

1 Curvature of 3-dimensional space curve:

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a parameterized curve in \mathcal{R}^3 then $r'(t)$ is a vector **tangent** to the curve.

The **unit tangent** is defined by

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

and the **length** of the curve is given by

$$\int_a^b |r'(t)| dt$$

The **arc length** function is given by

$$s(t) = \int_a^t |r'(u)| du$$

so that

$$s'(t) = |r'(t)|$$

The **curvature** is given by

$$\kappa = \left| \frac{dT}{ds} \right|$$

and parametrically, by the chain rule

$$\kappa(t) = \frac{|T'(t)|}{|s'(t)|}$$

We also have the **unit normal**

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

and **binormal**

$$B(t) = T(t) \times N(t)$$

2 Length of curves on a surface:

If we parameterize a surface $x : U \rightarrow \mathcal{R}^3$ by

$$\vec{x} = \langle x(u, v), y(u, v), z(u, v) \rangle$$

The length of a curve is determined by

$$ds^2 = Edu^2 + Fdudv + Gdv^2$$

where

$$E = \left| \frac{\partial x}{\partial u} \right|^2$$

$$F = \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v}$$

$$G = \left| \frac{\partial x}{\partial v} \right|^2$$

(called the **first fundamental form**).

The Gauss curvature can be written in terms of $\{E, F, G\}$ and their first and second partial derivatives

$$\begin{aligned} K = & \frac{1}{2g} \left[2 \frac{\partial^2 F}{\partial u \partial v} - \frac{\partial^2 E}{\partial v^2} - \frac{\partial^2 G}{\partial u^2} \right] \\ & - \frac{G}{4g^2} \left[\frac{\partial E}{\partial u} \left(2 \frac{\partial F}{\partial v} - \frac{\partial G}{\partial u} \right) - \left(\frac{\partial E}{\partial v} \right)^2 \right] \\ & + \frac{F}{4g^2} \left[\frac{\partial E}{\partial u} \frac{\partial G}{\partial v} - 2 \frac{\partial E}{\partial v} \frac{\partial G}{\partial u} + \left(2 \frac{\partial F}{\partial u} - \frac{\partial E}{\partial v} \right) \left(2 \frac{\partial F}{\partial v} - \frac{\partial G}{\partial u} \right) \right] \\ & - \frac{E}{4g^2} \left[\frac{\partial G}{\partial v} \left(2 \frac{\partial F}{\partial u} - \frac{\partial E}{\partial v} \right) - \left(\frac{\partial G}{\partial u} \right)^2 \right] \end{aligned}$$

and the determinant g is given by

$$g = EG - F^2$$