$\lim_{x\to 0} \cos x = 1$ $\quad \lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} \sin x = 0$ $\quad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

Math 151/171 **WEEK in REVIEW 5** 10/12/2016

1. Find the limit.

(a) $\lim_{t\to 0} \frac{\sin^2 3t}{t^2}$ $= \lim_{t\to 0} \left[\frac{3\sin 3t}{3t}\right]^2 = 3^2 \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right)^2 = \boxed{9}$

(b) $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x}$ $= \lim_{x\to 0} \frac{3\sin 3x}{3x} \cdot \frac{5x}{5\sin 5x} = \frac{3}{5} \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{5x}{\sin 5x} = \boxed{\frac{3}{5}}$

(c) $\lim_{x\to 0} \frac{(\cos x - 1)\sin 3x}{x^2}$ $= \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \frac{3\sin 3x}{3x} = \boxed{0}$

(d) $\lim_{x\to -2} \frac{\tan \pi x}{x+2}$ | substitution $u = x+2 \rightarrow x = u-2$, $u \to 0$ as $x \to -2$ | $= \lim_{u\to 0} \frac{\tan[\pi(u-2)]}{u} = \lim_{u\to 0} \frac{\tan \overbrace{[\pi u - 2\pi]}^{\tan \pi u}}{u}$

$= \lim_{u\to 0} \frac{\tan \pi u}{u} = \lim_{u\to 0} \frac{\sin \pi u}{u \cos \pi u} = \lim_{u\to 0} \frac{\pi \sin \pi u}{\pi u} \cdot \frac{1}{\cos \pi u} \to 1$

$= \pi \lim_{u\to 0} \frac{\sin \pi u}{\pi u} = \boxed{\pi}$

$(\sin x)' = \cos x$ $\quad$ $(\tan x)' = \sec^2 x$ $\quad$ $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

$(\cos x)' = -\sin x$ $\quad$ $(\sec x)' = \sec x \tan x$ $\quad$ $([u(x)]^n)' = n\,[u(x)]^{n-1} u'(x)$

2. Differentiate the function.

(a) $f(x) = \tan x + x \sec x$

$$f'(x) = (\tan x)' + (\underbrace{x \sec x}_{\text{product rule}})' = \sec^2 x + (x)' \sec x + x(\sec x)'$$

$$= \boxed{\sec^2 x + \sec x + x \sec x \tan x}$$

(b) $f(x) = (3x^3 - 2x^2 + 1)^6$

$$f'(x) = 6\,(3x^3 - 2x^2 + 1)^5 (3x^3 - 2x^2 + 1)'$$

$$= \boxed{6\,(3x^3 - 2x^2 + 1)^5 (9x^2 - 4x)}$$

(c) $f(x) = (1 + \cos^2 x)^3$

$$f'(x) = 3\,(1 + \cos^2 x)^2 (1 + \cos^2 x)'$$

$$= 3(1 + \cos^2 x)^2\, 2 \cos x (\cos x)'$$

$$= \boxed{3(1 + \cos^2 x)^2\, 2 \cos x\, (-\sin x)}$$

(d) $f(x) = \overset{\text{outer}}{\widetilde{\cos}}\sqrt{x}$

$$f'(x) = -\sin\sqrt{x}\ (\sqrt{x})' = -\sin\sqrt{x}\ \frac{1}{2}x^{-1/2}$$

$$= \boxed{-\frac{\sin\sqrt{x}}{2\sqrt{x}}}$$

(e) $f(x) = \left(\frac{x^4-1}{x^4+1}\right)^3$

$$f'(x) = 3\left(\frac{x^4-1}{x^4+1}\right)^2\left(\frac{x^4-1}{x^4+1}\right)' = 3\left(\frac{x^4-1}{x^4+1}\right)^2 \frac{(x^4-1)'(x^4+1)-(x^4-1)(x^4+1)'}{(x^4+1)^2}$$

$$= 3\left(\frac{x^4-1}{x^4+1}\right)^2 \frac{4x^3(x^4+1)-4x^3(x^4-1)}{(x^4+1)^2} = 3\left(\frac{x^4-1}{x^4+1}\right)^2 \frac{4x^3(\cancel{x^4}+1-\cancel{x^4}+1)}{(x^4+1)^2}$$

$$= 3\left(\frac{x^4-1}{x^4+1}\right)^2 \frac{8x^3}{(x^4+1)^2} = \boxed{\frac{24x^3(x^4-1)^2}{(x^4+1)^4}}$$

(f) $f(x) = \frac{2x+1}{\sqrt{x^2+3}} = \frac{2x+1}{(x^2+3)^{1/2}}$

$$f'(x) = \frac{2(x^2+3)^{1/2} - \frac{1}{2}(x^2+3)^{-1/2}(2x+1)}{x^2+3} = \frac{2\sqrt{x^2+3} - \frac{2x+1}{2\sqrt{x^2+3}}}{x^2+3}$$

$$= \frac{\frac{4(x^2+3)-(2x+1)}{2\sqrt{x^2+3}}}{x^2+3} = \frac{4x^2+12-2x-1}{2(x^2+3)^{3/2}} = \boxed{\frac{4x^2-2x+11}{2(x^2+3)^{3/2}}}$$

(g) $f(x) = (x^6 + 4x^5 - 11)^5(2 + x^8)^7$

$$f'(x) = 5(x^6+4x^5-11)^4(x^6+4x^5-11)'(2+x^8)^7 + (x^6+4x^5-11)^5\,7(2+x^8)^6(2+x^8)'$$

$$= 5(x^6+4x^5-11)^4(6x^5+20x^4)(2+x^8)^7 + 7(x^6+4x^5-11)^5(2+x^8)^6(8x^7)$$

$$= 5(x^6+4x^5-11)^4\,2x^4(3x+10)(2+x^8)^7 + 56x^7(x^6+4x^5-11)^5(2+x^8)^6$$

$$= (x^6+4x^5-11)^4(2x^4)(2+x^8)^6\left[5(3x+10)(2+x^8) + 28x^3(x^6+4x^5-11)\right]$$

$$= (x^6+4x^5-11)^4(2x^4)(2+x^8)^6\left[5(6x+3x^9+20+10x^8) + 28x^9 + 112x^8 - 308\right]$$

$$= (x^6+4x^5-11)^4(2x^4)(2+x^8)^6\left[30x+15x^9+100+50x^8+28x^9+112x^8-308\right]$$

$$= \boxed{(x^6+4x^5-11)^4(2x^4)(2+x^8)^6\left[43x^9+162x^8+30x-208\right]}$$

3. Functions f and g satisfy the properties as shown in the table. Find the indicated quantity.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-3	3	1	1
2	0	3	-5	10
3	2	5	0	4

(a) $h'(1)$, if $h(x) = f(g(x))$

$$h'(x) = [f(g(x))]' = f'(g(x))g'(x)$$

$$h'(1) = f'(\underbrace{g(1)}_{1})\underbrace{g'(1)}_{1} = f'(1) = \boxed{3}$$

2

(b) $z'(2)$, if $z(x) = [f(2x-1)]^4$

$$z'(x) = 4[f(2x-1)]^3[f(2x-1)]' = 4[f(2x-1)]^3 f'(2x-1)(2x-1)'$$

$$= 4[f(2x-1)]^3 f'(2x-1)(2)$$

$$z'(2) = 8[f(2\cdot 2-1)]^3 f'(2\cdot 2-1)$$

$$= 8[f(3)]^3 f'(3) = 8(2)^3(5) = 8(8)(5) = \boxed{160}$$

(c) $G'(1)$, if $G(x) = \left[x^2 - g(2x)\right]^3$

$$G'(x) = 3\left(x^2 - g(2x)\right)^2\left(x^2 - g(2x)\right)'$$

$$= 3\left(x^2 - g(2x)\right)^2\left(2x - g'(2x)(2x)'\right)$$

$$= 3\left(x^2 - g(2x)\right)^2\left(2x - g'(2x)(2)\right)$$

$$G'(1) = 3\left(1 - g(2)\right)^2\left(2 - 2g'(2)\right)$$

$$= 3\left(1 - (-5)\right)^2\left(2 - 2(10)\right) = 3(6)^2(-18)$$

$$= 3(36)(-18) = \boxed{-1944}$$

4. Find the equation of the tangent line to the curve $y = x\sqrt{1+x^2}$ at the point where $x = 1$.

Equation of a tangent line to $y=f(x)$ @ $x=a$ is

$$y = f'(a)(x-a)+f(a)$$

$$f(x) = x\,(1+x^2)^{1/2}, \quad f(1) = 1\,(1+1)^{1/2} = \sqrt{2}$$

$$f'(x) = (x)'(1+x^2)^{1/2} + x\left[(1+x^2)^{1/2}\right]'$$

$$= (1+x^2)^{1/2} + x\,\frac{1}{2}\,(1+x^2)^{-1/2}\,(1+x^2)'$$

$$= (1+x^2)^{1/2} + x\,\frac{1}{\cancel{2}}\,(1+x^2)^{-1/2}\,(\cancel{2}x)$$

$$= (1+x^2)^{\frac{1}{2}} + \frac{x^2}{(1+x^2)^{1/2}} = \frac{1+x^2+x^2}{(1+x^2)^{1/2}} = \frac{1+2x^2}{\sqrt{1+x^2}}$$

$$f'(1) = \frac{1+2}{\sqrt{1+1}} = \frac{3}{\sqrt{2}}$$

$$\boxed{y = \frac{3}{\sqrt{2}}(x-1)+\sqrt{2}}$$

5. Find $\frac{dy}{dx}$ for the equation $\frac{d}{dx}\left(\cos(x-y)\right) = \frac{d}{dx}\left(y\sin x\right)$

$$-\sin(x-y)(x-y)' = y'\sin x + y(\sin x)'$$

$$-\sin(x-y)(1-y') = y'\sin x + y\cos x \qquad \text{solve for } y'$$

$$-\sin(x-y) + y'\sin(x-y) = y'\sin x + y\cos x$$

$$y'\sin(x-y) - y'\sin x = y\cos x + \sin(x-y)$$

$$\frac{y'[\sin(x-y) - \sin x]}{\sin(x-y) - \sin x} = \frac{y\cos x + \sin(x-y)}{\sin(x-y) - \sin x}$$

$$\boxed{y' = \frac{y\cos x + \sin(x-y)}{\sin(x-y) - \sin x}}$$

$x = x(y)$

6. Find $\frac{dx}{dy}$ for the equation $y^4 + x^2y^2 + yx^4 = y + 1$.

$$\frac{d}{dy}\left[y^4 + x^2y^2 + yx^4\right] = \frac{d}{dy}\left[y + 1\right]$$

$$\frac{d(y^4)}{dy} + \frac{d(x^2y^2)}{dy} + \frac{d(yx^4)}{dy} = \frac{d(y)}{dy}$$

$$4y^3 + \frac{d(x^2)}{dy}\,y^2 + x^2\frac{d(y^2)}{dy} + \frac{d(y)}{dy}\,x^4 + y\,\frac{d(x^4)}{dy} = 1$$

$$4y^3 + \underbrace{2x\frac{dx}{dy}\,y^2} + x^2(2y) + 1(x^4) + \underbrace{y\,4x^3\frac{dx}{dy}} = 1$$

$$\frac{dx}{dy}\left[2xy^2 + 4x^3y\right] = 1 - 4y^3 - 2x^2y - x^4$$

$$\boxed{\frac{dx}{dy} = \frac{1 - 4y^3 - 2x^2y - x^4}{2xy^2 + 4x^3y}}$$

7. Find the slope of the tangent line to the curve $2(x^2+y^2)^2 = 25(x^2-y^2)$ at the point (3,1).

slope of the tangent line is $\frac{dy}{dx}(3,1)$.

$$\frac{d}{dx}\left[2(x^2+y^2)^2\right] = \frac{d}{dx}\left[25(x^2-y^2)\right]$$

$$4(x^2+y^2)\frac{d}{dx}(x^2+y^2) = 25\left[\frac{d(x^2)}{dx} - \frac{d(y^2)}{dx}\right]$$

$$4(x^2+y^2)\left[\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx}\right] = 25\left[2x - 2y\frac{dy}{dx}\right]$$

$$4(x^2+y^2)\left[2x + 2y\frac{dy}{dx}\right] = 50x - 50y\frac{dy}{dx}$$

$$\frac{8x(x^2+y^2) + 8y\frac{dy}{dx}(x^2+y^2)}{2} = \frac{50x - 50y\frac{dy}{dx}}{2}$$

$4y(x^2+y^2)$ - [illegible] y y

7. Find the slope of the tangent line to the curve $2(x^2+y^2)^2 = 25(x^2-y^2)$ at the point (3,1).

slope of the tangent line is $\frac{dy}{dx}(3,1)$.

$$\frac{d}{dx}\left[2(x^2+y^2)^2\right] = \frac{d}{dx}\left[25(x^2-y^2)\right]$$

$$4(x^2+y^2)\frac{d}{dx}(x^2+y^2) = 25\left[\frac{d(x^2)}{dx} - \frac{d(y^2)}{dx}\right]$$

$$4(x^2+y^2)\left[\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx}\right] = 25\left[2x - 2y\frac{dy}{dx}\right]$$

$$4(x^2+y^2)\left[2x + 2y\frac{dy}{dx}\right] = 50x - 50y\frac{dy}{dx}$$

$$\frac{8x(x^2+y^2) + 8y\frac{dy}{dx}(x^2+y^2)}{2} = \frac{50x - 50y\frac{dy}{dx}}{2}$$

$$4x(x^2+y^2) + 4y\frac{dy}{dx}(x^2+y^2) = 25x - 25y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[4y(x^2+y^2) + 25y\right] = 25x - 4x(x^2+y^2)$$

$$\frac{dy}{dx} = \frac{25x - 4x(x^2+y^2)}{4y(x^2+y^2)+25y}$$

$$\frac{dy}{dx}(3,1) = \frac{25(3) - 4(3)(3^2+1)}{4(3^2+1)+25} = \frac{75-120}{40+25} = -\frac{45}{65} = \boxed{-\frac{9}{13}}$$

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$
$$\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$$

8. Find a tangent vector of unit length to the curve $\mathbf{r}(t) =< t\cos t, t\sin t >$ at the point where $t = \frac{\pi}{4}$.

tangent vector $\vec{v}(t) = \vec{r}\,'(t) = \langle (t\cos t)', (t\sin t)' \rangle$

$$= \langle (t)'\cos t + t(\cos t)', (t)'\sin t + t(\sin t)' \rangle$$

$$= \langle \cos t - t\sin t, \sin t + t\cos t \rangle$$

$$\vec{v}\left(\tfrac{\pi}{4}\right) = \langle \cos\tfrac{\pi}{4} - \tfrac{\pi}{4}\sin\tfrac{\pi}{4}, \sin\tfrac{\pi}{4} + \tfrac{\pi}{4}\cos\tfrac{\pi}{4} \rangle = \langle \tfrac{\sqrt{2}}{2}\left(1-\tfrac{\pi}{4}\right), \tfrac{\sqrt{2}}{2}\left(1+\tfrac{\pi}{4}\right) \rangle$$

$$\left|\vec{v}\left(\tfrac{\pi}{4}\right)\right| = \frac{\sqrt{2}}{2}\sqrt{\left(1-\tfrac{\pi}{4}\right)^2 + \left(1+\tfrac{\pi}{4}\right)^2} = \frac{\sqrt{2}}{2}\sqrt{1 - \cancel{\tfrac{2\pi}{4}} + \tfrac{\pi^2}{16} + 1 + \cancel{\tfrac{2\pi}{4}} + \tfrac{\pi^2}{16}} = \frac{\sqrt{2}}{2}\sqrt{2 + \frac{2\pi^2}{16}} = \frac{\sqrt{2}}{2}\sqrt{2\left(1+\frac{\pi^2}{16}\right)}$$

$$= \frac{\sqrt{2}}{2}\cdot\sqrt{2}\,\sqrt{\frac{16+\pi^2}{16}} = \frac{2}{2}\cdot\frac{\sqrt{16+\pi^2}}{4} = \frac{\sqrt{16+\pi^2}}{4}$$

$$\vec{u} = \frac{\vec{v}\left(\frac{\pi}{4}\right)}{\left|\vec{v}\left(\frac{\pi}{4}\right)\right|} = \frac{4}{\sqrt{16+\pi^2}}\langle \tfrac{\sqrt{2}}{2}\left(1-\tfrac{\pi}{4}\right), \tfrac{\sqrt{2}}{2}\left(1+\tfrac{\pi}{4}\right) \rangle = \frac{4\sqrt{2}}{2\sqrt{16+\pi^2}}\langle 1-\tfrac{\pi}{4}, 1+\tfrac{\pi}{4} \rangle = \boxed{\frac{2\sqrt{2}}{\sqrt{16+\pi^2}}\langle 1-\tfrac{\pi}{4}, 1+\tfrac{\pi}{4} \rangle}$$

9. Find the vector and parametric equations for the line tangent to the curve $\vec{r}(t) =< 1-4t, 2t-3t^2 >$ at the point $P(-11, -21)$.

Find t such that $\langle -11, -21 \rangle = \langle 1-4t, 2t-3t^2 \rangle$

$$\begin{cases} 1-4t = -11 \\ 2t - 3t^2 = -21 \end{cases} \longrightarrow 4t = 12 \longrightarrow t = 3$$

tangent vector: $\vec{v}(t) = \vec{r}\,'(t) = \langle -4, 2-6t \rangle$

$$\vec{v}(3) = \langle -4, 2-6(3) \rangle = \langle -4, -16 \rangle$$

vector equation: $\boxed{\vec{r}(t) = \langle -11, -21 \rangle + t\langle -4, -16 \rangle}$

parametric equations: $\boxed{\begin{array}{l} x(t) = -11 - 4t \\ y(t) = -21 - 16t \end{array}}$

10. The ball is tossed into the air. Its position at time t is given by $\mathbf{r}(t) = < 5t, 100t - 16t^2 >$.

(a) Find the velocity and the speed of the ball when $t = 2$.

velocity $\vec{v}(t) = \vec{r}\,'(t) = < 5, 100-32t >$

$\vec{v}(2) = < 5, 100-32(2) > = \boxed{< 5, 36 >}$

speed $= |\vec{v}(2)| = \sqrt{5^2+36^2} = \boxed{\sqrt{1321}}$

(b) How high does the ball go?

vertical component of the velocity $= 0$

$100 - 32t = 0 \longrightarrow t = \frac{100}{32} = \frac{25}{8}$

max height $h_{max} = 100\left(\frac{25}{8}\right) - 16\left(\frac{25}{8}\right)^2 = \frac{25(25)}{2} - 2 \cdot \frac{625}{8}$

$= \frac{625}{2} - \frac{625}{4} = \boxed{\frac{625}{4}}$

(c) With what speed does the ball hit the ground?

vertical component of $\vec{r}(t) = 0$

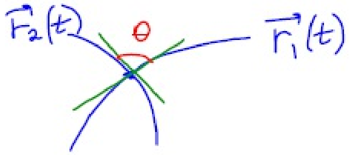
$100t - 16t^2 = 0 \rightarrow t(100-16t) = 0$

$t_1 = 0, \quad t_2 = \frac{100}{16} = \frac{25}{4}$

$\vec{v}\left(\frac{25}{4}\right) = < 5, 100 - 32\frac{(25)}{4} > = < 5, 100 - 200 > = < 5, -100 >$

speed $\left|\vec{v}\left(\frac{25}{4}\right)\right| = \sqrt{5^2 + (-100)^2} = \boxed{\sqrt{10025}}$

11. Find the angle of intersection of the curves traced by $\mathbf{r}_1(t) = < 1 - t, 3 + t^2 >$ and $\mathbf{r}_2(s) = < s - 2, s^2 >$.



Find an angle between tangents to the curves at the point of intersection.

1. Find the point of intersection.

$$\begin{cases} 1-t = s-2 \longrightarrow s = 1-t+2 = 3-t \\ 3+t^2 = s^2 \end{cases}$$

$$3+t^2 = (3-t)^2$$

$$3+\cancel{t^2} = 9-6t+\cancel{t^2}$$

$$6t = 6 \longrightarrow \underline{t=1}, \quad \underline{s = 3-1 = 2} \qquad \boxed{(0,4)}$$

$$<0,4> = \vec{r}_1(1) = \vec{r}_2(2)$$

2. Find tangent vectors to $\vec{r}_1(t)$ and $\vec{r}_2(s)$ @ $(0,4)$

$$\vec{v}_1(t) = \vec{r}_1{}'(t) = <-1, 2t>, \quad \boxed{\vec{v}_1(1) = <-1,2>}$$

$$\vec{v}_2(s) = \vec{r}_2{}'(s) = <1, 2s>, \quad \boxed{\vec{v}_2(2) = <1,4>}$$

3. Find an angle between $\vec{v}_1(1)$ and $\vec{v}_2(2)$.

$$\cos\theta = \frac{\vec{v}_1(1)\cdot\vec{v}_2(2)}{|\vec{v}_1(1)||\vec{v}_2(2)|} = \frac{<-1,2>\cdot<1,4>}{\sqrt{1+4}\cdot\sqrt{1+16}} = \frac{-1+8}{\sqrt{5}\sqrt{17}} = \frac{7}{\sqrt{85}}$$

$$\theta \approx \boxed{40.6^\circ}$$