/u^2}} C_{m,Q}(\tau_Q).$$

We first give an effective upper bound for  $|E_1(n)|$ . We have

$$(20) \quad \left| \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u)^{h(D_n/u^2)+1} h(D_n/u^2) \right| \leq \sum_{\substack{u>0 \\ u^2|D_n}} h(D_n/u^2) = H(D_n),$$

and thus Lemma 3.1 gives

$$\left| \pm a(0) \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u)^{h(D_n/u^2)+1} h(D_n/u^2) \right| \leq \frac{8}{3} \pi^4 H(D_n).$$

Next, using the definition of  $C_{m,Q}(z)$  we get

$$\left| \sum_{m=1}^{\infty} \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u)^{h(D_n/u^2)+1} \sum_{Q \in \mathcal{Q}_{D_n/u^2}} C_{m,Q}(\tau_Q) \right| \leq \sum_{m=1}^{\infty} \frac{|a(m)|}{\sqrt{m}} \sum_{\substack{u>0 \\ u^2|D_n}} \left| \sum_{Q \in \mathcal{Q}_{D_n/u^2}} \zeta_Q^{-1} e(m\tau_Q/h_Q) \right| + \sum_{m=1}^{\infty} \frac{|a(-m)|}{\sqrt{m}} \sum_{\substack{u>0 \\ u^2|D_n}} \left| \sum_{Q \in \mathcal{Q}_{D_n/u^2}} \zeta_Q e(-m\overline{\tau_Q}/h_Q) \right|.$$

Observe that

$$e(m\tau_Q/h_Q) = \zeta_{2a_Q h_Q}^{-b_Q m} \exp\left(-\frac{\pi m \sqrt{|D_n|/u^2}}{a_Q h_Q}\right),$$

where  $\zeta_{2a_Q h_Q}$  is the primitive  $2a_Q h_Q$ -th root of unity

$$\zeta_{2a_Q h_Q} := e\left(\frac{1}{2a_Q h_Q}\right).$$

Since  $Q \in \mathcal{Q}_{D_n/u^2}$  is *reduced*, the corresponding Heegner point  $\tau_Q$  lies in the standard fundamental domain  $\mathcal{F}$  for  $SL_2(\mathbb{Z})$ . In particular, we have

$$\text{Im}(\tau_Q) = \frac{\sqrt{|D_n|/u^2}}{2a_Q} \geq \sqrt{3}/2,$$

which implies that

$$a_Q \leq \frac{\sqrt{|D_n|/u^2}}{\sqrt{3}}.$$

Since  $h_Q \leq 6$ , we have

$$(21) \quad -\frac{\pi m \sqrt{|D_n|/u^2}}{a_Q h_Q} \leq -\frac{\pi m}{4\sqrt{3}}.$$

Then using (20) and (21), we get

$$\left| \sum_{Q \in \mathcal{Q}_{D_n/u^2}} \zeta_Q^{-1} e(m\tau_Q/h_Q) \right| \leq \sum_{Q \in \mathcal{Q}_{D_n/u^2}} \exp\left(-\pi m \frac{\sqrt{|D_n|/u^2}}{a_Q h_Q}\right) \leq h(D_n/u^2) \exp(-\pi m/4\sqrt{3}).$$

Similarly, since

$$e(-m\overline{\tau_Q}/h_Q) = \zeta_{2a_Q h_Q}^{b_Q m} \exp\left(-\pi m \frac{\sqrt{|D_n|/u^2}}{a_Q h_Q}\right),$$

the same argument gives

$$\left| \sum_{Q \in \mathcal{Q}_{D_n/u^2}} \zeta_Q e(-m\bar{\tau}_Q/h_Q) \right| \leq h(D_n/u^2) \exp(-\pi m/4\sqrt{3}).$$

Hence by Lemma 3.1, we have

$$\left| \sum_{m=1}^{\infty} \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u)^{h(D_n/u^2)+1} \sum_{Q \in \mathcal{Q}_{D_n/u^2}} C_{m,Q}(\tau_Q) \right| \leq H(D_n) \left\{ C_1 \sum_{m=1}^{\infty} m^{1/2} \exp(-\pi m/4\sqrt{3}) + C_2 \sum_{m=1}^{\infty} m^{1/2} \exp(4\pi\sqrt{m} - \pi m/4\sqrt{3}) \right\}.$$

Combining the preceding estimates yields

$$|E_1(n)| \leq \frac{8}{3} \pi^4 H(D_n) + H(D_n) \left\{ C_1 \sum_{m=1}^{\infty} m^{1/2} \exp(-\pi m/4\sqrt{3}) + C_2 \sum_{m=1}^{\infty} m^{1/2} \exp(4\pi\sqrt{m} - \pi m/4\sqrt{3}) \right\}.$$

We now estimate the infinite sums. First, we have

$$\sum_{m=1}^{\infty} m^{1/2} \exp(-\pi m/4\sqrt{3}) \leq \int_0^{\infty} x^{1/2} \exp(-\pi x/4\sqrt{3}) dx = \left( \frac{4\sqrt{3}}{\pi} \right)^{3/2} \Gamma(3/2) = \left( \frac{4\sqrt{3}}{\pi} \right)^{3/2} \frac{\sqrt{\pi}}{2}.$$

Next, write

$$4\pi\sqrt{m} - \pi m/4\sqrt{3} = -m \left( \frac{\pi}{4\sqrt{3}} - \frac{4\pi}{\sqrt{m}} \right),$$

and observe that

$$\frac{\pi}{4\sqrt{3}} - \frac{4\pi}{\sqrt{m}} > 0 \iff m > 768.$$

Then if  $m \geq 769$ , we have

$$\frac{\pi}{4\sqrt{3}} - \frac{4\pi}{\sqrt{m}} \geq \frac{\pi}{4\sqrt{3}} - \frac{4\pi}{\sqrt{769}} > 0$$

so that

$$-m \left( \frac{\pi}{4\sqrt{3}} - \frac{4\pi}{\sqrt{m}} \right) \leq -m \left( \frac{\pi}{4\sqrt{3}} - \frac{4\pi}{\sqrt{769}} \right).$$

Now, we split the sum into appropriate ranges and use the preceding bound to get

$$\begin{aligned} & \sum_{m=1}^{\infty} m^{1/2} \exp(4\pi\sqrt{m} - \pi m/4\sqrt{3}) \\ &= \sum_{m=1}^{768} m^{1/2} \exp(4\pi\sqrt{m} - \pi m/4\sqrt{3}) + \sum_{m=769}^{\infty} m^{1/2} \exp\left(-m \left( \frac{\pi}{4\sqrt{3}} - \frac{4\pi}{\sqrt{m}} \right)\right) \end{aligned}$$

$$\leq \sum_{m=1}^{768} m^{1/2} \exp\left(4\pi\sqrt{m} - \pi m/4\sqrt{3}\right) + \sum_{m=769}^{\infty} m^{1/2} \exp\left(-m\left(\frac{\pi}{4\sqrt{3}} - \frac{4\pi}{\sqrt{769}}\right)\right).$$

A calculation shows that

$$\sum_{m=1}^{768} m^{1/2} \exp\left(4\pi\sqrt{m} - \frac{\pi m}{4\sqrt{3}}\right) \leq \int_1^{768} x^{1/2} \exp\left(4\pi\sqrt{x} - \frac{\pi x}{4\sqrt{3}}\right) dx < 6.58 \times 10^{40}.$$

Also, if we let

$$\beta := \frac{\pi}{4\sqrt{3}} - \frac{4\pi}{\sqrt{769}}$$

then

$$\sum_{m=769}^{\infty} m^{1/2} \exp\left(-m\left(\frac{\pi}{4\sqrt{3}} - \frac{4\pi}{\sqrt{769}}\right)\right) \leq \int_{769}^{\infty} x^{1/2} \exp(-\beta x) dx = \beta^{-3/2} \Gamma(3/2, 769 \cdot \beta).$$

Combining things, we have shown that

$$|E_1(n)| \leq C(\beta)H(D_n),$$

where  $C(\beta)$  is the explicit positive constant defined by

$$C(\beta) := \frac{8}{3}\pi^4 + C_1 \left(\frac{4\sqrt{3}}{\pi}\right)^{3/2} \frac{\sqrt{\pi}}{2} + C_2 [(6.58 \times 10^{40}) + \beta^{-3/2} \Gamma(3/2, 769 \cdot \beta)].$$

We now find an explicit upper bound for  $C(\beta)$ . We have

$$\frac{8}{3}\pi^4 \approx 259.75 < 260.$$

Also, since

$$C(1/2) = \frac{4\sqrt{6}}{e^2 \log(2) \log(3)} \approx 1.74 < 2$$

and  $\zeta(3/2) \approx 2.61 < 3$ , we have

$$C_1 = 16\sqrt{2}(4\pi)^{3/2}C(1/2) + 16\pi^2\zeta^2(3/2) < 32\sqrt{2}(4\pi)^{3/2} + 144\pi^2 \approx 3437.17 < 3438,$$

so that

$$C_1 \left(\frac{4\sqrt{3}}{\pi}\right)^{3/2} \frac{\sqrt{\pi}}{2} < 3438 \left(\frac{4\sqrt{3}}{\pi}\right)^{3/2} \frac{\sqrt{\pi}}{2} \approx 9978.31 < 9979.$$

Similarly, we have

$$C_2 = \frac{32}{3}\sqrt{2}(4\pi)^{3/2}C(1/2) + 16\pi^2 e \zeta^2(3/2) < \frac{64}{3}\sqrt{2}(4\pi)^{3/2} + 144\pi^2 e \approx 5207.25 < 5208,$$

$$\beta^{-3/2} \approx 197437.10 < 197438$$

and  $\Gamma(3/2, 769 \cdot \beta) \approx 0.82 < 1$ , so that

$$C_2 [(6.58 \times 10^{40}) + \beta^{-3/2} \Gamma(3/2, 769 \cdot \beta)] < 5208 [(6.58 \times 10^{40}) + 197438] < 3.43 \times 10^{44}.$$

After combining estimates, we get

$$C(\beta) < 260 + 9979 + (3.43 \times 10^{44}) < 3.44 \times 10^{44}.$$



Summarizing, we have shown that

$$|E_1(n)| < (3.44 \times 10^{44})H(D_n).$$

It remains to analyze the main term. We write the main term as

$$\begin{aligned} & \sum_{\substack{u>0 \\ u^2|D_n}} \varepsilon(u)^{h(D_n/u^2)+1} \sum_{Q \in \mathcal{Q}_{D_n/u^2}} \zeta_Q [e(-\tau_Q/h_Q) - e(-\overline{\tau_Q}/h_Q)] \\ &= \sum_{Q \in \mathcal{Q}_{D_n}} \zeta_Q [e(-\tau_Q/h_Q) - e(-\overline{\tau_Q}/h_Q)] + E_2(n), \end{aligned}$$

where

$$E_2(n) := \sum_{\substack{u \geq 2 \\ u^2|D_n}} \varepsilon(u)^{h(D_n/u^2)+1} \sum_{Q \in \mathcal{Q}_{D_n/u^2}} \zeta_Q [e(-\tau_Q/h_Q) - e(-\overline{\tau_Q}/h_Q)].$$

Observe that for any form  $Q = [a_Q, b_Q, c_Q] \in \mathcal{Q}_{D_n/u^2}$ , we have

$$(22) \quad a_Q h_Q \equiv 0 \pmod{6}$$

and

$$(23) \quad e(-\tau_Q/h_Q) - e(-\overline{\tau_Q}/h_Q) = \zeta_{2a_Q h_Q}^{b_Q} \left[ \exp\left(\frac{\pi \sqrt{|D_n|/u^2}}{a_Q h_Q}\right) - \exp\left(-\frac{\pi \sqrt{|D_n|/u^2}}{a_Q h_Q}\right) \right].$$

Now, by [14, (4.2)] there are exactly 4 forms  $Q \in \mathcal{Q}_{D_n}$  with  $a_Q h_Q = 6$ , and these are given by

$$Q_1 = [1, 1, 6n], \quad Q_2 = [2, 1, 3n], \quad Q_3 = [3, 1, 2n], \quad Q_4 = [6, 1, n].$$

Moreover, the corresponding coset representatives  $\gamma_{Q_i} \in \mathcal{C}_6$  such that

$$[Q_i \circ \gamma_{Q_i}^{-1}] \in \mathcal{Q}_{D_n, 6, 1}^{\text{prim}}/\Gamma_0(6)$$

are given by

$$\gamma_{Q_1} = \gamma_{0,1}, \quad \gamma_{Q_2} = \gamma_{1/2,-1}, \quad \gamma_{Q_3} = \gamma_{1/3,0}, \quad \gamma_{Q_4} = \gamma_{\infty}.$$

Write

$$\sum_{Q \in \mathcal{Q}_{D_n}} \zeta_Q [e(-\tau_Q/h_Q) - e(-\overline{\tau_Q}/h_Q)] = \sum_{i=1}^4 \zeta_{Q_i} [e(-\tau_{Q_i}/h_{Q_i}) - e(-\overline{\tau_{Q_i}}/h_{Q_i})] + E_3(n),$$

where

$$E_3(n) := \sum_{\substack{Q \in \mathcal{Q}_{D_n} \\ Q \neq Q_i}} \zeta_Q [e(-\tau_Q/h_Q) - e(-\overline{\tau_Q}/h_Q)].$$

By (22) we have  $a_Q h_Q \geq 12$  for all  $Q \neq Q_i$ , hence using (23) we get

$$\begin{aligned} |E_3(n)| &\leq \sum_{\substack{Q \in \mathcal{Q}_{D_n} \\ Q \neq Q_i}} \left[ \exp(\pi \sqrt{|D_n|}/a_Q h_Q) - \exp(-\pi \sqrt{|D_n|}/a_Q h_Q) \right] \\ &\leq \sum_{\substack{Q \in \mathcal{Q}_{D_n} \\ Q \neq Q_i}} \exp(\pi \sqrt{|D_n|}/a_Q h_Q) \end{aligned}$$

$$\leq h(D_n)\exp(\pi\sqrt{|D_n|}/12).$$

Similarly, by (22) we have  $a_Q h_Q \geq 6$ , so that

$$\frac{\sqrt{|D_n|/u^2}}{a_Q h_Q} \leq \sqrt{|D_n|}/12$$

for  $u \geq 2$ . Hence by (23) we have

$$\begin{aligned} |E_2(n)| &\leq \sum_{\substack{u \geq 2 \\ u^2 | D_n}} \sum_{Q \in \mathcal{Q}_{D_n/u^2}} \left[ \exp\left(\frac{\pi\sqrt{|D_n|/u^2}}{a_Q h_Q}\right) - \exp\left(-\frac{\pi\sqrt{|D_n|/u^2}}{a_Q h_Q}\right) \right] \\ &\leq \sum_{\substack{u \geq 2 \\ u^2 | D_n}} \sum_{Q \in \mathcal{Q}_{D_n/u^2}} \exp\left(\frac{\pi\sqrt{|D_n|/u^2}}{a_Q h_Q}\right) \\ &\leq H(D_n)\exp(\pi\sqrt{|D_n|}/12). \end{aligned}$$

Next, using the identity

$$(24) \quad I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \frac{(\exp(z) - \exp(-z))}{2}$$

we have

$$e(-z) - e(-\bar{z}) = 2\pi\sqrt{y}I_{1/2}(2\pi y)e(-x), \quad z = x + iy.$$

Therefore, since  $a_{Q_i} h_{Q_i} = 6$  for  $i = 1, 2, 3, 4$ , we get

$$\sum_{i=1}^4 \zeta_{Q_i} [e(-\tau_{Q_i}/h_{Q_i}) - e(-\bar{\tau}_{Q_i}/h_{Q_i})] = \left[ 2\pi \frac{|D_n|^{1/4}}{\sqrt{12}} I_{1/2}(\pi\sqrt{|D_n|}/6) \right] \cdot \exp(\pi i/6) \cdot \sum_{i=1}^4 \zeta_{Q_i}.$$

Also, from the Fourier expansion of  $f(z)$  with respect to  $\gamma_{0,1}, \gamma_{1/2,-1}, \gamma_{1/3,0}$ , and  $\gamma_\infty$  given previously, we have

$$\zeta_{Q_1} = \zeta_6^{-1}, \quad \zeta_{Q_2} = \zeta_6^{3-2(-1)}, \quad \zeta_{Q_3} = (-1)^0, \quad \zeta_{Q_4} = 1.$$

Hence

$$\exp(\pi i/6) \cdot \sum_{i=1}^4 \zeta_{Q_i} = \exp(\pi i/6) \cdot \left( \zeta_6^{-1} + \zeta_6^{3-2(-1)} + (-1)^0 + 1 \right) = 4 \cos(\pi/6) = 2\sqrt{3}.$$

Then simplifying yields

$$\sum_{i=1}^4 \zeta_{Q_i} [e(-\tau_{Q_i}/h_{Q_i}) - e(-\bar{\tau}_{Q_i}/h_{Q_i})] = 2\pi |D_n|^{1/4} I_{1/2}(\pi\sqrt{|D_n|}/6).$$

By combining the preceding results, we get

$$S(n) = 2\pi |D_n|^{1/4} I_{1/2}(\pi\sqrt{|D_n|}/6) + E(n),$$

where  $E(n) := E_1(n) + E_2(n) + E_3(n)$  with

$$\begin{aligned} |E(n)| &\leq |E_1(n)| + |E_2(n)| + |E_3(n)| \\ &< 2H(D_n)\exp(\pi\sqrt{|D_n|}/12) + (3.44 \times 10^{44})H(D_n) \\ &< (3.45 \times 10^{44})H(D_n)\exp(\pi\sqrt{|D_n|}/12). \end{aligned}$$

To complete the proof, we require only a crude effective upper bound for the Hurwitz-Kronecker class number  $H(D_n)$ . Write  $D_n = d_n f_n^2$  with  $d_n < 0$  a fundamental discriminant and  $f_n \in \mathbb{Z}^+$ . Then we have the class number relation

$$(25) \quad H(D_n) = \sum_{\substack{u>0 \\ u^2|D_n}} h(D_n/u^2) = \sum_{\substack{u>0 \\ u|f_n}} h(u^2 d_n).$$

Inserting the formula (see e.g. [12, p. 233])

$$h(u^2 d_n) = u \prod_{p|u} \left(1 - \frac{\chi_{d_n}(p)}{p}\right) h(d_n)$$

into (25) yields

$$H(D_n) = \delta(n) h(d_n),$$

where

$$\delta(n) := \sum_{\substack{u>0 \\ u|f_n}} u \prod_{p|u} \left(1 - \frac{\chi_{d_n}(p)}{p}\right).$$

Now, a simple estimate yields

$$|\delta(n)| \leq \sqrt{|D_n|} \tau(|D_n|) 2^{\omega(|D_n|)}$$

where  $\omega(|D_n|)$  is the number of prime divisors of  $|D_n|$ . We have

$$\tau(|D_n|) < 2\sqrt{|D_n|},$$

and by [26, Théorème 13] we have

$$\omega(|D_n|) \leq \max \left\{ 1, \frac{\log(|D_n|)}{\log(\log(|D_n|)) - 1.1714} \right\} \leq \frac{\log(|D_n|)}{|\log(\log(|D_n|)) - 1.1714|} =: q(n).$$

Hence

$$|\delta(n)| \leq 2^{q(n)+1} |D_n|.$$

Using the class number formula

$$h(d_n) = \frac{\sqrt{|d_n|}}{\pi} L(\chi_{d_n}, 1)$$

and the evaluation

$$L(\chi_{d_n}, 1) = -\frac{\pi}{|d_n|^{3/2}} \sum_{t=1}^{|d_n|-1} \chi_{d_n}(t)t,$$

another simple estimate yields

$$h(d_n) \leq |d_n|.$$

Then combining the preceding estimates gives

$$H(D_n) \leq 2^{q(n)+1} |D_n|^2.$$

Finally, using the class number bound we get

$$|E(n)| < (3.45 \times 10^{44}) 2^{q(n)+1} |D_n|^2 \exp(\pi \sqrt{|D_n|}/12).$$

This completes the proof. □

#### 4. EFFECTIVE ASYMPTOTICS FOR $\text{SPT}(n)$

To prove our results, we will require an effective asymptotic formula for  $p(n)$  due to Lehmer [20]. For convenience, define

$$\mu(n) := \frac{\pi}{6} \sqrt{24n - 1}.$$

Inspired by the Hardy-Ramanujan asymptotic for  $p(n)$ , Rademacher [25] obtained the following exact formula:

$$p(n) = \frac{2\pi}{(24n - 1)^{3/4}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} I_{3/2} \left( \frac{\mu(n)}{c} \right),$$

where  $A_c(n)$  is the Kloosterman sum

$$A_c(n) := \sum_{\substack{d \pmod{c} \\ (d,c)=1}} e^{\pi i s(d,c)} e^{-\frac{2\pi i d n}{c}}$$

and  $s(d, c)$  is the Dedekind sum

$$s(d, c) := \sum_{r=1}^{c-1} \frac{r}{c} \left( \frac{dr}{c} - \left\lfloor \frac{dr}{c} \right\rfloor - \frac{1}{2} \right).$$

Using Rademacher's formula, Lehmer [20] proved the following result.

**Theorem 4.1** (Lehmer). *For all  $n \geq 1$ , we have*

$$p(n) = \frac{\sqrt{12}}{24n - 1} \sum_{c=1}^N \frac{A_c(n)}{\sqrt{c}} \left\{ \left( 1 - \frac{c}{\mu(n)} \right) e^{\mu(n)/c} + \left( 1 + \frac{c}{\mu(n)} \right) e^{-\mu(n)/c} \right\} + R_2(n, N),$$

where

$$|R_2(n, N)| < \frac{N^{-2/3} \pi^2}{\sqrt{3}} \left\{ \frac{N^3}{2\mu(n)^3} (e^{\mu(n)/N} - e^{-\mu(n)/N}) + \frac{1}{6} - \frac{N^2}{\mu(n)^2} \right\}.$$

We first use Theorem 4.1 to quickly deduce the following effective bound.

**Lemma 4.2.** *For all  $n \geq 1$ , we have*

$$p(n) = \frac{2\sqrt{3}}{24n - 1} \left( 1 - \frac{1}{\mu(n)} \right) e^{\mu(n)} + E_p(n),$$

where

$$|E_p(n)| \leq (1324) e^{\mu(n)/2}.$$

*Proof.* Using the identity

$$(26) \quad I_{3/2}(x) = \frac{1}{2} \sqrt{\frac{2}{\pi x}} \left[ \left( 1 - \frac{1}{x} \right) e^x + \left( 1 + \frac{1}{x} \right) e^{-x} \right],$$

we may write Theorem 4.1 (with the choice  $N = 2$ ) as

$$(27) \quad p(n) = \frac{2\pi}{(24n - 1)^{3/4}} \sum_{c=1}^2 \frac{A_c(n)}{c} I_{3/2} \left( \frac{\mu(n)}{c} \right) + R_2(n, 2),$$

where

$$|R_2(n, 2)| < \frac{\pi^2}{\sqrt{3}2^{2/3}} \left[ \left( \frac{2}{\mu(n)} \right)^3 \left\{ \frac{e^{\mu(n)/2} - e^{-\mu(n)/2}}{2} \right\} + 1/6 - \left( \frac{2}{\mu(n)} \right)^2 \right].$$

Now, using (27) and (26) we get

$$\begin{aligned} p(n) &= \frac{2\pi}{(24n-1)^{3/4}} I_{3/2}(\mu(n)) + \frac{2\pi}{(24n-1)^{3/4}} \frac{A_2(n)}{2} I_{3/2} \left( \frac{\mu(n)}{2} \right) + R_2(n, 2) \\ &= \frac{2\sqrt{3}}{24n-1} \left( 1 - \frac{1}{\mu(n)} \right) e^{\mu(n)} + E_p(n), \end{aligned}$$

where

$$E_p(n) := \frac{2\pi}{(24n-1)^{3/4}} \left[ \frac{1}{2} \sqrt{\frac{2}{\pi\mu(n)}} \left( 1 + \frac{1}{\mu(n)} \right) e^{-\mu(n)} + \frac{A_2(n)}{2} I_{3/2} \left( \frac{\mu(n)}{2} \right) \right] + R_2(n, 2).$$

By [21, (6.24)], we have the bound

$$(28) \quad I_{3/2}(x) < \left( \frac{x}{2} \right)^{3/2} \frac{1}{\Gamma(5/2)} \frac{1}{2} (1 + e^{-x}) e^x \leq x^{3/2} e^x, \quad x > 0.$$

Then an estimate using the trivial bound

$$|A_c(n)| < c$$

and the bound (28) yields

$$\left| \frac{2\pi}{(24n-1)^{3/4}} \frac{A_2(n)}{2} I_{3/2} \left( \frac{\mu(n)}{2} \right) \right| \leq 12e^{\mu(n)/2}.$$

Similarly, two straightforward estimates yield

$$\left| \frac{2\pi}{(24n-1)^{3/4}} \frac{1}{2} \sqrt{\frac{2}{\pi\mu(n)}} \left( 1 + \frac{1}{\mu(n)} \right) e^{-\mu(n)} \right| \leq 16e^{\mu(n)/2}$$

and

$$|R_2(n, 2)| < (1296)e^{\mu(n)/2}.$$

Hence

$$|E_p(n)| \leq (1324)e^{\mu(n)/2}.$$

□

**4.1. Proof of Theorem 1.1.** Using (4), the formula (6) can be written as

$$(29) \quad \text{spt}(n) = \frac{1}{12} [S(n) - (24n-1)p(n)].$$

Now, a straightforward calculation using (24) shows that

$$2\pi(24n-1)^{1/4} I_{1/2}(\mu(n)) = 2\sqrt{3}e^{\mu(n)} + E_4(n),$$

where

$$|E_4(n)| \leq (24n-1)e^{\mu(n)/2}.$$

Hence the asymptotic formula in Theorem 1.4 can be written as

$$(30) \quad S(n) = 2\sqrt{3}e^{\mu(n)} + \tilde{E}(n),$$

where  $\tilde{E}(n) := E(n) + E_4(n)$  with

$$|\tilde{E}(n)| \leq |E(n)| + |E_4(n)| < (3.46 \times 10^{44})2^{q(n)+1}(24n-1)^2e^{\mu(n)/2}.$$

Then using (29), (30), and Lemma 4.2, another straightforward calculation yields

$$\text{spt}(n) = \frac{\sqrt{3}}{\pi\sqrt{24n-1}}e^{\mu(n)} + E_s(n),$$

where the error term

$$E_s(n) := \frac{\tilde{E}(n)}{12} - \frac{24n-1}{12}E_p(n)$$

satisfies the bound

$$|E_s(n)| < (5.78 \times 10^{43})2^{q(n)}(24n-1)^2e^{\mu(n)/2}.$$

□

As pointed out by Bessenrodt and Ono [5], it is straightforward to obtain from Theorem 4.1 that

$$\frac{\sqrt{3}}{12n} \left(1 - \frac{1}{\sqrt{n}}\right) e^{\mu(n)} < p(n) < \frac{\sqrt{3}}{12n} \left(1 + \frac{1}{\sqrt{n}}\right) e^{\mu(n)}$$

for all  $n \geq 1$ . We will use Theorem 1.1 to prove the following analogous statement for  $\text{spt}(n)$ , where  $\sqrt{n}$  is replaced by any positive integral power of  $n$ .

**Theorem 4.3.** *For each  $\lambda \in \mathbb{Z}^+$  and  $k \in \mathbb{Z}^+$ , there is an effective positive integer  $B_k(\lambda)$  such that for all  $n \geq B_k(\lambda)$ , we have*

$$\frac{\sqrt{3}}{\pi\sqrt{24n-1}} \left(1 - \frac{1}{\lambda n^k}\right) e^{\mu(n)} < \text{spt}(n) < \frac{\sqrt{3}}{\pi\sqrt{24n-1}} \left(1 + \frac{1}{\lambda n^k}\right) e^{\mu(n)}.$$

*Proof.* By Theorem 1.1 we have the bounds

$$\frac{\sqrt{3}}{\pi\sqrt{24n-1}}e^{\mu(n)} - |E_s(n)| < \text{spt}(n) < \frac{\sqrt{3}}{\pi\sqrt{24n-1}}e^{\mu(n)} + |E_s(n)|,$$

where

$$|E_s(n)| < (5.78 \times 10^{43})2^{q(n)}(24n-1)^2e^{\mu(n)/2}.$$

Clearly, there is an effective positive integer  $B_k(\lambda)$  such that the inequality

$$(5.78 \times 10^{43})2^{q(n)}(24n-1)^2e^{\mu(n)/2} < \frac{\sqrt{3}}{\pi\sqrt{24n-1}} \cdot \frac{1}{\lambda n^k} e^{\mu(n)}$$

holds for all  $n \geq B_k(\lambda)$ . For instance, if  $\lambda = k = 1$  then  $B_1(1) = 13500$ . This completes the proof. □

## 5. PROOF OF THEOREM 1.3

By Theorem 1.1 and Lemma 4.2, we may write

$$(31) \quad \text{spt}(n) = \alpha(n)e^{\mu(n)} + E_s(n)$$

and

$$(32) \quad p(n) = \beta(n)e^{\mu(n)} + E_p(n),$$

where

$$\alpha(n) := \frac{\sqrt{3}}{\pi\sqrt{24n-1}}, \quad \beta(n) := \frac{2\sqrt{3}}{24n-1} \left(1 - \frac{6}{\pi\sqrt{24n-1}}\right).$$

Also, for  $\epsilon > 0$  we define

$$\gamma(n) := \frac{\sqrt{6}}{\pi}\sqrt{n}, \quad \gamma(n, \epsilon) := \left(\frac{\sqrt{6}}{\pi} + \epsilon\right)\sqrt{n}.$$

We must prove that there exists an effective positive constant  $N(\epsilon) > 0$  such that for all  $n \geq N(\epsilon)$ , we have

$$(33) \quad \gamma(n)p(n) < \text{spt}(n) < \gamma(n, \epsilon)p(n).$$

First, using (31) and (32) we find that the lower bound in (33) is equivalent to

$$(34) \quad c_1(n)e^{\mu(n)} > \gamma(n)E_p(n) - E_s(n),$$

where  $c_1(n) := \alpha(n) - \beta(n)\gamma(n)$ . Now, the error bounds in Lemma 4.2 and Theorem 1.1 imply that

$$|\gamma(n)E_p(n) - E_s(n)| \leq c_2(n)e^{\mu(n)/2},$$

where

$$c_2(n) := (1324)\gamma(n) + (5.78 \times 10^{43})2^{q(n)}(24n-1)^2.$$

Then noting that  $c_1(n) > 0$  for all  $n \geq 1$ , we find that (34) is implied by the bound

$$e^{\mu(n)/2} > c_3(n) := \frac{c_2(n)}{c_1(n)},$$

or equivalently, the bound

$$(35) \quad n > \frac{1}{24} \left[ \left( \frac{12}{\pi} \log(c_3(n)) \right)^2 + 1 \right].$$

A calculation shows that (35) holds for all  $n \geq N := 12788$ .

Similarly, using (31) and (32) we find that the upper bound in (33) is equivalent to

$$(36) \quad c_4(n, \epsilon)e^{\mu(n)} > E_s(n) - \gamma(n, \epsilon)E_p(n),$$

where  $c_4(n, \epsilon) := \beta(n)\gamma(n, \epsilon) - \alpha(n)$ . The error bounds in Lemma 4.2 and Theorem 1.1 imply that

$$|E_s(n) - \gamma(n, \epsilon)E_p(n)| \leq c_5(n, \epsilon)e^{\mu(n)/2},$$

where

$$c_5(n, \epsilon) := (1324)\gamma(n, \epsilon) + (5.78 \times 10^{43})2^{q(n)}(24n-1)^2.$$

Moreover, there exists an effective positive constant  $N_1(\epsilon) > 0$  such that  $c_4(n, \epsilon) > 0$  for all  $n \geq N_1(\epsilon)$ . Then arguing as above, we find that if  $n \geq N_1(\epsilon)$ , the bound (36) is implied by the bound

$$(37) \quad n > \frac{1}{24} \left[ \left( \frac{12}{\pi} \log(c_6(n, \epsilon)) \right)^2 + 1 \right],$$

where  $c_6(n, \epsilon) := c_5(n, \epsilon)/c_4(n, \epsilon)$ . Clearly, there exists an effective positive constant  $N_2(\epsilon) \geq N_1(\epsilon)$  such that (37) holds for all  $n \geq N_2(\epsilon)$ .

Let  $N(\epsilon) := \max\{N, N_2(\epsilon)\}$ . Then the inequalities (33) hold for all  $n \geq N(\epsilon)$ .  $\square$

## 6. PROOF OF THEOREM 1.2

**6.1. Proof of Conjecture (1).** Let  $\epsilon = 1 - \sqrt{6}/\pi$  in Theorem 1.3. We need to determine the constant  $N(1 - \sqrt{6}/\pi)$ . A calculation shows that the inequality  $c_4(n, 1 - \sqrt{6}/\pi) > 0$  holds if  $n \geq N_1(1 - \sqrt{6}/\pi)$  where  $N_1(1 - \sqrt{6}/\pi) = 4$ . Next, we need to find the smallest positive integer  $N_2(1 - \sqrt{6}/\pi) \geq 4$  such that the bound

$$n > \frac{1}{24} \left[ \left( \frac{12}{\pi} \log(c_6(n, 1 - \sqrt{6}/\pi)) \right)^2 + 1 \right]$$

holds for all  $n \geq N_2(1 - \sqrt{6}/\pi)$ . A calculation shows that this constant is given by  $N_2(1 - \sqrt{6}/\pi) = 11997$ . We now have

$$N(1 - \sqrt{6}/\pi) := \max\{N, N_2(1 - \sqrt{6}/\pi)\} = \max\{12788, 11997\} = 12788.$$

Therefore, the inequalities

$$\frac{\sqrt{6}}{\pi} \sqrt{np(n)} < \text{spt}(n) < \sqrt{np(n)}$$

hold for all  $n \geq 12788$ . Finally, one can verify with a computer that these inequalities also hold for  $5 \leq n < 12788$ .  $\square$

**6.2. Proof of Conjecture (2).** We follow closely the proof of [5, Theorem 2.1]. By taking  $\lambda = k = 1$  in Theorem 4.3 (recall that  $B_1(1) = 13500$ ), we find that

$$(38) \quad \frac{\sqrt{3}}{\pi\sqrt{24n-1}} \left(1 - \frac{1}{n}\right) e^{\mu(n)} < \text{spt}(n) < \frac{\sqrt{3}}{\pi\sqrt{24n-1}} \left(1 + \frac{1}{n}\right) e^{\mu(n)}$$

holds for all  $n \geq 13500$ . One can verify with a computer that (38) also holds for  $1 \leq n < 13500$ .

Now, assume that  $1 < a \leq b$ , and let  $b = Ca$  where  $C \geq 1$ . From (38) we get the inequalities

$$\text{spt}(a)\text{spt}(Ca) > \frac{3}{\pi^2\sqrt{24a-1}\sqrt{24Ca-1}} \left(1 - \frac{1}{a}\right) \left(1 - \frac{1}{Ca}\right) e^{\mu(a)+\mu(Ca)}$$

and

$$\text{spt}(a + Ca) < \frac{\sqrt{3}}{\pi\sqrt{24(a+Ca)-1}} \left(1 + \frac{1}{a+Ca}\right) e^{\mu(a+Ca)}.$$



Hence, for all but finitely many cases, it suffices to find conditions on  $a > 1$  and  $C \geq 1$  such that

$$(39) \quad e^{\mu(a)+\mu(Ca)-\mu(a+Ca)} > \frac{\pi\sqrt{24a-1}\sqrt{24Ca-1}}{\sqrt{3}\sqrt{24(a+Ca)-1}} \cdot \frac{\left(1+\frac{1}{a+Ca}\right)}{\left(1-\frac{1}{a}\right)\left(1-\frac{1}{Ca}\right)}.$$

For convenience, define

$$T_a(C) := \mu(a) + \mu(Ca) - \mu(a + Ca) \quad \text{and} \quad S_a(C) := \frac{\left(1+\frac{1}{a+Ca}\right)}{\left(1-\frac{1}{a}\right)\left(1-\frac{1}{Ca}\right)}.$$

Then by taking logarithms, we find that (39) is equivalent to

$$(40) \quad T_a(C) > \log\left(\frac{\pi\sqrt{24a-1}\sqrt{24Ca-1}}{\sqrt{3}\sqrt{24(a+Ca)-1}}\right) + \log S_a(C).$$

As functions of  $C$ , it can be shown that  $T_a(C)$  is increasing and  $S_a(C)$  is decreasing for  $C \geq 1$ , and thus

$$T_a(C) \geq T_a(1)$$

and

$$\log(S_a(1)) \geq \log(S_a(C)).$$

Hence it suffices to show that

$$T_a(1) > \log\left(\frac{\pi\sqrt{24a-1}\sqrt{24Ca-1}}{\sqrt{3}\sqrt{24(a+Ca)-1}}\right) + \log(S_a(1)).$$

Moreover, since

$$\frac{\sqrt{24Ca-1}}{\sqrt{24(a+Ca)-1}} \leq 1$$

for all  $C \geq 1$  and all  $a > 1$ , it suffices to show that

$$(41) \quad T_a(1) > \log\left(\frac{\pi\sqrt{24a-1}}{\sqrt{3}}\right) + \log(S_a(1)).$$

By computing the values  $T_a(1)$  and  $S_a(1)$ , we find that (41) holds for all  $a \geq 5$ .

To complete the proof, assume that  $2 \leq a \leq 4$ . For each such integer  $a$ , we calculate the real number  $C_a$  for which

$$T_a(C_a) = \log\left(\frac{\pi\sqrt{24a-1}}{\sqrt{3}}\right) + \log(S_a(C_a)).$$

The values  $C_a$  are listed in the table below.

$a$	$C_a$
2	27.87...
3	3.54...
4	1.79...

By the discussion above, if  $b = Ca \geq a$  is an integer for which  $C > C_a$ , then (40) holds, which in turn gives the theorem in these cases. Only finitely many cases remain, namely the pairs of integers where  $2 \leq a \leq 4$  and  $1 \leq b/a \leq C_a$ . We compute  $\text{spt}(a)$ ,  $\text{spt}(b)$ , and  $\text{spt}(a+b)$  in these cases to complete the proof.  $\square$

**6.3. Proof of Conjecture (3).** We require some lemmas and a proposition analogous to those of Desalvo and Pak [13] in order to prove the remaining conjectures.

The following is [13, Lemma 2.1].

**Lemma 6.1.** *Suppose  $h(x)$  is a positive, increasing function with two continuous derivatives for all  $x > 0$ , and that  $h'(x) > 0$  is decreasing, and  $h''(x) < 0$  is increasing for all  $x > 0$ . Then for all  $x > 0$ , we have*

$$h''(x-1) < h(x+1) - 2h(x) + h(x-1) < h''(x+1).$$

By Theorem 1.1, we may write

$$\text{spt}(n) = f(n) + E_s(n)$$

where

$$(42) \quad f(n) := \frac{\sqrt{3}}{\pi\sqrt{24n-1}} e^{\mu(n)}$$

and

$$|E_s(n)| \leq (5.78 \times 10^{43}) 2^{q(n)} (24n-1)^2 e^{\mu(n)/2}.$$

**Lemma 6.2.** *Let*

$$F(n) := 2\log(f(n)) - \log(f(n+1)) - \log(f(n-1)).$$

*Then for all  $n \geq 4$ , we have*

$$\frac{24\pi}{(24(n+1)-1)^{3/2}} - \frac{1}{n^2} < F(n) < \frac{24\pi}{(24(n-1)-1)^{3/2}} - \frac{288}{(24(n+1)-1)^2}.$$

*Proof.* We can write  $f(n)$  from (42) as

$$f(n) = \frac{1}{2\sqrt{3}\mu(n)} e^{\mu(n)},$$

so that

$$\log(f(n)) = \mu(n) - \log(\mu(n)) - \log(2\sqrt{3}).$$

Then we have

$$F(n) = 2\mu(n) - \mu(n+1) - \mu(n-1) - 2\log(\mu(n)) + \log(\mu(n+1)) + \log(\mu(n-1)).$$

Since the functions  $\mu(x)$  and  $\tilde{\mu}(x) := \log(\mu(x))$  satisfy the hypotheses of Lemma 6.1, we get

$$-\mu''(n+1) + \tilde{\mu}''(n-1) < F(n) < -\mu''(n-1) + \tilde{\mu}''(n+1).$$

Computing derivatives gives

$$\frac{24\pi}{(24(n+1)-1)^{3/2}} - \frac{288}{(24(n-1)-1)^2} < F(n) < \frac{24\pi}{(24(n-1)-1)^{3/2}} - \frac{288}{(24(n+1)-1)^2}$$

for all  $n \geq 2$ , from which we deduce that

$$\frac{24\pi}{(24(n+1)-1)^{3/2}} - \frac{1}{n^2} < F(n) < \frac{24\pi}{(24(n-1)-1)^{3/2}} - \frac{288}{(24(n+1)-1)^2}$$

for all  $n \geq 4$ . □

**Lemma 6.3.** Define the functions  $y_n := |E_s(n)|/f(n)$ ,

$$M(n) := 2\sqrt{3}(5.78 \times 10^{43})\mu(n)2^{q(n)}(24n-1)^2e^{-\mu(n)/2},$$

and

$$g(n) := \frac{M(n)}{1-M(n)}.$$

Then for all  $n \geq 2$ , we have

$$\log \left[ \frac{(1-y_n)^2}{(1+y_{n+1})(1+y_{n-1})} \right] > -2g(n) - M(n+1) - M(n-1)$$

and

$$\log \left[ \frac{(1+y_n)^2}{(1-y_{n+1})(1-y_{n-1})} \right] < 2M(n) + g(n+1) + g(n-1).$$

*Proof.* First observe that for all  $n \geq 1$ , we have

$$(43) \quad 0 < y_n := \frac{|E_s(n)|}{f(n)} < \frac{(5.78 \times 10^{43})2^{q(n)}(24n-1)^2e^{\mu(n)/2}}{\frac{1}{2\sqrt{3}\mu(n)}e^{\mu(n)}} = M(n).$$

Note also that  $y_n < 1$  for all  $n \geq 1$ . Then using (43) and the inequalities

$$\log(1-x) \geq -\frac{x}{1-x} \quad \text{for } 0 < x < 1$$

and

$$\log(1+x) < x \quad \text{for } x > 0,$$

we get

$$\begin{aligned} \log \left[ \frac{(1-y_n)^2}{(1+y_{n+1})(1+y_{n-1})} \right] &= 2\log(1-y_n) - \log(1+y_{n+1}) - \log(1+y_{n-1}) \\ &> -2\frac{y_n}{1-y_n} - y_{n+1} - y_{n-1} \\ &> -2g(n) - M(n+1) - M(n-1) \end{aligned}$$

for all  $n \geq 2$ . Similarly, we get

$$\begin{aligned} \log \left[ \frac{(1+y_n)^2}{(1-y_{n+1})(1-y_{n-1})} \right] &= 2\log(1+y_n) - \log(1-y_{n+1}) - \log(1-y_{n-1}) \\ &< 2y_n + \frac{y_{n+1}}{1-y_{n+1}} + \frac{y_{n-1}}{1-y_{n-1}} \\ &< 2M(n) + g(n+1) + g(n-1) \end{aligned}$$

for all  $n \geq 2$ . □

**Proposition 6.4.** Let

$$\text{spt}_2(n) := 2\log(\text{spt}(n)) - \log(\text{spt}(n+1)) - \log(\text{spt}(n-1)).$$

Then we have

$$\frac{1}{(24n)^{3/2}} < \text{spt}_2(n) < \frac{2}{n^{3/2}},$$

where the lower bound holds for all  $n \geq 14796$  and the upper bound holds for all  $n \geq 14641$ .

*Proof.* We first bound  $\text{spt}(n)$  by

$$f(n) \left(1 - \frac{|E_s(n)|}{f(n)}\right) < \text{spt}(n) < f(n) \left(1 + \frac{|E_s(n)|}{f(n)}\right).$$

Then recalling that

$$F(n) := 2 \log(f(n)) - \log(f(n+1)) - \log(f(n-1))$$

and  $y_n := |E_s(n)|/f(n)$ , we take logarithms in the preceding inequalities to get

$$F(n) + \log \left[ \frac{(1 - y_n)^2}{(1 + y_{n+1})(1 + y_{n-1})} \right] < \text{spt}_2(n) < F(n) + \log \left[ \frac{(1 + y_n)^2}{(1 - y_{n+1})(1 - y_{n-1})} \right].$$

It follows immediately from Lemmas 6.2 and 6.3 that for all  $n \geq 4$ , we have

$$\text{spt}_2(n) > \frac{24\pi}{(24(n+1)-1)^{3/2}} - \frac{1}{n^2} - 2g(n) - M(n+1) - M(n-1)$$

and

$$(44) \quad \text{spt}_2(n) < \frac{24\pi}{(24(n-1)-1)^{3/2}} - \frac{288}{(24(n+1)-1)^2} + 2M(n) + g(n+1) + g(n-1).$$

Finally, a calculation shows that

$$\frac{24\pi}{(24(n+1)-1)^{3/2}} - \frac{1}{n^2} - 2g(n) - M(n+1) - M(n-1) > \frac{1}{(24n)^{3/2}}$$

for all  $n \geq 14796$  and

$$\frac{24\pi}{(24(n-1)-1)^{3/2}} - \frac{288}{(24(n+1)-1)^2} + 2M(n) + g(n+1) + g(n-1) < \frac{2}{n^{3/2}}$$

for all  $n \geq 14641$ . This completes the proof.  $\square$

To prove Conjecture (3), we must show that

$$\text{spt}(n)^2 > \text{spt}(n-1)\text{spt}(n+1)$$

for  $n \geq 36$ . Taking logarithms, we see that this is equivalent to  $\text{spt}_2(n) > 0$ . By the lower bound in Proposition 6.4, we have  $\text{spt}_2(n) > 0$  for all  $n \geq 14796$ . Finally, one can verify with a computer that  $\text{spt}_2(n) > 0$  for all  $36 \leq n < 14796$ . This completes the proof.  $\square$

**6.4. Proof of Conjecture (4).** We follow closely the proof of [13, Theorem 5.1]. Recall that a sequence  $\{a(k)\}_{k=0}^{\infty}$  of non-negative integers is log-concave if

$$a(k)^2 \geq a(k-1)a(k+1)$$

for all  $k \geq 1$ . Moreover, it is known that log-concavity implies strong log-concavity

$$a(\ell-i)a(k+i) \geq a(k)a(\ell),$$

for all  $0 \leq k \leq \ell \leq n$  and  $0 \leq i \leq \ell - k$  (see e.g. [27]).

Now, we have proved that

$$\text{spt}(n)^2 > \text{spt}(n-1)\text{spt}(n+1)$$

for all  $n \geq 36$ . Therefore, if we take  $k = n - m$ ,  $\ell = n + m$ , and  $i = m$ , then

$$\text{spt}(n)^2 > \text{spt}(n-m)\text{spt}(n+m)$$

for all  $n > m > 1$  with  $n - m > 36$ .

We next consider the case  $n > m > 1$  with  $1 \leq n - m \leq 36$ . We will prove that

$$(45) \quad \text{spt}(n)^2 \geq \text{spt}(m+1)^2 > \text{spt}(36)\text{spt}(36+2m) \geq \text{spt}(n-m)\text{spt}(n+m)$$

for all  $1 \leq n - m \leq 36$  with  $m \geq 6290$ . On the other hand, one can verify with a computer that

$$\text{spt}(n)^2 > \text{spt}(n-m)\text{spt}(n+m)$$

for all  $1 \leq n - m \leq 36$  with  $m < 6290$ . This completes the proof of Conjecture (4), subject to verifying the inequalities (45).

Since  $n \geq m + 1$ , we have

$$\text{spt}(n)^2 \geq \text{spt}(m+1)^2.$$

Moreover, since  $n - m \leq 36$  we have

$$(46) \quad \text{spt}(n-m) < \text{spt}(36) < 90000,$$

and thus

$$\text{spt}(36)\text{spt}(36+2m) \geq \text{spt}(n-m)\text{spt}(n+m).$$

This verifies the first and third inequalities in (45).

It remains to prove that

$$(47) \quad \text{spt}(m+1)^2 > \text{spt}(36)\text{spt}(36+2m)$$

for all  $m \geq 6290$ . Taking logarithms in (47), we see that it suffices to prove

$$(48) \quad 2 \log(\text{spt}(m+1)) - \log(\text{spt}(36)) - \log(\text{spt}(36+2m)) > 0$$

for all  $m \geq 6290$ . By [18] and [15], respectively, we have the lower and upper bounds

$$\frac{e^{2\sqrt{m}}}{2\pi m e^{1/6m}} < p(m) < e^{\pi\sqrt{(2/3)m}}.$$

Then by Theorem 1.3 with the choice  $\epsilon = (2 - \sqrt{2})\sqrt{3}/\pi$ , we get

$$(49) \quad \frac{\sqrt{6m}}{\pi} \cdot \frac{e^{2\sqrt{m}}}{2\pi m e^{1/6m}} < \text{spt}(m) < \frac{\sqrt{12m}}{\pi} e^{\pi\sqrt{(2/3)m}}.$$

Using the inequalities (46) and (49), we see that (48) is implied by the lower bound

$$2 \log \left( \frac{\sqrt{6(m+1)}}{2\pi^2(m+1)e^{1/6(m+1)}} \right) + 4\sqrt{m+1} - \log(90000) - \log \left( \frac{\sqrt{12(36+2m)}}{\pi} \right) - \frac{\pi\sqrt{2(36+2m)}}{\sqrt{3}} > 0$$

for all  $m \geq 6290$ . A calculation shows this is true for the given range of  $m$ .  $\square$

**6.5. Proof of Conjecture (5).** Taking logarithms, we find that Conjecture (5) is equivalent to

$$\text{spt}_2(n) < \log \left( 1 + \frac{1}{n} \right)$$

for all  $n \geq 13$ . By the upper bound in Proposition 6.4 and some straightforward estimates, we have

$$\text{spt}_2(n) < \frac{2}{n^{3/2}} < \frac{1}{n+1} < \log \left( 1 + \frac{1}{n} \right)$$

for all  $n \geq 14641$ . Finally, one can verify with a computer that the conjectured inequality holds for all  $13 \leq n < 14641$ . This completes the proof.  $\square$

**6.6. Proof of Conjecture (6).** We follow closely the proof of [11, Conjecture 1.3]. Taking logarithms, we find that Conjecture (6) is equivalent to

$$\text{spt}_2(n) < \log \left( 1 + \frac{\pi}{\sqrt{24n^{3/2}}} \right)$$

for all  $n \geq 73$ . By (44) we have

$$\text{spt}_2(n) < \frac{24\pi}{(24(n-1)-1)^{3/2}} - \frac{288}{(24(n+1)-1)^2} + 2M(n) + g(n+1) + g(n-1)$$

for all  $n \geq 4$ . On the other hand, by [11, (2.3)] we have

$$\frac{24\pi}{(24(n+1)-1)^{3/2}} < \frac{24\pi}{(24n)^{3/2}} - \left( \frac{24\pi}{(24\pi)^{3/2}} \right)^2 + \frac{3}{2n^{5/2}}$$

for all  $n \geq 50$ , and by the second inequality following [11, (2.23)] we have

$$-\frac{288}{(24(n+1)-1)^2} < \frac{1}{6n^{5/2}} - \frac{1}{2n^2}$$

for all  $n \geq 50$ . Therefore, for all  $n \geq 50$  we have

$$\text{spt}_2(n) < \frac{24\pi}{(24n)^{3/2}} - \left( \frac{24\pi}{(24\pi)^{3/2}} \right)^2 + \frac{5}{3n^{5/2}} - \frac{1}{2n^2} + 2M(n) + g(n+1) + g(n-1).$$

Now, a calculation shows that

$$\frac{5}{3n^{5/2}} - \frac{1}{2n^2} + 2M(n) + g(n+1) + g(n-1) < 0$$

for all  $n \geq 15831$ . Hence

$$\text{spt}_2(n) < \frac{24\pi}{(24n)^{3/2}} - \left( \frac{24\pi}{(24\pi)^{3/2}} \right)^2 = \frac{24\pi}{(24n)^{3/2}} \left( 1 - \frac{24\pi}{(24n)^{3/2}} \right)$$

for all  $n \geq 15831$ . Then using the inequality

$$x(1-x) < \log(1+x) \quad \text{for } x > 0,$$

we get

$$\text{spt}_2(n) < \log \left( 1 + \frac{24\pi}{(24n)^{3/2}} \right) = \log \left( 1 + \frac{\pi}{\sqrt{24n^{3/2}}} \right)$$

for all  $n \geq 15831$ . Finally, one can verify with a computer that this inequality also holds for all  $73 \leq n < 15831$ . This completes the proof.  $\square$

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, EMORY UNIVERSITY, ATLANTA, GA 30322  
*E-mail address:* madeline.locus@emory.edu

DEPARTMENT OF MATHEMATICS, MAILSTOP 3368, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX  
77843-3368  
*E-mail address:* masri@math.tamu.edu