1. Show that the function is one-to-one and find the inverse.

(a) $f(x) = \dfrac{1+3x}{5-2x}$ There are x_1, x_2 such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$

$f(x_1) = \dfrac{1+3x_1}{5-2x_1}$, $f(x_2) = \dfrac{1+3x_2}{5-2x_2}$, $f(x_1) = f(x_2) \Rightarrow \dfrac{1+3x_1}{5-2x_1} = \dfrac{1+3x_2}{5-2x_2} \Leftrightarrow \begin{cases} \cancel{1}+\dfrac{3x_1}{3} = \cancel{1}+\dfrac{3x_2}{3} \Rightarrow x_1 = x_2 \\ \cancel{5}\dfrac{-2x_1}{-2} = \cancel{5}\dfrac{-2x_2}{-2} \Rightarrow x_1 = x_2 \end{cases}$

Contradiction with the initial assumption.

Thus, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$, so f is one-to-one

$(5-2x)\, y = \dfrac{1+3x}{5-2x}(5-2x)$ solve for x.

$(5-2x)y = 1+3x$

$5y - 2xy = 1 + 3x$

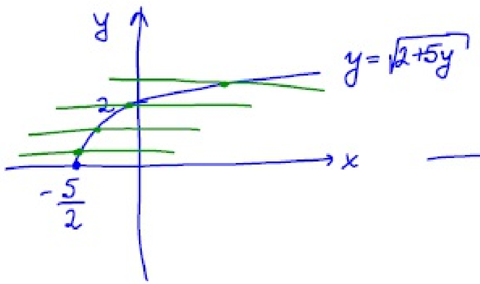
$5y - 1 = 3x + 2xy$

$5y - 1 = x(3+2y)$

$x = \dfrac{5y-1}{3+2y} = f^{-1}(y)$

$\boxed{f^{-1}(x) = \dfrac{5x-1}{3+2x}}$

(b) $g(x) = \sqrt{2+5x}$



only one point of intersection for a horizontal line and the graph of $y = \sqrt{2+5x}$

Thus, $y = \sqrt{2+5x}$ is one-to-one

$(y)^2 = (\sqrt{2+5x})^2$

$y^2 = 2 + 5x$

$5x = y^2 - 2$

$x = \frac{1}{5}(y^2 - 2) = g^{-1}(y)$

$\boxed{g^{-1}(x) = \frac{1}{5}(x^2 - 2)}$

$\boxed{g'(x) = \frac{1}{f'(g(x))}, \; g = f^{-1}}$ $\boxed{g'(a) = \frac{1}{f'(g(a))}}$

2. Find $g'(a)$, where g is the inverse function of the given function.

(a) $f(x) = x + x^2 + e^x$ at $a = 1$. $\;, f'(x) = 1 + 2x + e^x$

$g'(1) = \frac{1}{f'(g(1))}$ | $g(1) = ?$

$g(1) = b$, then $f(b) = 1$.

Find b such that $b + b^2 + e^b = 1$

plug $\underline{b=0}$: $0 + 0 + e^0 = 1$

$f(0) = 1 \Leftrightarrow g(1) = 0$

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{1+2(0)+e^0} = \boxed{\frac{1}{2}}$$

(b) $f(x) = 3 + x^2 + \tan\left(\frac{\pi x}{2}\right)$, $-1 < x < 1$ at $a = 3$.

$f'(x) = 2x + \sec^2\left(\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)$

$g'(3) = \frac{1}{f'(g(3))}$

| Find b such that $g(3) = b$ or $f(b) = 3$

$f(b) = 3 + b^2 + \tan\left(\frac{b\pi}{2}\right)$

plug $b = 0$: $f(0) = 3 + 0 + \tan 0 = 3$

$f(0) = 3 \Leftrightarrow g(3) = 0$

$$g'(3) = \frac{1}{f'(0)} = \frac{1}{2(0) + \sec^2(0)\left(\frac{\pi}{2}\right)} = \frac{1}{\cancel{\frac{1}{\cos^2 0}}\,1 \cdot \frac{\pi}{2}} = \boxed{\frac{2}{\pi}}$$

$\log_a(xy) = \log_a x + \log_a y \mid \log_a \frac{x}{y} = \log_a x - \log_a y \mid \log_a x^k = k \log_a x \mid \log_a a = 1$, $a^{\log_a x} = x$

3. Evaluate the following.

$\log = \log_{10}$, $\ln = \log_e$

(a) $\log_3 3^{\sqrt{5}} + \log .0001 + \ln e^4$

(a) $\log_3 3^{\sqrt{5}} + \log .0001 + \ln e^4$

$.0001 = 10^{-4}$

$= \sqrt{5} \cancel{\log_3 3}^{1} + \log 10^{-4} + 4 \cancel{\ln e}^{1}$

$= \sqrt{5} - 4 \cancel{\log 10}^{1} + 4 \quad = \sqrt{5} - 4 + 4 = \boxed{\sqrt{5}}$

(b) $2^{\log_2 3 + \log_2 5} = \cancel{2^{\log_2 3}}^{3} \cdot \cancel{2^{\log_2 5}}^{5} = (3)(5) = \boxed{15}$

4. Express the given quantity as a single logarithm.

(a) $\log_2 x + 5\log_2(x+1) + \frac{1}{2}\log_2(x-1)$

$$= \log_2 x + \log_2 (x+1)^5 + \log_2 (x-1)^{1/2}$$

$$= \boxed{\log_2 \left[x(x+1)^5 (x-1)^{1/2} \right]}$$

(b) $\frac{1}{3}\ln x - 4\ln(2x+3) = \ln x^{1/3} - \ln (2x+3)^4$

$$= \boxed{\ln \frac{x^{1/3}}{(2x+3)^4}}$$

5. Find the domain of the function $f(x) = \sqrt{x}\ln(x^2-1)$.

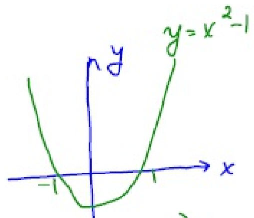
Domain: $\ln(x^2-1) \longrightarrow x^2-1>0$

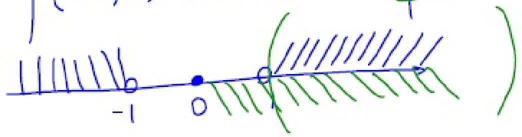
$\sqrt{x} \longrightarrow x \geq 0$

$x^2-1>0$

$(x-1)(x+1)>0$

$(-\infty,-1)\cup(1,\infty)$





Domain $\boxed{(1,\infty)}$.

$a>1:$ $\lim_{x\to\infty} \log_a(x) = \infty$, $\lim_{x\to 0^+} \log_a x = -\infty$ | $0<a<1:$ $\lim_{x\to\infty} \log_a x = -\infty$, $\lim_{x\to 0^+} \log_a x = \infty$

6. Find the limit.

(a) $\lim_{x\to -2^+} \log_2\left(\frac{x-3}{x+2}\right) = \boxed{DNE}$

$\lim_{x\to -2^+} \frac{x-3}{x+2} = \frac{-}{+} = -\infty$ $\quad x=-1.99$

the function is not defined for $\frac{x-3}{x+2} < 0$

(b) $\lim_{x\to\infty} \ln\frac{4}{x+2} = \lim_{y\to 0^+} \ln y = \boxed{-\infty}$

$\lim_{x\to\infty} \frac{4}{x+2} = 0$, $y = \frac{4}{x+2}$

(c) $\lim_{x\to\infty} [\ln(x+2) - \ln(1+x)] = \lim_{x\to\infty} \ln \frac{x+2}{x+1} = \ln 1 = \boxed{0}$

$\lim_{x\to\infty} \frac{x+2}{x+1} = 1$

7. Solve the equation for x.

(a) $2^{x-5} = 8$

$2^{x-5} = 2^3 \iff x - 5 = 3$

$\boxed{x = 8}$

(b) $3^{3x-4} = 2$

$\log_3 3^{3x-4} = \log_3 2$

$(3x-4)\log_3 3 = \log_3 2$

$3x - 4 = \log_3 2$

$\boxed{x = \frac{1}{3}(4 + \log_3 2)}$

(c) $\log_2 x = 3$

$x = 2^3 = \boxed{8}$

(d) $\ln(x+6) + \ln(x-3) = \ln 5 + \ln 2$

$\ln (x+6)(x-3) = \ln(5 \cdot 2)$

$(x+6)(x-3) = 10$

$x^2 - 3x + 6x - 18 - 10 = 0$

$x^2 + 3x - 28 = 0$

$(x+7)(x-4) = 0$

$x = -7$ or $\boxed{x = 4}$

not in the domain

Domain: $x+6 > 0 \Rightarrow x > -6$

$x - 3 > 0 \Rightarrow x > 3$

$\boxed{(3, \infty)}$

(e) $2^{3^x} = 5$

$$\log_2\left(2^{3^x}\right) = \log_2 5$$

$$3^x \cancel{\log_2 2}^{1} = \log_2 5$$

$$\log_3 3^x = \log_3(\log_2 5)$$

$$x \cancel{\log_3 3}^{1} = \log_3(\log_2 5)$$

$$\boxed{x = \log_3(\log_2 5)}$$

8. Find the inverse function.

(a) $y = \ln(x+3)$

$e \quad e$

$e^y = x+3 \quad \Rightarrow \quad x = e^y - 3 = f^{-1}(y)$

$$\boxed{f^{-1}(x) = e^x - 3}$$

(b) $y = 2^{10^x}$

$\log_2 y = \log_2 \left(2^{10^x}\right)$

$\log_2 y = 10^x \cancel{\log_2 2}^{\,1}$

$\log(\log_2 y) = \log(10^x)$

$\log(\log_2 y) = x \cancel{\log 10}^{\,1}$

$x = \log(\log_2 y) = f^{-1}(y)$

$$\boxed{f^{-1}(x) = \log(\log_2 x)}$$

(c) $\left[y = \dfrac{1+e^x}{1-e^x}\right](1-e^x)$

$(1-e^x)y = 1+e^x$

$y - ye^x = 1 + e^x$

$y - 1 = e^x + ye^x$

$y - 1 = e^x(1+y)$

$\ln(e^x) = \ln\left(\dfrac{y-1}{1+y}\right)$

$x = \ln\left(\dfrac{y-1}{1+y}\right) = f^{-1}(y)$

$$\boxed{f^{-1}(x) = \ln\left(\frac{x-1}{1+x}\right)}$$

$(\ln x)' = \frac{1}{x}$, $(\log_a x)' = \frac{1}{x \ln a}$, $(e^x)' = e^x$, $(a^x)' = a^x \ln a$

9. Find the derivative.

(a) $f(x) = \log(x^2 - x)$

$$f'(x) = \frac{1}{(x^2-x)\ln 10}(x^2-x)'$$

$$= \boxed{\frac{1}{(x^2-x)\ln 10}(2x-1)}$$

(b) $f(x) = 3^{\sin x}$

$$f'(x) = 3^{\sin x} \ln 3 \, (\sin x)'$$

$$= \boxed{3^{\sin x}(\ln 3)(\cos x)}$$

(c) $f(x) = x\sqrt{\ln x}$ $= x(\ln x)^{1/2}$

Product Rule:

$$f'(x) = (x)'(\ln x)^{1/2} + x[(\ln x)^{1/2}]'$$

$$= (\ln x)^{1/2} + x \frac{1}{2}(\ln x)^{-1/2}(\ln x)'$$

$$= (\ln x)^{1/2} + \cancel{x}\frac{1}{2}(\ln x)^{-1/2}\frac{1}{\cancel{x}} = (\ln x)^{1/2} + \frac{1}{2(\ln x)^{1/2}}$$

$$= \boxed{\sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}}}$$

(d) $f(x) = \ln(\ln(3x+1))$

$$f'(x) = \frac{1}{\ln(3x+1)}(\ln(3x+1))'$$

$$= \frac{1}{\ln(3x+1)} \cdot \frac{1}{3x+1}(3x+1)' = \boxed{\frac{3}{(3x+1)\ln(3x+1)}}$$

$(\ln|x|)' = \frac{1}{x}$

(e) $f(x) = \ln\left|\frac{x^2-4}{2x+5}\right| = \ln|x^2-4| - \ln|2x+5|$

$f'(x) = \frac{1}{x^2-4}(x^2-4)' - \frac{1}{2x+5}(2x+5)'$

$= \boxed{\frac{2x}{x^2-4} - \frac{2}{2x+5}}$

(f) $f(x) = (\cos x)^{\sin x}$ Logarithmic differentiation

$\ln f(x) = \ln(\cos x)^{\sin x}$

$\frac{d}{dx}(\ln f(x)) = \frac{d}{dx}(\sin x \ln(\cos x))$

$\frac{f'(x)}{f(x)} = \cos x \ln(\cos x) + \sin x(\ln(\cos x))'$

$\frac{f'(x)}{f(x)} = \cos x \ln(\cos x) + \sin x \cdot \frac{1}{\cos x}(\cos x)'$

$\frac{f'(x)}{f(x)} = \cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}$

$f'(x) = f(x)\left[\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right]$

$\boxed{f'(x) = (\cos x)^{\sin x}\left[\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right]}$

(g) $f(x) = \dfrac{\sqrt{x+1}\,(2-x^4)^5}{(x+3)^7(x^3-2x+1)^{10}}$

$$\ln f(x) = \ln \frac{\sqrt{x+1}\;(2-x^4)^5}{(x+3)^7\;(x^3-2x+1)^{10}}$$

$$= \ln\left[\sqrt{x+1}\;(2-x^4)^5\right] - \ln\left[(x+3)^7(x^3-2x+1)^{10}\right]$$

$$= \ln\sqrt{x+1} + \ln(2-x^4)^5 - \ln(x+3)^7 - \ln(x^3-2x+1)^{10}$$

$$\frac{d}{dx}\left(\ln f(x)\right) \frac{d}{dx}\left(\frac{1}{2}\ln(x+1) + 5\ln(2-x^4) - 7\ln(x+3) - 10\ln(x^3-2x+1)\right)$$

$$\frac{f'(x)}{f(x)} = \frac{1}{2}\,\frac{1}{x+1} + 5\,\frac{1}{2-x^4}\,(2-x^4)' - \frac{7}{x+3} - \frac{10}{x^3-2x+1}\,(x^3-2x+1)'$$

$$\frac{f'(x)}{f(x)} = \frac{1}{2(x+1)} + \frac{5(-4x^3)}{2-x^4} - \frac{7}{x+3} - \frac{10(3x^2-2)}{x^3-2x+1}$$

$$f'(x) = f(x)\left[\frac{1}{2(x+1)} - \frac{20x^3}{2-x^4} - \frac{7}{x+3} - \frac{10(3x^2-2)}{x^3-2x+1}\right]$$

$$\boxed{f'(x) = \frac{\sqrt{x+1}\;(2-x^4)^5}{(x+3)^7\;(x^3-2x+1)^{10}}\left[\frac{1}{2(x+1)} - \frac{20x^3}{2-x^4} - \frac{7}{x+3} - \frac{10(3x^2-2)}{x^3-2x+1}\right]}$$

10. A bacteria culture starts with 1000 bacteria and the growth rate is proportional to the number of bacteria. After 2 h the population is 9000.

(a) Find an expression for the number of bacteria after t hours.

(b) Find the number of bacteria after 3 h.

(c) In what period of time does the number of bacteria double?

$p(t)$ population of bacteria after t hours.

$\frac{dp}{dt} = kp(t)$, $p(t) = p(0)e^{kt}$, k is an unknown const

$p(0) = 1000$, $p(2) = 9000$

$p(t) = 1000e^{kt}$

$p(2) = \cancel{1000}e^{2k} = 9\cancel{000} \Rightarrow e^{2k} = 9 \Rightarrow 2k = \ln 9 \Rightarrow k = \frac{\ln 9}{2}$

$e^{\ln 9} = 9$

$9^{\frac{t}{2}} = (\sqrt{9})^t = 3^t$

$p(t) = 1000e^{\frac{\ln 9}{2}t} = 1000(9^{\frac{t}{2}})$

$p(t) = 1000(3^t)$

$p(3) = 1000(3^3) = \boxed{27000 = p(3)}$

Find t such that $p(t) = 2p(0) = 2000$

$2000 = 1000(3^t)$

$2 = 3^t \Rightarrow \ln 2 = t\ln 3$

$t = \frac{\ln 3}{\ln 2} \approx \boxed{1.58 \text{ (hours)}}$

11. An isotope of strontium, Sr^{90}, has a half-life of 25 years.

(a) Find the mass of Sr^{90} that remains from a sample of 18 mg after t years.

(b) How long will it take for the mass to decay to 2 mg?

$\frac{dm}{dt} = km$

$m(t)$ is the mass after t years

$m(t) = m(0)e^{kt}$, $m(0) = 18$

$m(25) = \frac{m(0)}{2} = \frac{18}{2} = 9$

$m(t) = 18e^{kt}$

$m(25) = 18e^{25k} = 9$

$e^{25k} = \frac{1}{2}$

$25k = \ln\frac{1}{2} = -\ln 2$

$k = -\frac{\ln 2}{25}$

$m(t) = 18e^{-t\frac{\ln 2}{25}}$

$\boxed{m(t) = 18\left(2^{-\frac{t}{25}}\right)}$

Find t such that $m(t) = 2$

$\frac{2}{18} = \frac{18}{18}\left(2^{-\frac{t}{25}}\right)$

$\frac{1}{9} = 2^{-\frac{t}{25}}$

$\ln\frac{1}{9} = \ln 2^{-\frac{t}{25}}$

$-\ln 9 = -\frac{t}{25}\ln 2$

$t = \frac{25\ln 9}{\ln 2} \approx \boxed{79 \text{ (years)}}$

12. A cup of coffee has a temperature of 200°F and is in a room that has a temperature of 70°F. After 10 min the temperature of the coffee is 150°F.

(a) What is the temperature of the coffee after 15 min?

(b) When will the coffee have cooled to 100°F?

$T(t)$ - temperature of the coffee after t min

Newton's Law of cooling:

$$\frac{dT}{dt} = k\,[70 - T(t)]$$

$$T(0) = 200, \quad T(10) = 150.$$

$$u(t) = 70 - T(t)$$

$$\frac{du}{dt} = -\frac{dT}{dt}$$

$$u(0) = 70 - 200 = -130$$

$u(t)$:
$$-\frac{du}{dt} = ku$$

$$\frac{du}{dt} = -ku$$

$$u(t) = u(0)e^{-kt}$$

$$u(t) = -130e^{-kt}$$

$$70 - T(t) = -130e^{-kt}$$

$$T(t) = 70 + 130e^{-kt}$$

$$T(10) = 70 + 130e^{-k(10)} = 150$$

$$130e^{-k(10)} = 80 \Rightarrow e^{-k(10)} = \frac{8}{13}$$

$$\ln\frac{8}{13} = -10k \Rightarrow k = \frac{1}{10}\ln\frac{8}{13} \approx -0.048$$

$$\boxed{T(t) = 70 + 130e^{\frac{t}{10}\ln\frac{8}{13}}}$$

$$T(15) = 70 + 130e^{\frac{15}{10}\ln\frac{8}{13}} \approx 133°\text{F}$$