CB #3

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \end{bmatrix} \xrightarrow{} \quad \text{rref to find null}(A)$$

$$\to \begin{bmatrix} 1 & 3 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $= r = 2$ $(r +)$
dim null $= 3$ $(n-r)$
$5 = (n)$

$x_2 = a$
$x_4 = b$
$x_5 = c$

solve for lead vars in terms of free.

$$x_1 + 3x_2 + 2x_4 + 3x_5 = 0$$
$$x_1 + 3a + 2b + 3c = 0$$
$$x_1 = -3a - 2b - 3c$$

$$x_3 + x_4 + x_5 = 0$$
$$x_3 + b + c = 0$$
$$x_3 = -b - c$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3a-2b-3c \\ a \\ -b-c \\ b \\ c \end{bmatrix} = \underbrace{a\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{basis for null}}$$

CA # 8

If S, T are subspaces of V.S. V then $S \cup T$ is a subspace. T/F ?

$S \cup T = \{ x \mid x \in S \text{ } \underline{\text{or}} \text{ } x \in T \}$

closure of "+": $x \in S \cup T$ $x \in V$
$y \in S \cup T$ $y \in V$

Q? $x+y \in S \cup T$?

no example ⇒ not closed under +

No!

$S = \{ (x,0) \mid x \in \mathbb{R} \} \subset \mathbb{R}^2$

$T = \{ (0,y) \mid y \in \mathbb{R} \} \subset \mathbb{R}^2$

$S \cup T = \{ (x,0) \text{ or } (0,y) \} =$ 

$(x,0) + (0,y) = (x,y)$

$S \cap T = \{ x \mid x \in S \text{ } \underline{\text{and}} \text{ } x \in T \}$

$x \in S \cap T \Rightarrow x \in S, x \in T$
$y \in S \cap T \Rightarrow y \in S, y \in T$

$x+y$ $x \in S, y \in S, \Rightarrow x+y \in S$
$x \in T, y \in T \Rightarrow x+y \in T$

$\Rightarrow x+y \in S \cap T$

→ closed under "+"

...

Yes!

CB-3 #8)

$$x_1 = \begin{bmatrix}1\\2\\2\end{bmatrix} \quad x_2 = \begin{bmatrix}1\\3\\3\end{bmatrix} \quad x_3 = \begin{bmatrix}1\\5\\5\end{bmatrix} \quad x_4 = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

a) $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ lin. indep?

At <u>most</u> 3 lin indep vectors in $\mathbb{R}^3 \Rightarrow$ one (or more) is dependent.

b) $\underbrace{\text{span}\{\vec{x}_1, \vec{x}_2\}}_{\dim \leq 2} = \mathbb{R}^3$? can't span dim=3 space.

c) $\{x_1, x_2, x_3\}$ span $\mathbb{R}^3$?

$$A = \underbrace{\begin{bmatrix}1&1&1\\2&3&5\\2&3&5\end{bmatrix}}_{\dim \leq 2} \to \text{rref} \begin{bmatrix}1&1&1\\2&3&5\\0&0&0\end{bmatrix}$$

$\Rightarrow$ dim rowspace = 2
$\Rightarrow$ dim col space = 2
$\Rightarrow$ not enough vectors

d) $\{x_1, x_2, x_4\}$

$$\begin{bmatrix}1&1&1\\2&3&2\\2&3&3\end{bmatrix} \to \begin{bmatrix}1&1&1\\2&3&2\\0&0&1\end{bmatrix} \to \begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

rank = 3 $\Rightarrow$ dim col space = 3 $\Rightarrow$ lin. indep, basis for $\mathbb{R}^3$ (span)

$$\mathcal{U} = \left\{ \overset{u_1}{\begin{bmatrix}1\\3\end{bmatrix}}, \overset{u_2}{\begin{bmatrix}2\\7\end{bmatrix}} \right\}$$

$$\mathcal{V} = \left\{ \underset{v_1}{\begin{bmatrix}5\\2\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}4\\9\end{bmatrix}} \right\}$$

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$$

$$\vec{e}_j = \begin{bmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{bmatrix} \leftarrow j^{th} \text{ row}$$

change of basis from $E = \{e_1, e_2\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

to $\mathcal{U} = \{\vec{u}_1, \vec{u}_2\}$

$$\begin{bmatrix}1 & 2\\3 & 7\end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix}7 & -2\\-3 & 1\end{bmatrix} = \begin{bmatrix}7 & -2\\-3 & 1\end{bmatrix} = \vec{u}^{-1}$$

$U \to E$: $E \cdot [x]_E = U[x]_U$

$E \to U$: $\boxed{[x]_U = U^{-1}E[x]_E = U^{-1}[x]_E}$

$$[x]_E = \begin{bmatrix}1\\1\end{bmatrix} \quad [x]_U = \begin{bmatrix}7 & -2\\-3 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5\\-2\end{bmatrix}$$

$$\vec{x} = \begin{bmatrix}1\\1\end{bmatrix} = a\begin{bmatrix}1\\3\end{bmatrix} + b\begin{bmatrix}2\\7\end{bmatrix} \Rightarrow \begin{matrix}a + 2b = 1\\3a + 7b = 1\end{matrix}$$

$$\begin{bmatrix}1 & 2\\3 & 7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 7\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}7 & -2\\-3 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

b) $[x]_v \to [x]_u$

$$\boxed{V \cdot [x]_v = U \cdot [x]_u}$$

$[x]_u = U^{-1} V [x]_v$ $\qquad U^{-1}V$ = t.m. from $V \to U$.

$[x]_v = V^{-1} U [x]_u$ $\qquad V^{-1}U$ = t.m. $U \to V$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 4 \\ 2 & 9 \end{bmatrix}$$

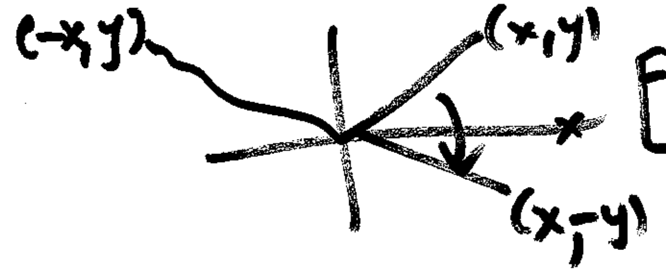
$[x]_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\quad (\vec{x} = 2v_1 + 3v_2)$

$[x]_u = ?$ $\quad [x]_u = U^{-1} V [x]_v = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 4 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

p. 209 #6.

① CC rotation by $\theta = 30^\circ = \frac{\pi}{6}$ $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ \frac{1}{2} & \sqrt{3}/2 \end{bmatrix}$

② reflect ~~by~~ about y - axis.



$\begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix} \begin{bmatrix} +x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ +y \end{bmatrix}$

$\vec{x}[A] = \cdots$

$$\begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

169 #13

$[A]_{5\times 3}$ $\quad r = 3$ $\qquad A = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$

$\{x_1, x_2, x_3\}$ = basis for $\mathbb{R}^3$ $\qquad$ rref $\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$y_1 = Ax_1$
$y_2 = Ax_2$
$y_3 = Ax_3$

$[y_1 | y_2 | y_3] = A[x_1 | x_2 | x_3]$ $\Rightarrow$ # lead = 3

$Y = AX$ $\qquad \Rightarrow$ # free = 0

$\Rightarrow N(A) = \{\vec{0}\} = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

$[A \mid y_1] \to \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \middle| x_1 \right]$

y_1, y_2, y_3 is lin indep $\Rightarrow$ $c_1y_1 + c_2y_2 + c_3y_3 = 0$ $\Rightarrow c_1, c_2, c_3 = 0!$

know x_1, x_2, x_3 are lin indep. $\Rightarrow A(c_1x_1 + c_2x_2 + c_3x_3) = 0 \Rightarrow c_1, c_2, c_3 = 0$

$c_1Ax_1 + c_2Ax_2 + c_3Ax_3 = A(0) = 0$

$c_1y_1 + c_2y_2 + c_3y_3 = 0$

173 #7 $\quad [A]_{6\times 4} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{bmatrix}$ $\quad$ rank = 4

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

\# lead = 4
\# free = 0

$\boxed{\dim \text{null}(A) = 0}$ (c)

$\boxed{\dim \text{col } A = r = 4}$ (a)

b) col vectors (4)
do nott span $\mathbb{R}^6$!
col vectors are l.i.

unique sol $\Leftrightarrow$ null $= \{0\}$

c) $Ax = b$ $\quad$ if $b \in$ col(A) $\Rightarrow$ $Ax = b$ has only one sol^n.

if null space $\neq \{0\}$

$x_1 - x_2 = a\vec{n}_1 + b\vec{n}_2 \cdots$

$x_1 = x_2 + \underbrace{a\vec{n}_1 + b\vec{n}_2 + \cdots}$
$=$

$Ax_1 = b$
$Ax_2 = b$

$A(x_1 - x_2) = b - b = 0$
$x_1 - x_2 \in \text{Null}(A)$
$x_1 - x_2 = 0$
$x_1 = x_2$

$$L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$L = A$$

$$L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \cdot x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} x_2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

not constant.

$x_1 \cdot x_2$ not linear.

$$L\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} (x_1+y_1)(x_2+y_2) \\ x_1 + y_1 \end{bmatrix} \neq \begin{bmatrix} x_1 x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} y_1 y_2 \\ y_1 \end{bmatrix}$$

#184.19

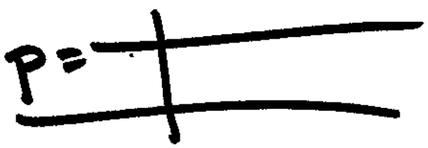
$L(p(x)) = x p'(x) = x \frac{d}{dx}(p(x))$ $\quad p \in \mathbb{P}^3 \quad L: \mathbb{P}^3 \to \mathbb{P}^3$

$L(p(x)+q(x)) = x \cdot (p'(x) + q'(x)) = xp' + xq' = L(p) + L(q)$

a) $L[p] = 0 \Rightarrow x \cdot p'(x) = 0 \Rightarrow p'(x) = 0 \;\; \forall x \neq 0$

p : constant

$\ker(L) = \{ p(x) \mid p(x) = C \}$ $\quad p =$



basis $= \{1\}$ $\quad$ dim(null) = 1

If $y \in$ range $L \Rightarrow L[p(x)] = y(x)$

$x p'(x) = y(x)$

if $y(x) \in$ range (L)

$\Rightarrow y(0) = 0$

$p(x) = \mathbb{P}^3$

$= p_0 + p_1 x + p_2 x^2$

$p' = p_1 + 2p_2 x$

$\boxed{xp' = p_1 \cdot x + 2p_2 x^2}$

$\Rightarrow$ 2 dim

dim(rang) = 2

\# 184. 19c)

$L[p(x)] = p(0)\cdot x + p(1) \in P^2$

$L[p]=0 \quad x\cdot p(0) + p(1) = 0$ for all x

$p(0) = p(1) = 0$

$p = \cancel{a} + bx + cx^2$

$p(0)=0 \Rightarrow a=0$

$p(1)=0 \quad b+c=0$

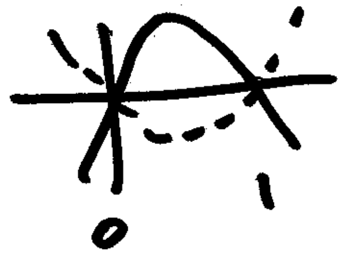
$c=-b$

$p = bx - bx^2$

$= b(x)(1-x)$

$p(x) \in P^3$

$p(0)=p(1)=0$



$\underbrace{Ax(1-x)} = p$

$\ker(L) = \text{span}\{x(1-x)\}$

dim ker = 1

dim $P^3 = 3$

range $(L) = y(x) \qquad p(0)x + p(1) = y(x) \Rightarrow$ linear f:

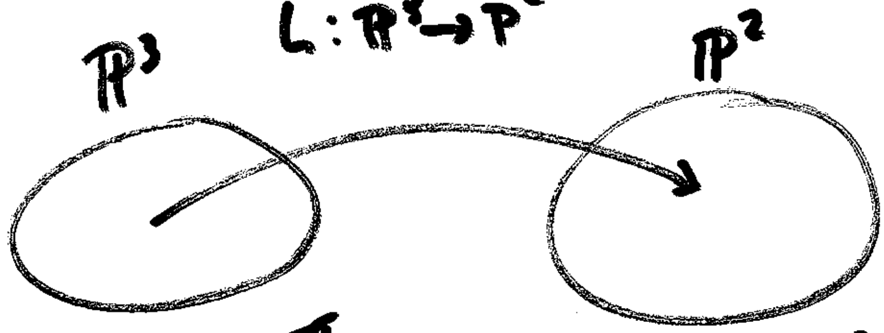
range $= P_2$

dim = 2
range

198 # 14

$L[p(x)] = p'(x) + p(0) \leftarrow$

$L : \mathbb{P}^3 \to \mathbb{P}^2$



$\{x^2, x, 1\}\ U$ $\qquad$ $V = \{2, 1-x\}$

$[A]:$ $L[x^2] = 2x + 0 = 2x$ $\qquad$ $2x = a\cdot 2 + b(1-x) \Rightarrow$ $\quad -b = 2 \quad b = -2$

$= 1\cdot 2 - 2(1-x) \to \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$2a + b - bx$ $\qquad$ $2a + b = 0$

$2a - 2 = 0$

$= 2 + -2(1-x)$ $\qquad$ $a = 1$

$\Rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$L[x] = 1$

$L[1] = 1$

$1 = a\cdot 2 + b(1-x)$ $\quad$ $b = 0$

$= 2a + b - bx$ $\quad$ $a = 1/2 \to \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$[A] = \begin{bmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{bmatrix}$

b) $[x^2+1]_E \to \begin{bmatrix}1\\0\\1\end{bmatrix}$

$$\begin{bmatrix}1 & 1/2 & 1/2\\ -2 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}3/2\\-2\end{bmatrix} = [y]_V$$

$$\Rightarrow y = \tfrac{3}{2}\cdot 2 - 2(1-x)$$

$$= 3 - 2 + 2x$$

$$= 1 + 2x$$

$$L[x^2+1] = \frac{d}{dx}(x^2+1) + (x^2+1)\Big|_{x=0}$$

$$2x+1$$