4.1 #19 kernel, range (P_3)

a) $L[p(x)] = xp'(x)$

kernel $L[p(x)]=0 \Rightarrow xp'(x)=0$ for all x $\Rightarrow p'(x)=0$ for all x.

$\Rightarrow p(x) = c$

kernel $= \{c\} = \text{span}\{1\}$ kernel = dim 1

$p(x) = a+bx+cx^2 \in P_3$

$\Rightarrow p' = b+2cx$

$xp'(x) = x(b+2cx)$

$= bx + 2cx^2 = \text{span}\{x, x^2\} \Rightarrow$ range = dim 2

b) $L[p(x)] = p(x) - p'(x)$ $[P_3 \to P_3]$

$L[p]=0 \Rightarrow p(x)-p'(x)=0$ for all x

$p'(x) = p(x) \Rightarrow p(x) = e^x$

No poly'l sol'n except for 0.

kernel $= \{0\}$ dim $= 0$

$p = a+bx+cx^2$

$p' = b+2cx$

$a+bx+cx^2 = b+2cx$

$\left.\begin{array}{l} a=b \\ b=2c \\ c=0 \end{array}\right\} \Rightarrow a=0, b=0, c=0$

$L[a+bx+cx^2] = a+bx+cx^2-(b+2cx)$

$= (a-b)+(b-2c)x+cx^2$

$= \text{span}\{1, x, x^2\} = P_3$ (dim $= 3$)

6.2 sample question (setup)

$y'' + 3y' - 2y = 0$

write as a 2x2 system

$y_1 = y$ $\Rightarrow \ldots$
$y_2 = y'$

6.3 Diagonalization = Similarity using basis of e-vectors

198 #14d) $L: \mathbb{P}_3 \to \mathbb{P}_2$ $\quad L[p(x)] = p'(x) + p(0)$

$\{x^2, x, 1\}$ $\quad \{2, 1-x\}$

$L[x^2] = (x^2)' + (x^2)\big|_{x=0} = 2x + 0 = 2x = -2(1-x) + 1 \cdot 2$

$\to \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$2x = a \cdot 2 + b(1-x)$
$= 2a + b - bx$
$b = -2 \quad 2a + b = 0$
$2a - 2 = 0$
$a = 1$

$L[x] = (x)' + x\big|_{x=0}$
$= 1 + 0 = 1$
$= \frac{1}{2} \cdot 2 + 0 \cdot (1-x)$
$\to \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$L[1] = 1' + 1\big|_{x=0} = +1$ $\quad = +\frac{1}{2} \cdot 2 + 0$
$= \begin{bmatrix} +1/2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1/2 & +1/2 \\ -2 & 0 & 0 \end{bmatrix}$

d) $4x^2 + 2x = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$

$L[4x^2 + 2x]$
$= (4x^2 + 2x)' + (4x^2 + 2x)\big|_{x=0}$
$= 8x + 2$

$= 5 \cdot 2 + (-8)(1-x)$
$= 10 - 8 + 8x$
$= 2 + 8x$

NOTE: $\langle u(x), v(x) \rangle = \int_0^1 u(x)v(x)\,dx \leftarrow$

$p = 1+x$

$q = 2x+3$

$$\Rightarrow \langle p,q \rangle = \int_0^1 (1+x)(2x+3)\,dx$$

$$= \int_0^1 2x + 2x^2 + 3 + 3x$$

$$= \int_0^1 3 + 5x + 2x^2$$

$$= 3 + 5/2 + \frac{2}{3}$$

$p \to [1, 1]$

$q \to [3, 2]$

$$[1,1]\begin{bmatrix}3\\2\end{bmatrix} = 3+2=5$$

$\|u\| = \langle u, u \rangle^{1/2}$

208 #2 $\quad v_1 = \begin{bmatrix}1\\1\end{bmatrix}, \quad v_2 = \begin{bmatrix}-1\\2\end{bmatrix}, \quad v_3 = \begin{bmatrix}1\\7\end{bmatrix}$

$L[v_1] = \begin{bmatrix}2\\5\end{bmatrix}, \quad L[v_2] = \begin{bmatrix}-3\\1\end{bmatrix}$

$L[v_3]$?

$$\begin{bmatrix}1\\7\end{bmatrix} = v_3 = a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}a-b\\a+2b\end{bmatrix}$$

$a - b = 1$

$a + 2b = 7$

$-a + b = -1$

$a + 2b = 7$

$3b = 6$

$b = 2$

$a = 1 + b$

$= 3$ $\quad a = 3$

$$\begin{bmatrix}1\\7\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}-1\\2\end{bmatrix}$$

$$L\begin{bmatrix}1\\7\end{bmatrix} = 3L\begin{bmatrix}1\\1\end{bmatrix} + 2L\begin{bmatrix}-1\\2\end{bmatrix}$$

$$= 3\begin{bmatrix}2\\5\end{bmatrix} + 2\begin{bmatrix}-3\\1\end{bmatrix} = \ldots$$

296 #5

$$Q^TQ = I$$

Show angles are preserved

$$z = Qx$$
$$w = Qy$$

i angle between x, y ?

$$\cos\theta_1 = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}$$

ii angle between z, w

$$\cos\theta_2 = \frac{\langle z, w\rangle}{\|z\|\,\|w\|} = \frac{\langle Qx, Qy\rangle}{\|Qx\|\,\|Qy\|} = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}$$

$$\|Qx\|^2 = (Qx)^T(Qx)$$
$$= x^T\underbrace{Q^TQ}x = x^Tx = \|x\|$$

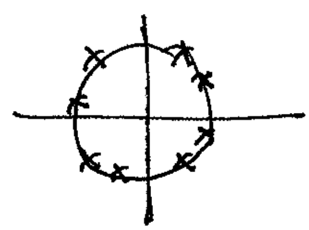
$$\langle Qx, Qy\rangle = (Qx)^T Qy = x^TQ^TQy$$
$$= x^Ty = \langle x, y\rangle$$

$$\cos\theta_1 = \cos\theta_2$$
$$\theta_1 = \theta_2 \;!$$

Q orthog.

$\Rightarrow$ Q preserves angles + lengths.

$\Rightarrow$ pure rotation $\quad |\lambda(Q)| = 1$

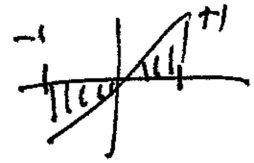


297 #12

$$\langle f, g\rangle = \int_{-1}^{+1} f(x)\,g(x)\,dx$$

$\left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}x\right\}$ are orthonormal

$$0 = \left\langle \frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}x \right\rangle = \int_{-1}^{+1}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{6}}{2}x\right)dx$$



$$= \frac{\sqrt{3}}{2}\int_{-1}^{+1} x\,dx = 0\,!$$

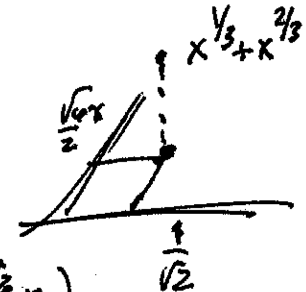
$0 = \langle 1, x\rangle$

$$1 \overset{?}{=} \left\|\frac{1}{\sqrt{2}}\right\| = \int_{-1}^{+1}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)dx = \int_{-1}^{1}\frac{1}{2}\,dx = \frac{1}{2}\int_{-1}^{+1}dx = \frac{1}{2}\cdot 2 = 1 \checkmark$$

$$1 = \left\|\frac{\sqrt{6}}{2}x\right\| = \int_{-1}^{+1}\left(\left(\frac{\sqrt{6}}{2}\right)x\right)\left(\frac{\sqrt{6}}{2}x\right) = \frac{6}{4}\int_{-1}^{+1}x^2\,dx = \frac{6}{4}\left.\frac{x^3}{3}\right|_{-1}^{+1} = \frac{6}{4}\cdot\frac{2}{3} = 1$$

b) $x^{1/3} + x^{2/3} \cong \cancel{a + bx}$

$$= a\left(\frac{1}{\sqrt{2}}\right) + b\left(\frac{\sqrt{6}}{2}x\right)$$



$$\left\langle x^{1/3}+x^{2/3}, \frac{1}{\sqrt{2}}\right\rangle = a\underbrace{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle}_{1} + b\underbrace{\left\langle \frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}x\right\rangle}_{0}$$

$$= a \cdot 1 = a$$

$$\left\langle x^{1/3}+x^{2/3}, \frac{\sqrt{6}}{2}x\right\rangle = b$$

$$a = \int_{-1}^{+1}\left(x^{1/3}+x^{2/3}\right)\cdot\frac{1}{\sqrt{2}}\,dx =$$

$$b = \int_{-1}^{+1}\left(x^{1/3}+x^{2/3}\right)\left(\frac{\sqrt{6}}{2}x\right)dx$$

$$f(x) \cong \sum_{k=1}^{N} c_k u_k(x)$$

if $\{u_k\}$ ortho-normal

$$\Rightarrow c_k = \langle f(x), u_k(x)\rangle$$