1. Find y'' for the function $f(x) = (1 + x^2)\tan x$. Product Rule.

$$y' = (1+x^2)'\tan x + (1+x^2)(\tan x)'$$
$$= 2x\tan x + (1+x^2)\sec^2 x$$
$$y'' = (2x\tan x + (1+x^2)\sec^2 x)'$$
$$= 2(x)'\tan x + 2x(\tan x)' + (1+x^2)'\sec^2 x + (1+x^2)(\sec^2 x)'$$
$$= 2\tan x + 2x\sec^2 x + 2x\sec^2 x + (1+x^2)\,2\sec x\,(\sec x)'$$
$$= 2\tan x + 4x\sec^2 x + 2(1+x^2)\sec x\sec x\tan x$$
$$= \boxed{2\tan x + 4x\sec^2 x + 2(1+x^2)\sec^2 x\tan x}$$

2. Find the n-th derivative of the function $f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$$f'(x) = -1(1+x)^{-2} \underbrace{(1+x)'}_{1}$$

$$= -1(1+x)^{-2}$$

$$f''(x) = \overbrace{(-1)}^{(-1)(1)}\overbrace{(-2)}^{(-1)(2)}(1+x)^{-3}$$

$$= (-1)^2(1)(2)(1+x)^{-3}$$

$$f'''(x) = (-1)^2(1)(2)(-3)(1+x)^{-4}$$

$$= (-1)^3\underbrace{(1)(2)(3)}_{3!}(1+x)^{-4}$$

$$\boxed{f^{(n)}(x) = (-1)^n\, n!\, (1+x)^{-n-1}}$$

3. Find the 46-th derivative of the function $\cos\left(\frac{x}{3}\right)$.

$46 = 44 + 2$

$f^{(44)}(x)$

$f^{(45)}(x)$

$f^{(46)}(x)$

$f(x) = \cos\frac{x}{3}$

$f'(x) = -\sin\frac{x}{3}\left(\frac{x}{3}\right)'$

$= -\frac{1}{3}\sin\frac{x}{3}$

$f''(x) = -\frac{1}{3}\cos\frac{x}{3}\left(\frac{x}{3}\right)'$

$= -\frac{1}{3}\frac{1}{3}\cos\frac{x}{3}$

$= -\frac{1}{3^2}\cos\frac{x}{3}$

$f'''(x) = -\frac{1}{3^2}\left(-\sin\frac{x}{3}\right)\left(\frac{x}{3}\right)'$

$= \frac{1}{3^3}\sin\frac{x}{3}$

$f^{IV}(x) = \frac{1}{3^3}\cos\frac{x}{3}\left(\frac{x}{3}\right)'$

$= \frac{1}{3^4}\cos\frac{x}{3}$

$\boxed{f^{(46)}(x) = -\frac{1}{3^{46}}\cos\frac{x}{3}}$

$f^{(53)}(x) = -\frac{1}{3^{53}}\sin\frac{x}{3}$

4. Find $\frac{d^2y}{dx^2}$ if $\sqrt{x}+\sqrt{y}=8$.

$$\frac{d}{dx}\left(x^{1/2}+y^{1/2}\right)=\frac{d}{dx}(8)$$

$$(2)\left(\frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}\frac{dy}{dx}=0\right)$$

$$x^{-1/2}+y^{-1/2}\frac{dy}{dx}=0 \implies \frac{dy}{dx}=-\frac{x^{-1/2}}{y^{-1/2}}=-\frac{y^{1/2}}{x^{1/2}}$$

$$\frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\frac{y^{1/2}}{x^{1/2}}\right)=-\frac{\frac{d}{dx}(y^{1/2})\,x^{1/2}-\frac{d}{dx}(x^{1/2})\,y^{1/2}}{(x^{1/2})^2}$$

$$=-\frac{\frac{1}{2}y^{-1/2}\frac{dy}{dx}x^{1/2}-\frac{1}{2}x^{-1/2}y^{1/2}}{x}$$

$$=-\frac{y^{-1/2}\frac{dy}{dx}x^{1/2}-x^{-1/2}y^{1/2}}{2x}$$

$$=-\frac{y^{-1/2}\left(-\frac{y^{1/2}}{\cancel{x^{1/2}}}\right)\cancel{x^{1/2}}-x^{-1/2}y^{1/2}}{2x}$$

$$=+\frac{1+x^{-1/2}y^{1/2}}{2x}=\frac{1+\frac{1}{x^{1/2}}y^{1/2}}{2x}$$

$$x^{1/2}+y^{1/2}=8$$

$$=\frac{\frac{x^{1/2}+y^{1/2}}{x^{1/2}}}{2x}=\frac{x^{1/2}+y^{1/2}}{2x\,x^{1/2}}=\frac{8}{2x^{3/2}}=\boxed{\frac{4}{x^{3/2}}}$$

5. If $\mathbf{r}(t) = < t^3, t^2 >$ represents the position of a particle at time t, find the angle between the velocity and the acceleration vector at time $t = 1$.

velocity $\vec{v}(t) = \vec{r}\,'(t) = < (t^3)', (t^2)' >$

$\vec{v}(t) = < 3t^2, 2t >$, $\vec{v}(1) = <3,2>$

acceleration $\vec{a}(t) = \vec{v}\,'(t) = < (3t^2)', (2t)' >$

$= < 6t, 2 >$, $\vec{a}(1) = < 6, 2 >$

$$\cos\theta = \frac{\vec{v}(1)\cdot\vec{a}(1)}{|\vec{v}(1)|\cdot|\vec{a}(1)|} = \frac{<3,2>\cdot<6,2>}{|<3,2>|\cdot|<6,2>|}$$

$$= \frac{3(6)+2(2)}{\sqrt{3^2+2^2}\cdot\sqrt{6^2+2^2}} = \frac{22}{\sqrt{13}\,\underbrace{\sqrt{40}}_{\sqrt{4\cdot 10} = 2\sqrt{10}}} = \frac{22}{2\sqrt{130}} = \frac{11}{\sqrt{130}}$$

$$\theta = \arccos\left(\frac{11}{\sqrt{130}}\right) \approx \boxed{15^\circ}$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

6. Consider the curve $x = t^2 - 10t - 3$, $y = 5t^2 + t$.

(a) Find the equation of the tangent line at the point (8,4).

Find t such $\begin{cases} t^2 - 10t - 3 = 8 \\ 5t^2 + t = 4 \end{cases}$

$t^2 - 10t - 11 = 0$

$(t+1)(t-11) = 0$

$t_1 = -1, \quad t_2 = 11$

Find t that satisfies the 2nd eqn:

$t = -1: \quad 5(-1)^2 + (-1) = 4$

$t = 11: \quad 5(11)^2 + 11 \neq 4$ — not valid

slope: $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{(5t^2+t)'}{(t^2-10t-3)'} = \dfrac{10t+1}{2t-10}$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{10(-1)+1}{2(-1)-10} = \frac{-9}{-12} = \frac{3}{4}$$

$\boxed{y = \frac{3}{4}(x-8)+4}$ tangent line

(b) At what point(s) is the tangent line to the graph parallel to the line $7x + 2y = 19$.

$2y = 19 - 7x$

$y = \frac{19}{2} - \frac{7}{2}x \Rightarrow$ slope $= -\frac{7}{2}$

Find t such that $\dfrac{dy}{dt} = -\dfrac{7}{2}$

$$\frac{10t+1}{2t-10} = -\frac{7}{2}$$

$2(10t+1) = -7(2t-10)$

$20t + 2 = -14t + 70$

$34t = 68 \Rightarrow t = 2$

Find the point on the curve that corresponds to $t = 2$.

$x(2) = 2^2 - 10(2) - 3 = 4 - 20 - 3 = -19$

$y(2) = 5(2)^2 + 2 = 22$

$\boxed{(-19, 22)}$

7. Find the point(s) on the curve $x = 1 - 2\cos t$, $y = 2 + 3\sin t$ where the tangent is horizontal or vertical.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(2+3\sin t)'}{(1-2\cos t)'} = \frac{\cos t}{\sin t}$$

horizontal tangent $\Rightarrow$ slope is zero $\Rightarrow$ $y'(t) = \cos t = 0$

$t = \frac{\pi}{2} + \pi n, \; n = 0, \pm 1, \pm 2, \ldots$

$$x\left(\frac{\pi}{2} + \pi n\right) = 1 - 2\underbrace{\cos\left(\frac{\pi}{2} + \pi n\right)}_{0} = 1$$

$$y\left(\frac{\pi}{2} + \pi n\right) = 2 + 3\sin\left(\frac{\pi}{2} + \pi n\right)$$

if n is even, then $\sin\left(\frac{\pi}{2} + \pi n\right) = \sin\frac{\pi}{2} = 1$

if n is odd, then $\sin\left(\frac{\pi}{2} + \pi n\right) = \sin\frac{3\pi}{2} = -1$

$$= 2 + 3(-1)^n = \begin{cases} 5, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases}$$

horizontal tangent $\boxed{(1,5), \; (1,-1)}$

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vertical tangent $\Rightarrow$ $x'(t) = 0$

$\sin t = 0 \Longrightarrow t = \pi n, \; n = 0, \pm 1, \pm 2, \ldots$

$$x(\pi n) = 1 - 2\cos(\pi n)$$

$\cos(\pi n) = \cos 0 = 1$, if n is even

$\cos(\pi n) = \cos \pi = -1$, if n is odd

$$x(\pi n) = \begin{cases} 1 - 2(1), & \text{if } n \text{ is even} \\ 1 - 2(-1), & \text{if } n \text{ is odd} \end{cases} = \begin{cases} -1, & n \text{ is even} \\ 3, & n \text{ is odd} \end{cases}$$

$$y(\pi n) = 2 + 3\underbrace{\sin(\pi n)}_{0} = 2$$

vertical tangent @ $\boxed{(-1,2)}$ and $\boxed{(3,2)}$

8. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?

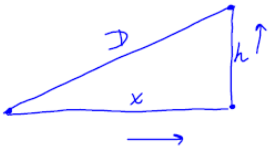
D is the distance between the boy and the balloon

h is the height of the balloon

x is distance traveled by the boy

$\frac{dx}{dt}$ is the speed of the boy

$\frac{dx}{dt} = 15$, $\frac{dh}{dt} = 5$ (speed of the balloon)



$x = 3(15) = 45$

$h = 45 + 3(5) = 60$

$D = \sqrt{x^2+h^2} = \sqrt{45^2+60^2} = 75$

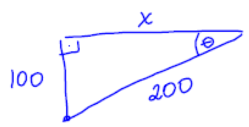
$$\frac{d}{dt}D^2 = \frac{d}{dt}(x^2+h^2)$$

$$\cancel{2}D\frac{dD}{dt} = \cancel{2}x\frac{dx}{2t} + \cancel{2}h\frac{dh}{dt}$$

$$\frac{dD}{dt} = \frac{1}{\underbrace{D}_{75}}\left(\overbrace{x}^{45}\underbrace{\frac{dx}{dt}}_{15} + \overbrace{h}^{60}\underbrace{\frac{dh}{dt}}_{5}\right)$$

$$= \frac{1}{75}\left(45(15) + 60(5)\right) = \frac{975}{75} = \boxed{13 \text{ ft/s}}$$

9. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?



$\sin\theta = \frac{100}{200} = \frac{1}{2}$

$\frac{dx}{dt} = 8$

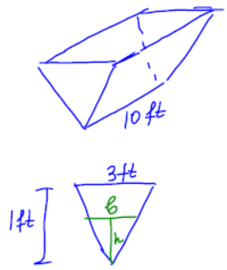
$\frac{d}{dt}(\cot\theta) = \frac{d}{dt}\left(\frac{x}{100}\right)$

$-\csc^2\theta \, \frac{d\theta}{dt} = \frac{1}{100} \frac{dx}{dt}$

$\frac{d\theta}{dt} = -\frac{\sin^2\theta}{100} \frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{-(1/2)^2}{100}(8) = -\frac{2}{100} = \boxed{-\frac{1}{50} \text{ rad/s}}$

10. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across the top and have a height of 1 ft. If the trough is filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 inches deep?



h is the height of the water

V is the amount of the water in the tank

$\frac{dV}{dt} = 12 \text{ ft}^3/\text{min}$

Find $\frac{dh}{dt}$, when $h = 6 \text{ inches} = \frac{1}{2} \text{ ft}$

$V = \frac{1}{2} bh(10) = 5bh$

Eliminate b.

Similar triangles: $\frac{3}{1} = \frac{b}{h} \Rightarrow b = 3h$

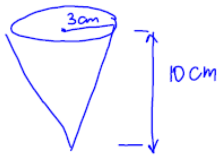
$$\frac{d}{dt}\left(V\right) = 5\overbrace{(3h)}^{b}h = \frac{d}{dt}\left(15h^2\right)$$

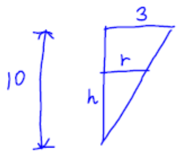
$$\frac{dV}{dt} = 15(2h)\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{30h}\frac{dV}{dt}$$

$$\frac{dV}{dt} = 30h\frac{dh}{dt} \qquad = \frac{1}{30(1/2)}(12) = \frac{12}{15}$$

$$= \boxed{\frac{4}{5} \text{ ft/min}}$$

11. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If the water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?





h is the height of the water

V is the amount of the water

$$\frac{dV}{dt} = 2$$

Find $\frac{dh}{dt}$ when $h = 5$

$$V = \frac{1}{3}\pi r^2 h$$

Eliminate r.

Use similar triangles: $\frac{3}{10} = \frac{r}{h}$

$$r = \frac{3h}{10}$$

$$\frac{d}{dt}(V) = \frac{1}{3}\pi\left(\frac{3h}{10}\right)^2 h = \frac{1}{3}\pi \frac{9h^2}{100} h = \frac{d}{dt}\left(\frac{3\pi}{100} h^3\right)$$

$$\frac{dV}{dt} = \frac{3\pi}{100} 3h^2 \frac{dh}{dt} = \frac{9\pi}{100} h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{100}{9\pi h^2} \frac{dV}{dt}$$

$$= \frac{100}{9\pi (5)^2} \cdot 2 = \frac{100}{9\pi (25)} (2) = \boxed{\frac{8}{9\pi} \text{ cm/s}}$$

Water is leaking out of an inverted conical tank at a rate of 9,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 600 cm and the diameter at the top is 400 cm. If the water level is rising at a rate of 20 cm/min when the height of the water is 200 cm, find the rate at which water is being pumped into the tank. (Round your answer to the nearest integer.)

$$\frac{dV}{dt} = [\text{rate in}] - \underbrace{[\text{rate out}]}_{9{,}000}$$

$$[\text{rate in}] = \frac{dV}{dt} + [\text{rate out}]$$