Since chaotic attractors consist of infinite numbers of points, we need a way to measure the size of the attractor.

It is simplest to talk about the size of an open set, for example a ball, or a cube. We can then proceed to a finite union of disjoint open sets, and then to a countable union of disjoint open sets.

A "measure" is simply a function that is positive

$$\mu(S) \geq 0 \text{ (for any set)}$$

and coutably additive

$$\mu\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} \mu(S_n),\ S_n \text{ disjoint}$$

If the measure of the entire space Ω is finite, we can normalize the measure by

$$\upsilon(s) = \frac{\mu(S)}{\mu(\Omega)} \geq 0$$

which implies that $\upsilon(\Omega) = 1$, that is, we can define a probability measure.

Invariance can be thought of as $\mu \circ f^{-1} = \mu$, or $\mu = \mu \circ f$, which means that the action of f does not affect the measure (i.e. it leaves it invariant).

One natural way of measure the size of a closed set is to take the limit of measures of open sets that contain it (such as countable unions of balls).

When we consider an orbit (or trajectory) of a dynamical system, we can consider the probability that the orbit lies in a set S. The simplest is to count the percentage of orbital points that lie in S as a function of n

$$\frac{\#\{x_n \in S\}}{n} = \frac{\#\{f^n(x_0) \in S\}}{n}$$

and then take the limit as n becomes infinite

$$F(x_0, S) = \lim_{n\to\infty} \frac{\#\{f^n(x_0) \in S\}}{n}$$

You can think of this as the probability that the orbit (trajectory) lies in S.

There is a problem with boundaries and closed sets. If S is the attracting basin, then $F(x_0, S) = 1$ if x_0 lies in S. However, if x_0 is on the **boundary** of S, the orbit may not be in S! We get around this problem by "fattening" up the set S by defining

$$N(r,S) = \{x \mid dist(x,S) \le r\}$$

That is, we consider all points that are within a distance "r" of S. We can define the proportion (probability) of points in the trajectory which lie in $N(r,S)$.

$$F(x_0, N(r,S))$$

This will have different values, depending on r. It is, however, monotone in r and bounded below (since it is positive) so we look at the limit as r approaches 0 (which is guaranteed to exist)

$$\mu_f(x_0, S) = \lim_{r \to 0} F(x_0, N(r,S))$$

If every choice of x_0 gives rise to the same measure, then we can unambiguously define the quantity $\mu_f(S)$ since it will be independent of x_0. This is the so called "natural measure generated by f." Remember f defines the dynamics of the underlying discrete dynamical system.

This is an example of what often occurs in mathematics. If we cannot define something directly, we definite it as a limit of things (each of which can be presumably computed easily).