1. Find the linear and quadratic approximation for the function $f(x) = \sin x$ at $a = \pi/6$.

$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$ linear approximation for $f(x)$ near a.

$\boxed{f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$ Quadratic approxim. for $f(x)$ near a.

$f(x) = \sin x$	$f(\frac{\pi}{6}) = \sin\frac{\pi}{6} = \frac{1}{2}$
$f'(x) = \cos x$	$f'(\frac{\pi}{6}) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
$f''(x) = -\sin x$	$f''(\frac{\pi}{6}) = -\sin\frac{\pi}{6} = -\frac{1}{2}$

Linear Approximation: $\boxed{f(x) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)}$

Quadratic Approximation: $\sin x \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2\cdot 2}\left(x - \frac{\pi}{6}\right)^2$

$\boxed{\sin x \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2}$

2. Use differentials to approximate $(1.97)^6$.

$$f(a+\Delta x) \approx f(a) + f'(a)\Delta x$$

$$f(x) = x^6, \quad a = 2, \quad \Delta x = -(2-1.97) = -0.03$$

$$f'(x) = 6x^5, \quad f'(2) = 6 \cdot 2^5 = 6 \cdot 32 = 192$$

$$f(2) = 2^6 = 64$$

$$(1.97)^6 \approx \underbrace{64}_{f(2)} + \underbrace{192}_{f'(2)}\underbrace{(-0.03)}_{\Delta x}$$

$$64 - 5.76 = \boxed{58.24}$$

3. Use differentials to estimate the amount of paint needed to apply a coat of paint .05 cm thick to a hemispherical dome with diameter 50 m.

$$\Delta r = 0.05 \text{ cm} = 0.0005 \text{ m}, \quad D = 50, \quad r = \frac{50}{2} = 25 \text{ m}$$

$$V = \frac{1}{2} \cdot \frac{4}{3}\pi r^3$$

$$V = \frac{2}{3}\pi r^3$$

Find the differential of V

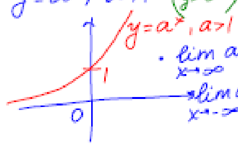
$$dV = \left(\frac{2}{3}\pi r^3\right)' dr$$

$$dV = \frac{2}{\cancel{3}}\pi(\cancel{3}r^2)\,dr$$

$$dV = 2\pi r^2 dr = 2\pi\underbrace{(25)}_{r}^2\underbrace{(0.0005)}_{dr} = \boxed{.625\pi m^3}$$

4. Find the limit

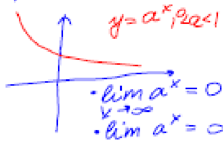
$y = a^x,\ a > 1$ $(y = e^x)$



$\cdot\ \lim_{x \to \infty} a^x = \infty$

$\cdot\ \lim_{x \to -\infty} a^x = 0$

$y = a^x,\ 0 < a < 1$ $(y = e^{-x})$



$\cdot\ \lim_{x \to \infty} a^x = 0$

$\cdot\ \lim_{x \to -\infty} a^x = \infty$

(a) $\lim_{x \to -\infty} \pi^x$

$\pi > 1$

$= \boxed{0}$

(b) $\lim_{x \to \infty} \left(\frac{3}{\pi}\right)^x = \boxed{0}$

$\frac{3}{\pi} = \frac{3}{3.14\ldots} < 1$

(c) $\lim_{x \to -\infty} \left(\frac{3}{\pi}\right)^x = \boxed{\infty}$

(d) $\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x\to\infty} \frac{\frac{1}{e^{-3x}} - e^{-3x}}{\frac{1}{e^{-3x}} + e^{-3x}} = \lim_{x\to\infty} \frac{\frac{1 - e^{-3x}\cdot e^{-3x}}{\cancel{e^{-3x}}}}{\frac{1 + e^{-3x}\cdot e^{-3x}}{\cancel{e^{-3x}}}}$

$e^{-3x} \cdot e^{-3x} = e^{-3x-3x} = e^{-6x}$

$\lim_{x\to\infty} e^{3x} = \infty$ – get rid of e^{3x}

$\lim_{x\to\infty} e^{-3x} = 0$

$= \lim_{x\to\infty} \frac{1 - \cancel{e^{-6x}}^{0}}{1 + \cancel{e^{-6x}}^{0}} = \boxed{1}$

(e) $\lim_{x\to 3^-} \left(\frac{1}{7}\right)^{\frac{x}{3-x}} = \lim_{t\to\infty} \left(\frac{1}{7}\right)^{t} = \boxed{0}$, since $\frac{1}{7} < 1$

sub. $t = \frac{x}{3-x}$

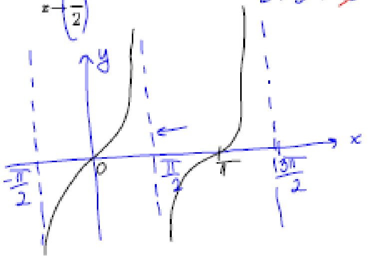
$\lim_{x\to 3^-} \frac{x}{3-x} = \infty$

$(x<3)$

$\lim_{x\to 3^+} \left(\frac{1}{7}\right)^{\frac{x}{3-x}} = \lim_{t\to-\infty} \left(\frac{1}{7}\right)^{t} = \boxed{\infty}$

$\lim_{x\to 3^+} \frac{x}{3-x} = -\infty$

(f) $\lim_{x \to \left(\frac{\pi}{2}\right)^+} \frac{2}{1+e^{\tan x}}$ $= \lim_{t \to -\infty} \frac{2}{1+\cancel{e^t}\,0}$ $= \boxed{2}$



$\lim_{x \to \left(\frac{\pi}{2}\right)^+} \tan x = -\infty$

$t = \tan x$

$\lim_{t \to -\infty} e^t = 0$

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\tan x}{x}=1,\quad \lim_{x\to 0}\cos x=1,\quad \lim_{x\to 0}\sin x=0,\quad \lim_{x\to 0}\frac{\cos x-1}{x}=0$$

(g) $\displaystyle\lim_{t\to 0}\frac{\sin^2 3t}{t^2\cos 5t}$ $\displaystyle=\lim_{t\to 0}\left(\frac{\sin 3t}{t}\right)^2\cdot\cancel{\lim_{t\to 0}\frac{1}{\cos 5t}}^{\,1}$

$$=\lim_{t\to 0}\left(\frac{3\sin 3t}{3t}\right)^2=\lim_{t\to 0}3^2\left(\cancel{\frac{\sin 3t}{3t}}^{\,1}\right)^2=3^2=\boxed{9}$$

$$\lim_{x\to 0}\frac{\tan x}{x}=1$$

(h) $\displaystyle\lim_{x\to 0}\frac{\tan \boxed{3}x}{\tan \boxed{5}x}$ $\displaystyle=\lim_{x\to 0}\frac{3\tan 3x}{3x}\cdot\frac{5x}{5\tan 5x}=\frac{3}{5}\lim_{x\to 0}\cancel{\frac{\tan 3x}{3x}}^{\,1}\cdot\cancel{\frac{5x}{\tan 5x}}^{\,1}=\boxed{\frac{3}{5}}$

(i) $\displaystyle\lim_{x\to -2}\frac{\tan \pi x}{x+2}$

$t=x+2,\quad x=t-2$

$$\lim_{x\to -2}t=\lim_{x\to -2}(x+2)=0$$

$$=\lim_{t\to 0}\frac{\tan \pi(t-2)}{t}=\lim_{t\to 0}\frac{\tan[\pi t-2\pi]}{t}=\lim_{t\to 0}\frac{\pi\cancel{\tan \pi t}^{\,1}}{\cancel{\pi t}}=\boxed{\pi}$$

$$\boxed{(e^x)' = e^x} \quad \boxed{\left(e^{u(x)}\right)' = u'(x)e^{u(x)}}$$

5. Find an equation of the tangent line to the curve $2e^{xy} = x + y$ at the point $(0,2)$.

$$2\frac{d}{dx}\left(e^{xy}\right) = \frac{d}{dx}(x+y)$$

Product Rule

$$2e^{xy}\frac{d}{dx}(xy) = 1 + y'$$

$$2e^{xy}\left(y + xy'\right) = 1 + y'$$

$$2ye^{xy} + 2xe^{xy}y' = 1 + y'$$

$$2xe^{xy}y' - y' = 1 - 2ye^{xy}$$

$$y'\left(2xe^{xy} - 1\right) = 1 - 2ye^{xy}$$

$$y' = \frac{1-2ye^{xy}}{2xe^{xy}-1}, \quad y'(0,2) = \frac{1-2\cdot 2\cdot e^{0\cdot 2}}{2\cdot 0\cdot e^{0\cdot 2}-1}$$

$$= \frac{1-4}{-1} = \frac{-3}{-1} = 3$$

Tangent line: $y = 3(x-0)+2$

$$\boxed{y = 3x+2}$$

6. Find the values of a and b that make the function

$$f(x) = \begin{cases} ax^2 + x + 1, & \text{if } x \le 1 \\ bx - 1, & \text{if } x > 1 \end{cases} \quad , \quad f'(x) = \begin{cases} 2ax+1, & \text{if } x \le 1 \\ b, & \text{if } x > 1 \end{cases}$$

differentiable everywhere. Find $f'(x)$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$ ($f(x)$ should be continuous everywhere)

$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x)$ [$f'(x)$ should be continuous for all x]

$$\left.\begin{aligned} &\lim_{x\to 1^-} f(x) = \lim_{x\to 1} (ax^2+x+1) = a+1+1 = a+2 \\ &\lim_{x\to 1^+} f(x) = \lim_{x\to 1} (bx-1) = b-1 \end{aligned}\right\} \Rightarrow a+2 = b-1$$

$$\left.\begin{aligned} &\lim_{x\to 1^-} f'(x) = 2a+1 \\ &\lim_{x\to 1^+} f'(x) = b \end{aligned}\right\} \quad b = 2a+1$$

$$\begin{cases} a+2 = b-1 \\ b = 2a+1 \end{cases} \Rightarrow \begin{aligned} a+2 &= (2a+1)-1 \\ a+2 &= 2a \end{aligned} \Rightarrow \boxed{a=2}$$

$$b = 2(2)+1 = \boxed{5=b}$$

$$\boxed{f'(x) = \begin{cases} 4x+1, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}}$$

7. If $f(x) = \sin(g(x))$, find $f'(2)$ given that $g(2) = \frac{\pi}{3}$ and $g'(2) = \frac{\pi}{4}$.

$$f'(x) = (\sin g(x))' = \cos(g(x)) \cdot g'(x)$$

$$f'(2) = \cos(g(2)) \cdot g'(2)$$

$$= \cos\frac{\pi}{3} \cdot \frac{\pi}{4}$$

$$= \frac{1}{2} \cdot \frac{\pi}{4} = \boxed{\frac{\pi}{8}}$$

8. At what point on the curve $f(x) = 36\sqrt{x}$ is the tangent line parallel to the line $9x - y + 2 = 0$?

$y = 9x + 2$, slope $m = 9$

Find x such that $f'(x) = 9$

$f'(x) = 36\left(\frac{1}{2}\right)x^{-1/2} = \underbrace{\frac{18}{\sqrt{x}} = 9}$

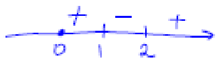
$\frac{2}{\sqrt{x}} = 1$ or $\sqrt{x} = 2$

$\boxed{x = 4}$

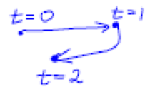
9. An object is moving along a straight path. The position of the object at time t is given by $s(t) = 2t^3 - 9t^2 + 12t + 1$, where t is measured in seconds and $s(t)$ is measured in feet. Find the total distance traveled in the first 2 seconds.

$v(t) = 6t^2 - 18t + 12 > 0$

$(t-1)(t-2) > 0$



$t=3 \quad (3-1)(3-2) > 0$



$t=1.5 \quad (1.5-1)(1.5-2) < 0$

$t=0.5 \quad (0.5-1)(0.5-2) > 0$

$v(t) > 0$ on $[0,1)$

$v(t) < 0$ on $[1,2]$

$$\text{total distance} = \underbrace{|s(1)-s(0)|}_{\text{distance traveled when } v(t)>0} + |\overbrace{s(2)-s(1)}^{\text{distance traveled when } v(t)<0}|$$

$$= |\overbrace{2-9+12+\cancel{1}}^{s(1)} \overbrace{-\cancel{1}}^{s(0)}| + |\underbrace{2(2)^3 - 9(2^2) + 12(2) + \cancel{1}}_{s(2)} - \underbrace{(2-9+12+\cancel{1})}_{s(1)}|$$

$$= 5 + |8 - 36 + 24 - 5| = 5 + |-9| = 5 + 9 = \boxed{14}$$

10. Find the derivative

(a) $f(x) = x^2 \cot(3x)$ Product Rule

$$f'(x) = (x^2)' \cot(3x) + x^2 (\cot(3x))'$$
$$= 2x \cot(3x) + x^2(-\csc^2(3x))(3x)'$$
$$= \boxed{2x \cot(3x) - x^2 \csc^2(3x)(3)}$$

(b) $f(x) = \dfrac{e^{\sqrt{x}}}{x + \sqrt{x}}$ Quotient Rule

$$f'(x) = \frac{(e^{\sqrt{x}})'(x+\sqrt{x}) - e^{\sqrt{x}}(x+\sqrt{x})'}{(x+\sqrt{x})^2} = \frac{e^{\sqrt{x}}(\sqrt{x})'(x+\sqrt{x}) - e^{\sqrt{x}}\left(1 + \frac{1}{2}x^{-1/2}\right)}{(x+\sqrt{x})^2}$$
$$= \frac{e^{\sqrt{x}}\left(\frac{1}{2}x^{-1/2}\right)(x+\sqrt{x}) - e^{\sqrt{x}} - \frac{1}{2}e^{\sqrt{x}}x^{-1/2}}{(x+\sqrt{x})^2}$$
$$= \frac{\frac{1}{2}e^{\sqrt{x}}\overbrace{x^{-1/2}x}^{x^{1/2}} + \frac{1}{2}e^{\sqrt{x}}\cancel{x^{-1/2}}\cancel{\sqrt{x}} - e^{\sqrt{x}} - \frac{1}{2}e^{\sqrt{x}}x^{-1/2}}{(x+\sqrt{x})^2}$$
$$= \frac{\frac{1}{2}e^{\sqrt{x}}x^{1/2} + \frac{1}{2}e^{\sqrt{x}} - e^{\sqrt{x}} - \frac{1}{2}e^{\sqrt{x}}x^{-1/2}}{(x+\sqrt{x})^2}$$
$$= \frac{e^{\sqrt{x}}x^{1/2} - e^{\sqrt{x}} - e^{\sqrt{x}}x^{-1/2}}{2(x+\sqrt{x})^2} = \boxed{\frac{e^{\sqrt{x}}\left(x^{1/2} - 1 - x^{-1/2}\right)}{2(x+\sqrt{x})^2}}$$

(c) $f(x) = \tan^3(e^{-x} + ex - x^e)$

$$f'(x) = 3\tan^2(e^{-x}+ex-x^e)\left(\tan(e^{-x}+ex-x^e)\right)'$$
$$= 3\tan^2(e^{-x}+ex-x^e)\sec^2(e^{-x}+ex-x^e)(e^{-x}+ex-x^e)'$$
$$= 3\tan^2(e^{-x}+ex-x^e)\sec^2(e^{-x}+ex-x^e)\left(e^{-x}(-x)'+e-ex^{e-1}\right)$$
$$\boxed{= 3\tan^2(e^{-x}+ex-x^e)\sec^2(e^{-x}+ex-x^e)\left(-e^{-x}+e-ex^{e-1}\right)}$$

(d) $f(x) = \left(\dfrac{x^3+3x}{x^2-4x+1}\right)^{5/2}$

$$f'(x) = \frac{5}{2}\left(\frac{x^3+3x}{x^2-4x+1}\right)^{3/2}\left(\frac{x^3+3x}{x^2-4x+1}\right)'$$
$$= \boxed{\frac{5}{2}\left(\frac{x^3+3x}{x^2-4x+1}\right)^{3/2}\frac{(3x^2+3)(x^2-4x+1)-(2x-4)(x^3+3x)}{(x^2-4x+1)^2}}$$

11. Find $f^{(58)}(x)$ if $f(x) = e^{-2x} + \cos(3x)$.

$f'(x) = -2e^{-2x} - 3\sin(3x)$

$f''(x) = 4e^{-2x} - 9\cos(3x)$

$f'''(x) = -8e^{-2x} + 27\sin(3x)$

$f^{iv}(x) = 16e^{-2x} + 81\cos(3x)$

$58 = \overbrace{56}^{14 \cdot 4} + \underbrace{2}$

the remainder is 2

$f^{(58)}(x)$ is going to be similar to $f''(x)$.

$$f^{(58)}(x) = (-2)^{58} e^{-2x} - 3^{58}\cos(3x)$$
$$= \boxed{2^{58} e^{-2x} - 3^{58}\cos(3x)}$$

12. The vector function $\mathrm{r}(t) =< t + e^{4t}, -t\cos(2t) >$, $0 \le t \le 2\pi$, represents the position of a particle at time t. Find the velocity and acceleration vectors of the object at $t = \dfrac{\pi}{4}$.

$$\vec{v}(t) = \vec{r}\,'(t) = < 1 + 4e^{4t}, -\cos(2t) - t\underbrace{(-\sin(2t))(2t)'}_{[\cos(2t)]'} >$$

$$= < 1 + 4e^{4t}, -\cos(2t) + t\sin(2t)(2) >$$

$$\vec{v}\left(\tfrac{\pi}{4}\right) = < 1 + 4e^{4\cdot\frac{\pi}{4}}, -\cos\left(2\cdot\tfrac{\pi}{4}\right) + 2\cdot\tfrac{\pi}{4}\cdot\sin\left(2\tfrac{\pi}{4}\right) >$$

$$\boxed{\vec{v}\left(\tfrac{\pi}{4}\right) = < 1 + 4e^{\pi}, \tfrac{\pi}{2} >}$$

$$\vec{a}(t) = \vec{v}\,'(t) = < 4(4e^{4t}), -\sin(2t)(2t)' + 2\sin(2t) + 2t\cos(2t)(2t)' >$$

$$= < 16e^{4t}, -2\cancel{\sin(2t)} + 2\cancel{\sin(2t)} + 4t\cos(2t) >$$

$$= < 16e^{4t}, 4t\cos(2t) >$$

$$\vec{a}\left(\tfrac{\pi}{4}\right) = < 16e^{4\cdot\frac{\pi}{4}}, 4\tfrac{\pi}{4}\cos\left(2\cdot\tfrac{\pi}{4}\right) >$$

$$\boxed{\vec{a}\left(\tfrac{\pi}{4}\right) = < 16e^{\pi}, 0 >}$$

13. At what point(s) does the curve parametrized by $x = t^2 - 6t + 5$, $y = t^2 + 4t + 3$ have a horizontal or vertical tangent?

horizontal tangent: $y'(t) = 0$

$y'(t) = 2t + 4 = 0 \Rightarrow t = -2$

$x(2) = (2)^2 - 6(2) + 5$
$= 4 + 12 + 5 = 21$
$y(2) = (-2)^2 + 4(-2) + 3$
$= 4 - 8 + 3 = -1$

$\boxed{H.T. \text{ @ } (21, -1)}$

vertical tangent: $x'(t) = 0$

$x'(t) = 2t - 6 = 0, \quad t = 3$

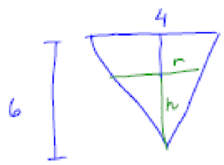
$x(3) = 9 - 18 + 5 = -4$
$y(3) = 9 + 12 + 3 = 24$

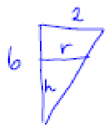
$\boxed{V.T. \text{ @ } (-4, 24)}$

14. Water is leaking out of an inverted conical tank at a rate of 1 m^3/min. The tank has height 6 m and the diameter at the top is 4 m. At what rate is the water level changing when the height of the water is 3 m?

V is the volume of the water in the tank

$$\frac{dV}{dt} = -1 \, (m^3/min)$$





h is the height of the water

r is the radius of the cone at height h.

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{6}{2} = \frac{h}{r} \Rightarrow r = \frac{h}{3}$$

$$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h$$

$$\frac{d}{dt} V = \frac{\pi}{27} \frac{d}{dt}(h^3)$$

$$\frac{dV}{dt} = \frac{\pi}{27}(3h^2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{\pi h^2}\frac{dV}{dt}$$

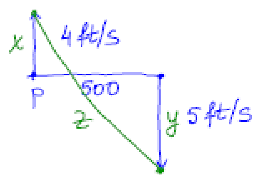
$$= \frac{9}{\pi(3^2)}(-1) = \boxed{-\frac{1}{\pi} \ m/min}$$

15. A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 min after woman starts walking.

x distance traveled by the man

y distance traveled by the woman

z is the distance between the people



$$\frac{dx}{dt} = 4$$

$$\frac{dy}{dt} = 5$$

$$x = 4(20)(60) = 4800$$

$$y = 5(15)(60) = 4500$$

$$z = \sqrt{500^2 + (4800 + 4500)^2}$$

$$= 100\sqrt{8674}$$

$$\frac{d}{dt}(z^2) = \frac{d}{dt}\left(500^2 + (x+y)^2\right)$$

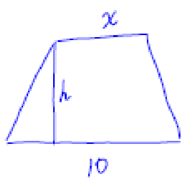
$$\cancel{2} z \frac{dz}{dt} = \cancel{2}(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$\frac{dz}{dt} = \frac{48\cancel{00} + 45\cancel{00}}{\cancel{100}\sqrt{8674}}(4+5)$$

$$= \frac{(93)(9)}{\sqrt{8674}} = \boxed{\frac{837}{\sqrt{8674}} \text{ ft/s}}$$

16. A trapezoid has a base with length 10 inches. The length of the top of the trapezoid is decreasing at a rate of 1 in/min while the height of the trapezoid is increasing at a rate of 2 in/min. At what rate is the area of the trapezoid changing when the height is 3 inches and the area of the trapezoid is 24 in^2?



$24 = A = \frac{1}{2}(x+10)(3)$

$48 = 3(x+10)$

$16 = x+10$

$x = 6$

$\frac{dx}{dt} = -1, \quad \frac{dh}{dt} = 2$

Find $\frac{dA}{dt}$ if $h = 3, \; A = 24$

$A = \frac{1}{2}(x+10)h = \frac{1}{2}xh + 5h$

$\frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt}h + \frac{dh}{dt}x\right) + 5\frac{dh}{dt}$

$\frac{dA}{dt} = \frac{1}{2}\left((-1)(3) + 2(6)\right) + 5(2)$

$= 10 + \frac{9}{2} = \boxed{\frac{29}{2} \text{ in}^2/\text{min}}$