1. Calculate the following limits or state why the limit does not exist. Do not use the L'Hospital's Rule.

(a) $\lim_{x\to 4}\sqrt{x+\sqrt{x}} = \sqrt{4+\sqrt{4}} = \sqrt{4+2} = \boxed{\sqrt{6}}$

(b) $\lim_{x\to 5}\frac{5x-x^2}{x^2-4x-5} = \frac{0}{0} = \lim_{x\to 5}\frac{x(5-x)}{(x-5)(x+1)} = \lim_{x\to 5}\frac{-x\cancel{(x-5)}}{\cancel{(x-5)}(x+1)} = \lim_{x\to 5}\frac{-x}{x+1} = \boxed{-\frac{5}{6}}$

(c) $\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h} = \frac{0}{0} = \lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h} = \lim_{h\to 0}\frac{\frac{3-(3+h)}{3(3+h)}}{h} = \lim_{h\to 0}\frac{\cancel{3}-\cancel{3}-h}{h(3)(3+h)}$

$= \lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}(3)(3+h)} = \lim_{h\to 0}\frac{-1}{3(3+h)} = \boxed{-\frac{1}{9}}$

$(a-b)(a+b)=a^2-b^2$

(d) $\lim_{x\to 3}\frac{(x-\sqrt{4x-3})}{(x^2-9)}\frac{(x+\sqrt{4x-3})}{(x+\sqrt{4x-3})} = \lim_{x\to 3}\frac{x^2-(\sqrt{4x-3})^2}{(x^2-9)(x+\sqrt{4x-3})} = \lim_{x\to 3}\frac{x^2-4x+3}{(x^2-9)(x+\sqrt{4x-3})}$

$= \lim_{x\to 3}\frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x+3)(x+\sqrt{4x-3})} = \frac{3-1}{(3+3)(3+\sqrt{\underbrace{4(3)-3}_{9}})} = \frac{2}{6(6)} = \frac{2}{36} = \boxed{\frac{1}{18}}$

(e) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{|x|}\right)$ DNE

$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x<0 \end{cases}$

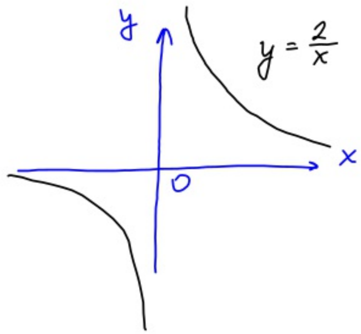
$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right) = \lim_{x\to 0^+}\left(\cancel{\frac{1}{x}}-\cancel{\frac{1}{x}}\right) = 0$

$(x>0)$

$\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right) = \lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{-x}\right) = \lim_{x\to 0^-}\frac{2}{x} = -\infty$

$(x<0)$

$0 \neq -\infty$



(f) $\lim_{x\to 2} \frac{x^2+x-6}{|x-2|}$ DNE

$$|x-2| = \begin{cases} x-2, & \text{if } x-2 \geq 0 \quad (x \geq 2) \\ -(x-2), & \text{if } x-2 < 0 \quad (x<2) \end{cases}$$

$$\lim_{\substack{x\to 2^- \\ (x<2)}} \frac{x^2+x-6}{|x-2|} = \lim_{x\to 2^-} \frac{x^2+x-6}{-(x-2)} = \lim_{x\to 2^-} \frac{\cancel{(x-2)}(x+3)}{-\cancel{(x-2)}} = -5$$

$$\lim_{\substack{x\to 2^+ \\ x>2}} \frac{x^2+x-6}{|x-2|} = \lim_{x\to 2^+} \frac{\cancel{(x-2)}(x+3)}{\cancel{x-2}} = 5$$

$(5 \neq -5)$

(g) $\lim_{t\to 5} \left\langle \frac{2t-10}{t-5}, \frac{5-t}{t^2-4t-5} \right\rangle = \left\langle \lim_{t\to 5} \frac{2t-10}{t-5}, \lim_{t\to 5} \frac{5-t}{t^2-4t-5} \right\rangle$

$$= \left\langle \lim_{t\to 5} \frac{2\cancel{(t-5)}}{\cancel{t-5}}, \lim_{t\to 5} \frac{-1\cancel{(t-5)}}{\cancel{(t-5)}(t+1)} \right\rangle = \left\langle \lim_{t\to 5} 2, \lim_{t\to 5} \frac{-1}{t+1} \right\rangle = \boxed{\left\langle 2, -\frac{1}{6} \right\rangle}$$

(h) $\lim_{x\to 0} x^4 \cos\frac{1}{x^2}$

Squeeze Thm.

$(-1)x^4 \le x^4\cos\frac{1}{x^2} \le (1)(x^4)$

$-x^4 \le x^4\cos\frac{1}{x^2} \le x^4$

$\lim_{x\to 0}(-x^4) = 0$, $\lim_{x\to 0} x^4 = 0$

$0 \le \lim_{x\to 0} x^4\cos\frac{1}{x^2} \le 0$

Thus, $\lim_{x\to 0} x^4\cos\frac{1}{x^2} = \boxed{0}$

2. If $\underbrace{2x - 2 \le f(x) \le x^2 - 2x + 2}$ for $x \ge 0$, find $\lim_{x\to 2} f(x)$.

The Squeeze Thm.

$\lim_{x\to 2}(2x-2) = 2$, $\lim_{x\to 2}(x^2-2x+2) = 4-2(2)+2 = 2$

$2 \le \lim_{x\to 2} f(x) \le 2$, thus $\lim_{x\to 2} f(x) = \boxed{2}$

3. Let

$$f(x) = \begin{cases} \sqrt{-x}, & \text{if } x < 0 \\ 3 - x, & \text{if } 0 \leq x < 3 \\ (x-3)^2, & \text{if } x \geqslant 3 \end{cases}$$

Evaluate each limit if exists.

(a) $\lim_{x \to 0} f(x)$ $\boxed{DNE}$

$\lim_{\substack{x \to 0^- \\ (x<0)}} f(x) = \lim_{x \to 0^-} \sqrt{-x} = 0$, $\lim_{\substack{x \to 0^+ \\ x>0}} f(x) = \lim_{x \to 0^+} (3-x) = 3$

$0 \neq 3$

(b) $\lim_{x \to 3} f(x)$

$\lim_{\substack{x \to 3^- \\ x<3}} f(x) = \lim_{x \to 3^-} (3-x) = 0$, $\lim_{\substack{x \to 3^+ \\ x>3}} f(x) = \lim_{x \to 3^+} (x-3)^2 = 0$

$0 = 0$

$\lim_{x \to 3} f(x) = \boxed{0}$

4. Find the x-value at which f is discontinuous and determine whether f is continuous from the right, or from the left, or neither.

$$f(x) = \begin{cases} 1+x^2, & \text{if } x \le 0 \\ 4-x, & \text{if } 0 < x \le 4 \\ (x-4)^2, & \text{if } x > 4 \end{cases}$$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1+x^2) = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (4-x) = 4$

$f(0) = 1+0^2 = 1$

$1 \neq 4$ jump discontinuity @ $x=0$

$f(0) = \lim_{x\to 0^-} f(x)$, continuous from the left

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (4-x) = 0$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (x-4)^2 = 0$

$f(4) = 4-4 = 0$

f is continuous @ $x=4$.

5. Find the value(s) of x where the function $f(x)$ is discontinuous. If the discontinuity, $x = a$, is removable, find a function g that agrees with f for all values of x and is continuous at $x = a$.

(a) $f(x) = \dfrac{x-4}{x^2+x-20} = \dfrac{x-4}{(x-4)(x+5)} = \dfrac{1}{x+5}$

removable discontinuity @ $\overset{(x-4=0)}{x=4}$, $g(x) = \dfrac{1}{x+5}$

$\lim\limits_{x\to -5^+} f(x) = \lim\limits_{x\to -5^+} \dfrac{1}{x+5} = \infty$

infinity discontinuity @ $x=-5$

(b) $f(x) = \dfrac{x^2-2x-8}{x^2-x-6} = \dfrac{\cancel{(x+2)}(x-4)}{\cancel{(x+2)}(x-3)} = \dfrac{x-4}{x-3}$

removable discontinuity @ $\overset{(x+2=0)}{x=-2}$, $g(x) = \dfrac{x-4}{x-3}$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} \dfrac{x-4}{x-3} = -\infty$

infinity discontinuity @ $x=3$

6. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{if } x < 2 \\ ax^2 - bx + 3, & \text{if } 2 \le x < 3 \\ 4x - a + b, & \text{if } x \ge 3 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \lim_{x\to 2}(x+2) = 4$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2}(ax^2 - bx + 3) = 4a - 2b + 3$$

$$\Rightarrow \quad 4 = 4a - 2b + 3$$

$$4a - 2b = 1$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3}(ax^2 - bx + 3) = 9a - 3b + 3$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3}(4x - a + b) = 12 - a + b$$

$$\Rightarrow \quad 9a - 3b + 3 = 12 - a + b$$

$$10a - 4b = 9$$

$$\begin{cases} 4a - 2b = 1 \\ 10a - 4b = 9 \end{cases} \rightarrow 2b = 4a - 1$$

$$10a - 2(4a-1) = 9$$

$$10a - 8a + 2 = 9 \rightarrow 2a = 7$$

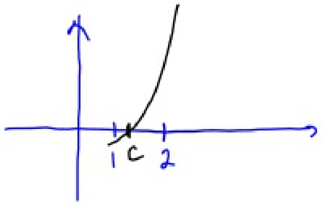
$$\boxed{a = 7/2}$$

$$2b = (4)\frac{7}{2} - 1 = 13$$

$$\boxed{b = 13/2}$$

7. Use the Intermediate Value Theorem to show that there is a root of the equation $x^4 + x = 3$ in the interval $(1, 2)$.

$$x^4 + x - 3 = 0$$



$f(x) = x^4 + x - 3$ – continuous for all $-\infty < x < \infty$

$f(1) = 1 + 1 - 3 = -1 < 0$

$f(2) = 2^4 + 2 - 3 = 16 - 1 = 15 > 0$

By the Intermediate Value Theorem, there is a number $1 < c < 2$ such that $f(c) = 0$

$x = c$ is a root of the equation $x^4 + x - 3 = 0$

if r is a positive rational number, then $\lim_{x\to\infty}\frac{1}{x^r} = \lim_{x\to-\infty}\frac{1}{x^r} = 0$

8. **Find the limit.**

(a) $\lim\limits_{x\to\infty}\dfrac{x^2-5x+1}{3x+7} = \lim\limits_{x\to\infty}\dfrac{x^{\cancel{2}}\,\frac{x^2-5x+1}{x^2}}{\cancel{x}\,\frac{3x+7}{x}} = \lim\limits_{x\to\infty}\dfrac{x\left(\frac{x^2}{x^2}-\frac{5x}{x^2}+\frac{1}{x^2}\right)}{\frac{3x}{x}+\frac{7}{x}}$

$= \lim\limits_{x\to\infty}\dfrac{x\left(1-\cancel{\frac{5}{x}}^{\,0}+\cancel{\frac{1}{x^2}}^{\,0}\right)}{3+\cancel{\frac{7}{x}}^{\,0}} = \lim\limits_{x\to\infty}\dfrac{x}{3} = \boxed{\infty}$

(b) $\lim\limits_{x\to\infty}\dfrac{x^2+x-4}{x^3-2x+1} = \lim\limits_{x\to\infty}\dfrac{\cancel{x^2}\,\frac{x^2+x-4}{x^2}}{x^{\cancel{3}}\,\frac{x^3-2x+1}{x^3}} = \lim\limits_{x\to\infty}\dfrac{\overset{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{4}{x^2}}{1+\cancel{\frac{1}{x}}^{\,0}-\cancel{\frac{4}{x^2}}^{\,0}}}{\underset{\frac{x^3}{x^3}-\frac{2x}{x^3}+\frac{1}{x^3}}{x\left(1-\cancel{\frac{2}{x^2}}^{\,0}+\cancel{\frac{1}{x^3}}^{\,0}\right)}} = \lim\limits_{x\to\infty}\frac{1}{x} = \boxed{0}$

(c) $\lim\limits_{x\to-\infty}\dfrac{\boxed{2}x^3+3x^2-3x+7}{x^3-16x+5} = 2$ (power of x on the top = power of x on the bottom)

(d) $\lim_{x\to\infty} \frac{(\sqrt{x^2+x-1}-\sqrt{x^2-x})(\sqrt{x^2+x-1}+\sqrt{x^2-x})}{(\sqrt{x^2+x-1}+\sqrt{x^2-x})} = \lim_{x\to\infty} \frac{(\sqrt{x^2+x-1})^2-(\sqrt{x^2-x})^2}{\sqrt{x^2+x-1}+\sqrt{x^2-x}} = \lim_{x\to\infty} \frac{x^2+x-1-(x^2-x)}{\sqrt{x^2+x-1}+\sqrt{x^2-x}}$

$= \lim_{x\to\infty} \frac{2x-1}{\sqrt{x^2(1+\cancel{\frac{1}{x}}^{0}-\cancel{\frac{1}{x^2}}^{0})}+\sqrt{x^2(1-\cancel{\frac{1}{x}}^{0})}} = \lim_{x\to\infty} \frac{2x-1}{\sqrt{x^2}+\sqrt{x^2}} = \lim_{x\to\infty} \frac{\boxed{2}x-1}{\boxed{2}x} = \frac{2}{2} = \boxed{1}$

(e) $\lim_{x\to-\infty} (x+\sqrt{x^2+2x}) = \lim_{x\to-\infty} \frac{(x+\sqrt{x^2+2x})(x-\sqrt{x^2+2x})}{x-\sqrt{x^2+2x}} = \lim_{x\to-\infty} \frac{x^2-(x^2+2x)}{x-\sqrt{x^2(1+\cancel{\frac{2}{x}}^{0})}}$

$= \lim_{x\to-\infty} \frac{-2x}{x-\sqrt{x^2}} = \lim_{x\to-\infty} \frac{-2x}{x+x} = \lim_{x\to-\infty} \frac{\cancel{-2x}}{\cancel{2x}} = \boxed{-1}$

$\sqrt{x^2} = -x$

9. Find the vertical and horizontal asymptotes (if any) for the function $f(x) = \frac{x^2 - 2x - 8}{x^2 - x - 6}$. $= \frac{(x-4)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}}$

removable discontinuity @ $x = -2$

V.A. $x - 3 = 0$ or $\boxed{x = 3}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x-4}{x-3} = \lim_{x\to\infty} \frac{x(1-\cancel{\frac{4}{x}}^{0})}{x(1-\cancel{\frac{3}{x}}^{0})} = \lim_{x\to\infty} \frac{x}{x} = 1$$

H.A. $\boxed{y = 1}$