Chaos and all that ...

1 Chaos

1.1 Definitions

Chaos is a qualitative term, generally used to describe systems which exhibit aperiodic, random behavior and extreme sensitivity to initial conditions. The important features are:

1. Determinism. The process is describable by a **deterministic** set of equations, that is there are no explicit random components.

2. Aperiodicity. Trajectories in phase space do not converge to fixed points or periodic orbits.

3. Extreme sensitivity. Trajectories that are nearby at time t_0 do not remain close for subsequent time.

Determinism is the belief that every action is the result of preceding actions, and that consequently, if one knows the initial conditions infinitely accurately, one could predict the behavior of systems infinitely far into the future. An example would be predicting the behavior of the solar system.

If one could show that given an initial error (ϵ), subsequent motions remained uniformly bounded (in terms of ϵ), then the system was uniformly stable. In the 1800's a prize was offered to anyone who could show that the Solar System was stable. No one could, and in fact the system is unstable. (Systems of more than two bodies are almost always not stable).

Given the law of gravity and initial velocities and positions of three bodies in space, their subsequent motion is **uniquely determined**. However, Poincare discovered that small differences in initial conditions were magnified, that is the system was not stable. He was the first person to observe chaotic behavior in a deterministic system.

1.2 Difference between Chaos and Randomness

Chaotic systems are not truly random, nor disorerly. If one applies a test for randomness, [see *http://stat.fsu.edu/pub/diehard/cdrom/linux/tests.txt* for example] it will typically fail for a chaotic system. Furthermore, the nonlinearities that give rise to chaos may give rise to **pattern formation** and **self-similarity** as is the case with fractals.

A good starting point is this web page, written by a student, and cited by numerous journals (like the New York Times) because of its clarity of exposition, is

http://www.duke.edu/~mjd/chaos/chaos.html

Another excellent starting point is

http://www.wfu.edu/~petrejh4/chaosind.htm

The other classical reference (a New York Times bestseller) is the book by James Glieck, “Chaos: Making a New Science.”

1.3 Lorenz Attractor

Although the three-body problem of classical mechanics predates the Lorenz attractor problem (which comes from a model problem in gas dynamics), the first equation which was extensively studied (numerically) and is in some sense the **father of experimental chaos**, is the following

$$\frac{dx}{dt} = a.(y(t) - x(t)) \quad (1)$$

$$\frac{dy}{dt} = c.x(t) - y(t) - x(t).z(t) \quad (2)$$

$$\frac{dx}{dt} = x(t).y(t) - b.z(t) \quad (3)$$

$$a = 10$$

$$b = 28$$

$$c = 8/3$$

2 Fractals

A fractal is by definition any geometrical object whose *Hausdorff* dimension strictly exceeds its topological dimension. The Hausdorff dimension is almost always a non-integer.

2.1 Non-integer Dimension

The **Hausdorff s-measure** s of a set A is defined as

$$\lim_{\epsilon \longrightarrow 0} \inf_{A_\epsilon} \sum_{A \in A_\epsilon} [\mathrm{diam}(A)]^s$$

In english, this means the following: Take a set of “balls” of diameter ϵ and take the minimum number necessary to cover A. Call this number $N(\epsilon)$. Obviously this number approaches infinity. We therefore weight it by multiplying by $(\epsilon)^s$. We look at the limit

$$\lim_{\epsilon \longrightarrow 0} N(\epsilon)\epsilon^s$$

If s is too large, then the measure will be zero. The Hausdorff dimension is the inf of the set of s’s where $H^s(A) = 0$. Roughly speaking, If $N(\epsilon) \approx 1/\epsilon^D$ then the dimension is D.

The similarity (or fractal) dimension is defined somewhat differently. If you take a line segment, and divide it into two equal pieces, there are $N = 2$ pieces, with length scale $r = 1/2$. If you take a unit square, and divide it into 4 equal squares, then there are $N = 4$ pieces with length scale $r = 1/2$. Similarly, a unit cube gives $N = 8$ and $r = 1/2$. We note that

$$Nr^D = 1, D = \log N / \log(1/r)$$

where D is the dimension of the figure.

This is generalized to

$$D = \lim_{\epsilon \to 0} \frac{\log N(A, \epsilon)}{\log 1/\epsilon},$$

a result very similar to the Hausdorff dimension.

By the way, the **Koch snowflake** has dimension equal to $D = log4/log3 \approx 1.26 > 1$ Fractals are also used to generate artificial terrains (such as coastlines and mountain ranges).

3 Iterated Function Systems

3.1 The Logistics Equation

In 1845, a biologist named Verhulst arrived at the equation

$$x_{n+1} = ax_n(1 - x_n) = F(x_n)$$

to describe the population of a species as a function of time, with constraints. x_n is the normalized population at time t_n.

In terms of the initial population, $x_1 = F(x_0)$, $x_2 = F(x_1) = F(F(x_0)) = F^2(x_0)$, ... $x_n = F^n(x_0)$. Clearly, if F is continuous, then this has a limit, β, only if

$$\beta = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n) = F(\lim_{n\to\infty} x_n) = F(\beta)$$

that is, β is a **fixed point** of F.

The existence of fixed points depends on the behavior of F and other paramters (such as a in the system). A nice java applet illustrating the behavior of this system can be found at

http://www.math.pitt.edu/~gartside/math0450/chaos/expm.html

If we graph the behavior of the system for $a \in (2, 4)$ we observe the phenomena of **bifurcation** and **period doubling**.

Let $\mu_1 = 3$ be the first period doubling, μ_2 the second etc. The values "pile up" around 3.57 where the cycles become dense. Period doubling occurs geometrically, and converges to a point at roughly a=3.5699456 The successive ratios converge to a well-defined limit

$$\lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}} = \delta = 4.669201609102990...$$

which is Feigenbaum's **universal constant**

Period doubling has been put forward as one possible explanation for the **onset of turbulence** in viscous fluids.

3.2 Mandelbrot and Julia Sets

If instead of $F(x) = ax(1-x)$ we student

$$z_{n+1} = z_n^2 + c = F_c(z_n)$$

for complex values of c. If, at some point, $|z_n| > 2$ then the sequence will diverge. If we plot the iteration number for which this occurs, and assign it a color map, then wonderful graphs emerge ...

The **Mandelbrot set** is the set of c for which the iteration remains bounded given $z_0 = 0$. The **Julia set** is the boundary of the Mandelbrot set.

4 Numerical Integration of Evolution Equations

Consider the first order non linear o.d.e

$$\frac{dx}{dt} = f(t, x)$$

Euler's method gives

$$x_{n+1} = x_n + hf(t_n, x_n) = x_n + hf(t_0 + nh, x_n) = F_h(x_n)$$

which is an **iterated map**! Consequently, if f is nonlinear, it can exhibit **bifurcation**, **chaos**, **instability** and even **pattern formation**. [Of course, one hopes that undesirable behavior is a product of a poor choice for the numerical scheme!]