Phase Plane Analysis

1 Nonlinear System

Consider the following set of autonomous nonlinear ordinary differential equations

$$\frac{dx}{dt} = f_1(x, y)$$
$$\frac{dy}{dt} = f_2(x, y)$$

If there is a critical point (x_0, y_0), then $f_1(x_0, y_0) = 0, f_2(x_0, y_0) = 0$ and we can expand in a Taylor series about the critical point:

$$\begin{aligned}
\frac{dx}{dt} &= f_1(x_0, y_0) + \frac{\partial f_1}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f_1}{\partial y}(x_0, y_0)(y - y_0) + ... \\
&= \frac{\partial f_1}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f_1}{\partial y}(x_0, y_0)(y - y_0) + ... \\
&= a_{11}(x - x0) + a_{12}(y - y0) + ... \\
\frac{dy}{dt} &= f_2(x_0, y_0) + \frac{\partial f_2}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f_2}{\partial y}(x_0, y_0)(y - y_0) + ... \\
&= \frac{\partial f_2}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f_2}{\partial y}(x_0, y_0)(y - y_0) + ... \\
&= a_{21}(x - x0) + a_{22}(y - y0) + ...
\end{aligned}$$

In terms of new variables $\tilde{x} = x - x_0, \tilde{y} = y - y_0$, and neglecting higher order terms, we have

$$\frac{d\tilde{x}}{dt} = a_{11}\tilde{x} + a_{12}\tilde{y}$$
$$\frac{d\tilde{y}}{dt} = a_{21}\tilde{x} + a_{22}\tilde{y}$$

that is

$$\frac{dx}{dt} = Ax \tag{1}$$

where A is a 2×2 constant matrix (called the Jacobian). Consequently, the behavior in the neighborhood of a critical point will be that of the system (1).

2 Behavior of Linear System

The behavior of solutions of (1) in the neighborhood of the origin (0,0) is completely determined by the eigenvalues of A. This follows from the fact that solutions of (1) are of the form $\overrightarrow{x}(t) = \overrightarrow{v}e^{\lambda t}$.

Substituting this expression into (1) leads to the requirement that

$$\lambda\overrightarrow{v}e^{\lambda t} = A\overrightarrow{v}e^{\lambda t} \tag{2}$$

which can only be true if

$$A\overrightarrow{v} = \lambda\overrightarrow{v} \tag{3}$$

that is, λ is an eigenvalue of A, and $\overrightarrow{v}$ is the associated eigenvector.

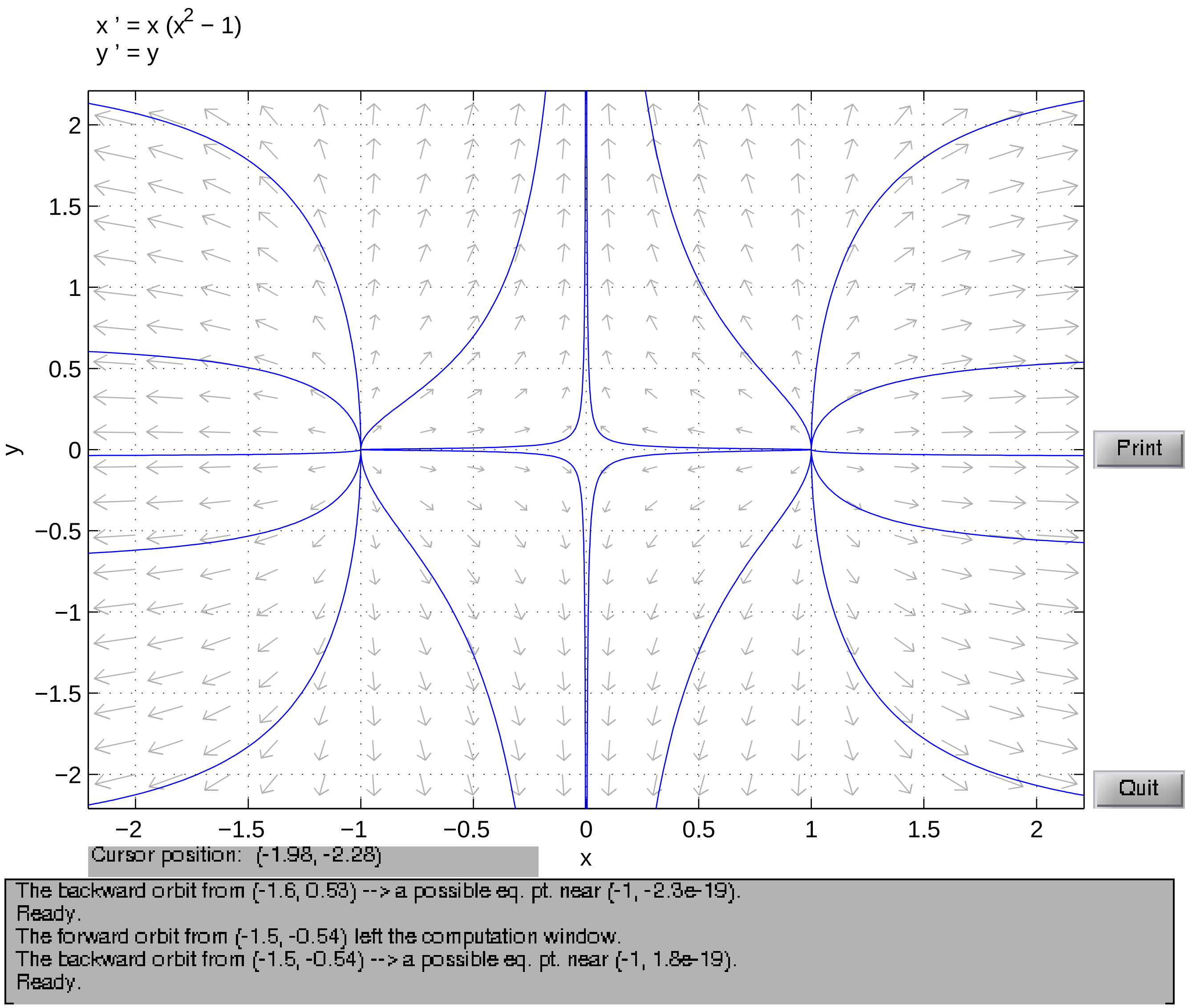
The eigenvalues λ_1, λ_2 of a 2×2 real matrix are one of the following cases:

Eigenvalues	Description	Behavior	Stability
$\lambda_1 > \lambda_2 > 0$	real, distinct, both positive	node, outgoing	unstable
$\lambda_1 > 0 > \lambda_2$	real, distinct, opposite sign	saddlepoint	unstable
$0 > \lambda_1 > \lambda_2$	real, distinct, both negative	node, ingoing	stable
$\lambda_1 = \lambda_2 > 0$	real, repeated, positive	node (degenerate)	unstable
$\lambda_1 = \lambda_2 < 0$	real, repeated, negative	node (degenerate)	stable
$\lambda = a \pm ib, a > 0$	complex, positive real part	spiral, outgoing	unstable
$\lambda = a \pm ib, a < 0$	complex, negative real part	spiral, ingoing	stable
$\lambda = \pm ib, a = 0$	pure imaginary	ellipse	stable

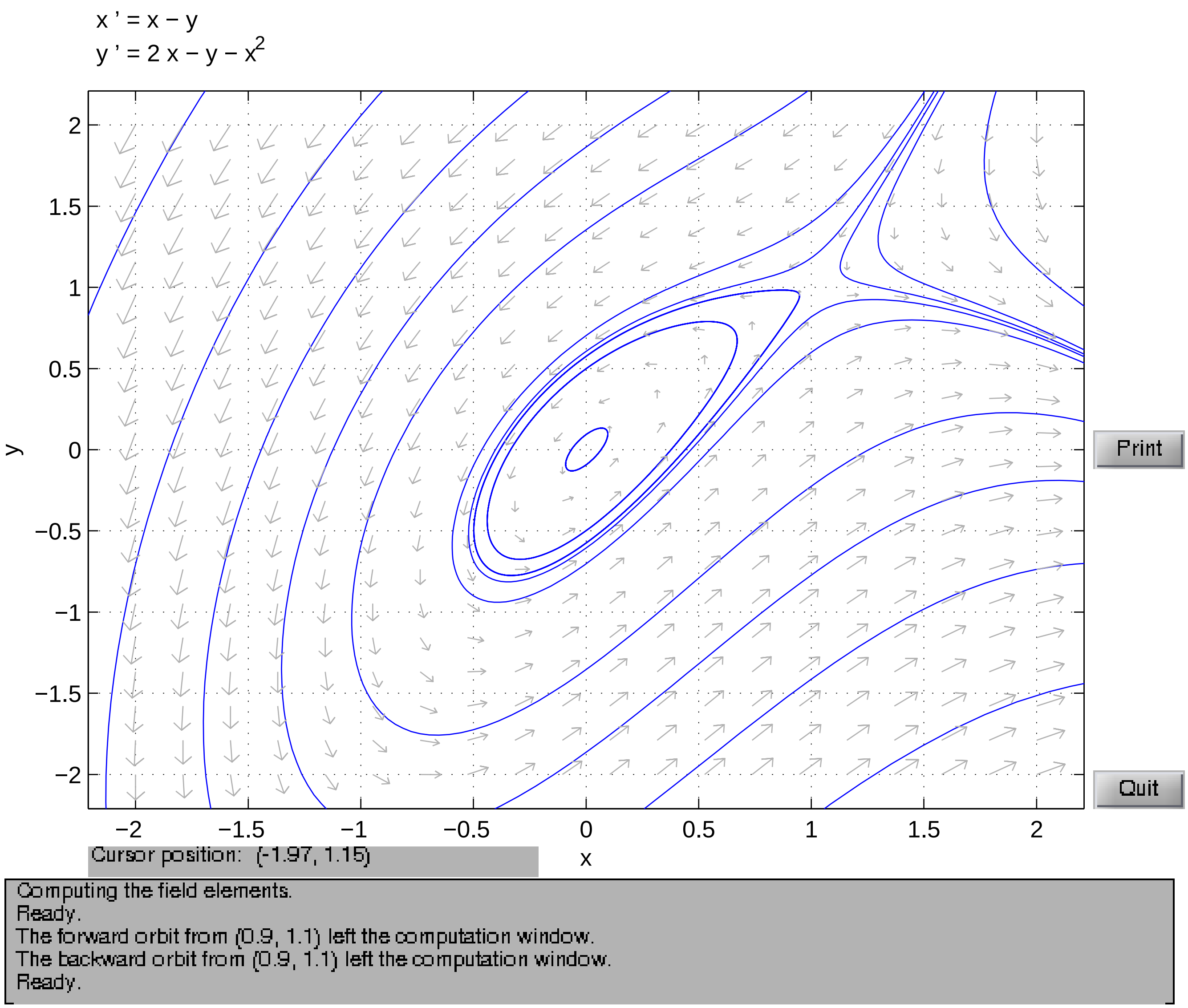
3 Global Behavior

Section 2 governs the local behavior in the vicinity of a critical point. Away from the critical points, the behavior can become much more complex.

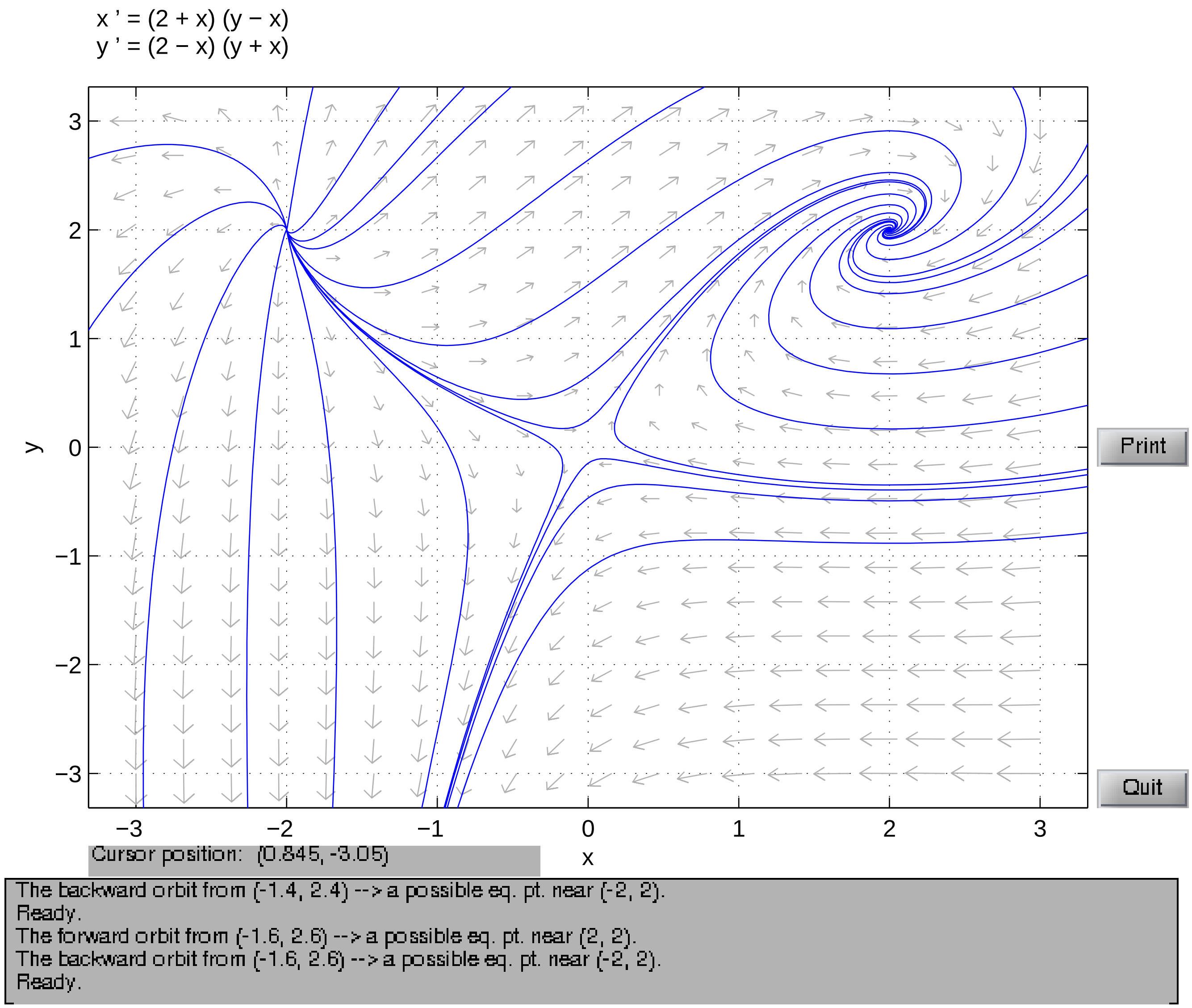
For example, if one has two outgoing nodes, there needs to be a saddle point connecting them - as shown below



A periodic orbit and a saddle point can co-exist nicely



Here is a combination of outgoing node, saddle point, ingoing spiral:



All of the figures above were generated using the pplane8.m Matlab interface written by J. Polking.