Fall 2004 Math 151
Overview: Derivative and Integral
Thu, 15/Jul ©2004, Art Belmonte

Summary

We'll look at various geometrical and analytical connotations of **derivative** and **integral**. These two concepts are seminal in the study of calculus. Indeed, we will spend an entire semester developing them! Our limited purpose in this summary is twofold:

- to provide a broad overview of these essential concepts;
- to give you mechanistic ways to compute derivatives and integrals (by hand and with MATLAB).

Once you have assimilated this material, you will put it to *immediate* use in your Physics class.

Derivative

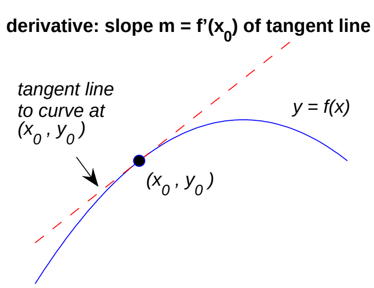
Here are five connotations of derivative from the textbook *Differential Equations* by Polking, Boggess, and Arnold.

1. *Rate of change* (modeling definition) The manner in which a dependent entity $\mathbf{x}(t)$ changes with respect to another t is its **rate of change** or **derivative**. For example, velocity is the rate of change of position $\mathbf{x}(t)$ with respect to time t. We write

$$\mathbf{v}(t) = \mathbf{x}'(t) \quad \text{or} \quad \mathbf{v}(t) = \frac{d\mathbf{x}}{dt}.$$

Or again, the rate of change of population P with respect to time t is the derivative dP/dt. The derivative has wide applicability.

2. *Slope of the tangent line* (geometric definition) The **slope of the tangent line** to the graph of a function $y = f(x)$ at a point (x_0, y_0) is its **derivative**.

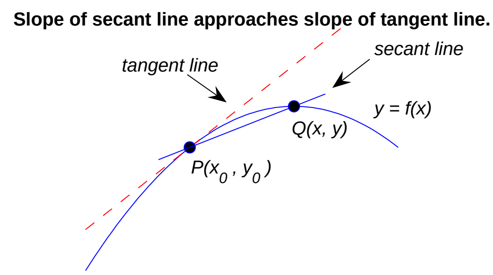


3. *Best linear approximation* (algebraic definition) The equation of the tangent line mentioned above is given by

$$\begin{aligned} L(x) &= f(x_0) + f'(x_0)(x - x_0) \\ L(x) &= f'(x_0)x + \left(f(x_0) - f'(x_0)x_0\right) \\ L(x) &= mx + b \end{aligned}$$

Thus the tangent line gives the best straight-line approximation (linear; first power in x) to the nonlinear curve $y = f(x)$. The **derivative** $f'(x_0)$ is perhaps the most important piece of information in the equation for $L(x)$.

4. *Limit of difference quotients* (limit quotient definition) During the first month of Math 151 calculus class, we'll see that the **derivative** $f'(x_0)$ mentioned above may be computed by taking the limiting value of slopes of secant lines through $P(x_0, y_0)$; i.e., slopes of lines through P and a nearby point $Q(x, y)$ as Q approaches P along the curve $y = f(x)$.



$$f'(x_0) = m_{\tan} = \lim_{Q \to P} m_{\sec} = \lim_{x \to x_0} \frac{y - y_0}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

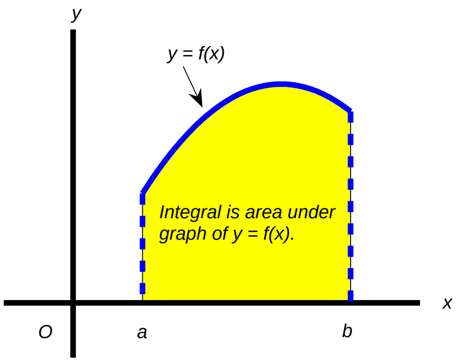
5. Table of formulas (formulaic definition) Of course, the derivative or slope of the tangent line at another point, say (x_1, y_1), is a separate computation. Indeed, this collection of slopes of tangent lines may be regarded as a function in itself, the **derivative function**. Now computing derviatives via the limit definition is almost as much fun as pulling your teeth out with a pair of pliers! Fortunately, rules for general classes of functions may be established, tabulated, and memorized. This makes hand computation of derivatives mechanistic. Herewith a short table of function-derivative pairs (a_k, c: constants; n: positive integer).

Rule	$f(x)$	$f'(x)$
1	c	0
2	x	1
3	x^n	nx^{n-1}
4	$\sin x$	$\cos x$
5	$\cos x$	$-\sin x$
6	e^x	e^x
7	$\ln x$	$1/x$
8	$cf(x)$	$cf'(x)$
9	$f(x) + g(x)$	$f'(x) + g'(x)$
10	$\sum_{k=0}^{n} a_k x^k$	$\sum_{k=0}^{n} k a_k x^{k-1}$

Integral

Here are three connotations of integral from the textbook *Differential Equations* by Polking, Boggess, and Arnold.

1. The area under the graph (definite integral) For a positive function $f(x)$ that is continuous (unbroken) on the interval $a \le x \le b$, the **definite integral** $\int_a^b f(x)\,dx$ represents the area between the curve $y = f(x)$ and the x-axis between vertical lines $x = a$ and $x = b$.



2. The antiderivative (indefinite integral) The function $F(x)$ is an **antiderivative** (or **indefinite integral**) of the continuous function $f(x)$ provided that (C: constant)

$$F'(x) = f(x) \quad \text{if and only if} \quad \int f(x)\,dx = F(x) + C.$$

Whereas f is obtained from F via differentiation, F is obtained from f via the inverse operation: antidifferentiation or indefinite integration. In the **Fundamental Theorem of Calculus**, one sees the connection between the definite and indefinite integrals.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

3. Table of formulas (formulaic definition) Toward the end of Math 151, we'll see that the definite integral is defined via yet another limiting operation.

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k$$

Again, computing a definite integral in this manner is a world of hurt! Accordingly, we'll resort to finding an antiderivative $F(x)$ of $f(x)$, then applying the Fundamental Theorem of Calculus. Toward this end, here is a short table of function-antiderivative pairs (a_k: constants; n: positive integer).

Rule	$f(x)$	$\int f(x)\,dx$
1	0	0
2	1	x
3	x^n	$\frac{1}{n+1}x^{n+1}$
4	$\cos x$	$\sin x$
5	$\sin x$	$-\cos x$
6	e^x	e^x
7	$1/x$	$\ln x$
8	$cf(x)$	$c\int f(x)\,dx$
9	$f(x)+g(x)$	$\int f(x)\,dx + \int g(x)\,dx$
10	$\sum_{k=0}^{n} a_k x^k$	$\sum_{k=0}^{n} \frac{1}{k+1} a_k x^{k+1}$

Note that this more or less amounts to reading the earlier table from right-to-left instead of left-to-right. OBSERVATION: For a given row the most general antiderivative is obtained by adding a **constant of integration** C; e.g., $\int \cos x\,dx = \sin x + C$.

Hand Examples

The two tables introduced provide you with a mechanistic way of computing derviatives and integrals by hand. This will help you early on in your Physics class. Refer back to the tables if needed.

DERIVATIVE EXAMPLES

Example A

The derivative of $f(x) = 47$ is $f'(x) = 0$ [derivative rule 1].

Example B

The derivative of $w(t) = t^3$ is $w'(t) = 3t^2$ [derivative rule 3].

Example C

The derivative of $f(x) = 4\sin x + 7e^x$ is $f'(x) = 4\cos x + 7e^x$ [derivative rules 9, 8, 4, and 6], as follows.

$$\begin{aligned} f'(x) &= (4\sin x + 7e^x)' \\ &= (4\sin x)' + (7e^x)' \\ &= 4(\sin x)' + 7(e^x)' \\ &= 4\cos x + 7e^x \end{aligned}$$

Example D

If $x(t) = \frac{1}{4}t^4 - 3t^2 + 17t - 9$, then $v(t) = x'(t) = t^3 - 6t + 17$ [derivative rule 10].

Example E

"The derivative of the vector is the vector of derivatives." Observe:

$$\left[2t^2, 4\cos t - 8\right]' = \left[\left(2t^2\right)', (4\cos t - 8)'\right] = [4t, -4\sin t].$$

INTEGRAL EXAMPLES

Example F

An indefinite integral (or an antiderivative) of $f(x) = 22$ is $\int f(x)\,dx = 22x$. The most *general* antiderivative of $f(x)$ in this instance is $\int f(x)\,dx = 22x + C$, where C is an arbitrary constant of integration. [Here we employed integral rules 8 and 2.]

Example G

If $f(x) = 18\cos x + 10x^3$, then the most general antiderivative of $f(x)$ is $\int f(x)\,dx = 18\sin x + \frac{5}{2}x^4 + C$, as follows [integration rules 9, 8, 4, and 3].

$$\begin{aligned}
\int f(x)\,dx &= \int 18\cos x + 10x^3\,dx \\
&= \int 18\cos x\,dx + \int 10x^3\,dx \\
&= 18\int \cos x\,dx + 10\int x^3\,dx \\
&= (18\sin x + C_1) + \left(10\cdot\tfrac{1}{4}x^4 + C_2\right) \\
&= 18\sin x + \tfrac{5}{2}x^4 + C \qquad (\text{where } C = C_1 + C_2)
\end{aligned}$$

Example H

The area under the curve $f(t) = t^2 + 4t + 6$ between $t = 0$ and $t = 3$ is 45 as follows. First, an antiderivative of $f(t)$ is $F(t) = \int t^2 + 4t + 6\,dt = \frac{1}{3}t^3 + 2t^2 + 6t$ [integration rule 10]. Then apply the Fundamental Theorem of Calculus.

$$\begin{aligned}
\int_0^3 f(t)\,dt &= F(3) - F(0) \\
&= (9 + 18 + 18) - (0 + 0 + 0) \\
&= 45
\end{aligned}$$

Example I

"The integral of the vector is the vector of integrals." Observe:

$$\begin{aligned}
\int \left[3t, 8 + 5e^t\right]\,dt &= \left[\int 3t\,dt, \int 8 + 5e^t\,dt\right] \\
&= \left[\tfrac{3}{2}t^2 + K_1, 8t + 5e^t + K_2\right] \\
&= \left[\tfrac{3}{2}t^2, 8t + 5e^t\right] + \mathbf{K}, \quad \mathbf{K} = [K_1, K_2]
\end{aligned}$$

Example J

The preceding also holds for definite integrals. Verify the following by applying the Fundamental Theorem of Calculus to each element of the middle vector expression.

$$\int_0^1 \left[z^2, z^3\right]\,dz = \left[\int_0^1 z^2\,dz, \int_0^1 z^3\,dz\right] = \left[\frac{1}{3}, \frac{1}{4}\right]$$

MATLAB Examples

To compute derivatives, use **diff**. To compute integrals, use **int**. Symbolic variables are declared with **syms**. We revisit our ten hand examples. Here is a transcript.

```
%
delete s00d.txt; diary s00d.txt
clear; clc; close all; echo on
%-----------------------------------------------------
% Percent sign begins a comment.
% Semicolon suppresses output.
% Default name of answer: ans.
% pretty: makes symbolic output easier to read.
%
% DERIVATIVE EXAMPLES
syms t x
% A
diff(47, x)

ans =

0


% B
diff(tˆ3, t); pretty(ans)

                                     2
                                  3 t
% C
diff(4*sin(x) + 7*exp(x), x); pretty(ans)

                           4 cos(x) + 7 exp(x)
% D
diff(tˆ4/4 - 3*tˆ2 + 17*t - 9, t); pretty(ans)

                                 3
                                t  - 6 t + 17
% E
diff([2*tˆ2, 4*cos(t)-8], t); pretty(ans)

                           [4 t    -4 sin(t)]
%
% INTEGRAL EXAMPLES
% (NOTE: MATLAB doesn't add a constant of integration
% to an antiderivative. YOU need to remember it your-
self!)
% F
int(22, x); pretty(ans)

                                  22 x
% G
int(18*cos(x) + 10*xˆ3, x); pretty(ans)

                                                 4
                           18 sin(x) + 5/2 x
% H
int(tˆ2 + 4*t + 6, t, 0, 3)

ans =

45
```

```
% I
int([3*t, 8+5*exp(t)], t); pretty(ans)

                           [     2                  ]
                           [3/2 t     8 t + 5 exp(t)]
% J
syms z
int([z^2, z^3], z, 0, 1)

ans =

[ 1/3, 1/4]


%
echo off; diary off
```