Picard's Method - Project

1 Derivation of the Method

If we start with the general, first order **nonlinear** ordinary differential equation

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0$$

we can integrate both sides of the equation for $x_0 \leq x \leq x_1$ to get

$$y(x_1) - y(x_0) = \int_{x_0}^{x_1} dy = \int_{x_0}^{x_1} f(x, y(x))dx$$

Using t as the variable for integration, instead of x, and setting $x = x_1$ to be the arbitrary final point

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t))dt \tag{1}$$

Although a correct representation of the solution, this is not completely satisfatory. We proceed to try to solve this by iteration. Substituting an *initial guess* of $y(x) = \phi_0(x)$ into the right hand side of (1), we get

$$\phi_1(x) = y(x_0) + \int_{x_0}^{x} f(t, \phi_0(t))dt \tag{2}$$

In the unlikely event that $\phi_1(x) = \phi_0(x)$, we are finished, we have found a solution. Otherwise, we keep on iterating:

$$\phi_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f(t, \phi_n(t))dt$$

If no further information on the solution is available, we normally choose $\phi_0(x) = y_0$.

1.1 Linear Example

To start off with a simple example let $f(x,y) = y$, and $y(0) = 1$ so we are trying to solve:

$$\frac{dy}{dx} = y(x), \qquad y(0) = 1$$

In this case, we actually know the solution:

$$y(x) = e^x$$

Now let's see what Picard's method is doing, given the initial guess $\phi_0(x) = 1$.

$$\phi_1(x) = 1 + \int_0^x \phi_0(t)dt = 1 + \int_0^x 1dt = 1 + x$$

$$\phi_2(x) = 1 + \int_0^x \phi_1(t)dt = 1 + \int_0^x 1 + tdt = 1 + x + \frac{1}{2}x^2$$

$$\phi_3(x) = 1 + \int_0^x \phi_2(t)dt = 1 + \int_0^x 1 + t + \frac{1}{2}t^2dt = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$\phi_4(x) = 1 + \int_0^x \phi_3(t)dt = 1 + \int_0^x 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3dt = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

These are precisely the first four terms in the Maclaurin (Taylor) series expansion for e^x about $x = 0$.

1.2 Non linear Example with Unique Solution

For the next case, consider

$$\frac{dy}{dx} = x - y^2(x), \qquad y(0) = 0$$

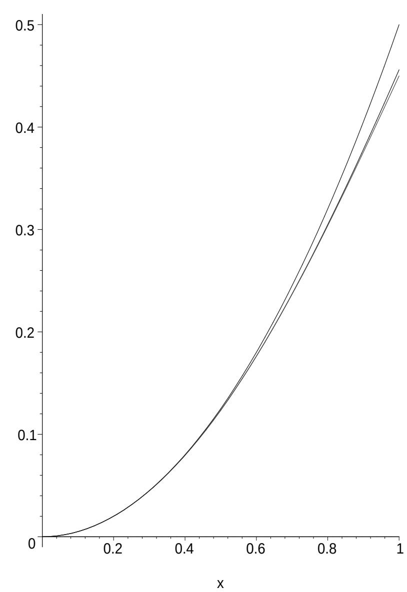
Although this can be solved (see attached Maple worksheet), it is in terms of very complicated functions! A simpler method for getting an approximate solution is Picard's method. With $\phi_0(x) = 0$

$$\phi_1(x) = \int_0^x (t - \phi_0^2(t))dt = \int_0^x t\,dt = \frac{1}{2}x^2$$

$$\phi_2(x) = \int_0^x (t - \phi_1^2(t))dt = \int_0^x (t - (\frac{1}{2}t^2)^2)dt = \frac{1}{2}x^2 - \frac{1}{2}x^5$$

$$\phi_3(x) = \int_0^x (t - \phi_2^2(t))dt = \int_0^x (t - (\frac{1}{2}t^2 - \frac{1}{20}t^5)^2)dt = \frac{1}{2}x^2 - \frac{1}{20}x^5 + \frac{1}{160}x^8 - \frac{1}{4400}x^{11}$$

The iterations rapidly converge, as seen below:



1.3 Non linear Example with Non Unique Solution

$$\frac{dy}{dx} = 3[y(x)]^{3/2}, \qquad y(2) = 0$$

If we start with $\phi_0(x) = 0$ then the iterations satisfy

$$\phi_1(x) = \int_2^x 3[\phi_0(t)]^{3/2} dt = \int_2^x 0 dt = 0$$

and

$$\phi_2(x) = \int_2^x 3[\phi_1(t)]^{2/3} dt = \int_2^x 0 dt = 0$$

Each iterate is identically 0. This means that $y(x) = 0$ is a solution of the initial value problem. Is it the only one, however?

If we start with an alternative initial guess, $\phi_0(x) = x - 2$, then we have

$$\phi_1(x) = \int_2^x 3[\phi_0(t)]^{2/3} dt = \int_2^x 3[t-2]^{2/3} dt = \frac{9}{5}[x-2]^{5/3}$$

$$\phi_2(x) = \int_2^x 3[\phi_1(t)]^{2/3} dt = \int_2^x 3[\frac{9}{5}(t-2)^{5/3}]^{2/3} dt = \int_2^x 3[\frac{9}{5}]^{2/3}[(t-2)^{10/9}] dt$$

Although this looks complicated (and it is!) we can gain some insight by setting

$$\phi_n(x) = c_n[x-2]^{r_n}$$

which means that

$$\phi_{n+1}(x) = \int_2^x 3[c_n[t-2]^{r_n}]^{2/3} dt$$

$$= 3c_n^{2/3} \int_2^x [t-2]^{2r_n/3} dt$$

$$= 3\frac{c_n^{2/3}}{2r_n/3+1}[x-2]^{2r_n/3+1}$$

$$= c_{n+1}[x-2]^{r_{n+1}}$$

if $r_{n+1} = 1 + 2r_n/3$ and $c_{n+1} = 3\frac{c_n^{2/3}}{2r_n/3+1}$

If $r_n \to r$ exists, then r must satisfy $r = 1 + 2r/3$, which implies $r = 3$. If the limit of c_n exists, then c must satisfy $c = c^{2/3}$, which means that $c = 0, 1$

$c = 0$ gives the trivial solution, so we take $c = 1$. The limit of the iterations is therefore $(x-2)^3$.