Math 304
Exam #3B - Solutions
Sections 4.1-5.5

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1. [5 pts] Show that the operator $L : R^{n\times n} \to R^{n\times n}$ defined by

$$L[A] = \frac{A - A^T}{2}$$

is linear. (An example is not sufficient!)

Solution:

$$L[A+B] = \frac{(A+B) - (A+B)^T}{2} = \frac{(A - A^T) + (B - B^T)}{2} = L[A] + L[B]$$

$$L[cA] = \frac{(cA) - (cA)^T}{2} = \frac{cA - cA^T}{2} = cL[A]$$

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2. [10 pts] Find the matrix representation of $L : P_3 \to P_3$ (using the standard basis $\{1, x, x^2\}$) where L is given by

$$L[p(x)] = xp''(x) + xp'(x) + p(x) - p(1)$$

Solution: From the definition of L, we can compute $L[1] = 0$, $L[x] = 2x - 1$, $L[x^2] = 3x^2 + 2x - 1$. In terms of the standard basis, this is $[0, 0, 0]^T, [-1, 2, 0]^T, [-1, 2, 3]^T$ so the matrix is

$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

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3. [5 pts] Show that the only matrix similar to the two by two identity

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the matrix I_2 itself.

Solution: If A is similar to the identity I, then we must have $A = S^{-1}IS$ for some S. Since $S^{-1}IS = S^{-1}S = I$, we must have $A = I$.

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4. [15 pts] Given $u = x+x^2$ and $v = 1+x^3$, and the usual inner product $\langle u, v\rangle = \int_0^1 u(x)v(x)dx$,

   (a) Find the **scalar projection** of u onto v

   Solution: $\langle u, v\rangle = \int_0^1 (x+x^2)(1+x^3)dx = 6/5$, $\|v\|^2 = \int_0^1 (1+x^3)^2 = 23/14$. Therefore the scalar projection is given by

   $$\frac{\frac{6}{5}}{\sqrt{\frac{23}{14}}}$$

   (b) Find the **vector projection** of u onto v

   Solution: The vector projection is given by

   $$\frac{\langle u, v\rangle}{\|v\|^2}v = \frac{6/5}{23/14}(1+x^3) = \frac{84}{115}(1+x^3)$$

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5. [10 pts] Find the distance of the point $(1,2,3,4)$ to the plane $x_1 + 2x_2 + x_3 + 2x_4 = 0$ in R^4.

   Solution: The direction of the normal is given by $[1,2,1,2]^T$. The unit normal is therfore $\hat{n} = [1,2,1,2]/\sqrt{10}$ The distance is given by the scalar projection

   $$d = \frac{[1,2,3,4].[1,2,1,2]^T}{\sqrt{10}} = \frac{16}{\sqrt{10}}$$

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6. [5 pts] If S is a subspace of R^5 of dimension 2, what is the dimension of the orthogonal complement?

   Solution: Since the sum of the dimension of S and its orthogonal complement add up to 5, it must be dimension 3.

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7. [10 pts] Find the least squares line which best fits the data

x	0	1	2	3	4	5	6	7	8	9	10
y	1	1	0	0	1	1	0	0	1	1	0

Solution: The matrices A and the vector b is given by

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \\ 1 & 8 \\ 1 & 9 \\ 1 & 10 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

This gives the 2×2 system of equations

$$A^T A = \begin{bmatrix} 11 & 55 \\ 55 & 385 \end{bmatrix} x = \begin{bmatrix} 6 \\ 27 \end{bmatrix} = A^T b$$

which has the solution $[15/22, -3/110]^T$, or $\hat{y} = (15/22) - (3/110)x$.

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8. [15 pts] Let the inner product on $R^{2\times 2}$ (the space of 2×2 matrices) be defined by

$$\langle [A], [B] \rangle = \sum_{i=1}^{2} \sum_{j=1}^{2} A_{i,j} B_{i,j}$$

find the angle between the two matrices

$$[A] = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, [B] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution: The inner product of A and B is given by $\langle A, B \rangle = 1*0+2*1+3*1+4*0 = 5$ Similarly $\|A\|^2 = \langle A, A \rangle = 1^2+2^2+3^2+4^2 = 30$, $\|B\|^2 = \langle B, B \rangle = 0^2+1^2+0^2+1^2 = 2$ so $\|A\| = \sqrt{30}$, $\|B\| = \sqrt{2}$. Consequently, the angle is given by

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{30}\sqrt{2}}\right) = 0.8691 \text{ rad} = 49.797 \text{ deg}$$

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9. [15 pts] Let S be the subspace of R^3 spaned by the orthogonal vectors $(1, 1, 1), (1, -2, 1)$. Find the best approximation (closest point) to $(2, 1, 4)$ in S.

   Solution: Since the vectors are already orthogonal, we can find the best approximation to $u(x) = c_1[1, 1, 1]^T + c_2[1, -2, 1]^T$ by taking the inner products

$$c_1 = \frac{[2, 1, 4].[1, 1, 1]^T}{\|[1, 1, 1]^T\|^2} = 7/3$$

$$c_1 = \frac{[2, 1, 4].[1, -2, 1]^T}{\|[1, -2, 1]^T\|^2} = 2/3$$

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10. [10 pts] Find the linear function which best approximates e^x on [0,1] with the usual inner product $\langle u, v\rangle = \int_0^1 u(x)v(x)dx$. Do this in two steps

    (a) Find two functions $u_1(x)$ and $u_2(x)$ which are an orthonormal basis for Span$[\{1, x\}]$.
    Solution: The function $u_1(x) = 1$ is already normalized. This means we have to find a linear function $u_2(x) = a + bx$ which is orthogonal to u_1 and has unit length. From

$$\langle u_1, u_2\rangle = \int_0^1 1 * (a + bx)dx = a + b/2 = 0$$

    we conclude that $b = -2a$, so $u_2 = a - 2ax = a(1 - 2x)$. In order for it to have unit length $a = \sqrt{3}$. We now have an orthonormal basis $\{1, \sqrt{3}(1 - 2x)\}$.

    (b) Find the coefficients c_1, c_2 in $e^x \approx c_1u_1(x) + c_2u_2(x)$.
    Solution: The coefficients are given by

$$c_1 = \langle e^x, u_1(x)\rangle = \int_0^1 e^x dx = e^1 - 1$$

$$c_2 = \langle e^x, u_2(x)\rangle = \int_0^1 e^x\sqrt{3}(1 - 2x)dx = \sqrt{3}(e - 3)$$
