10.7: Taylor and Maclaurin Series

- The Taylor series for \( f(x) \) about \( x = a \):
  \[
  f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \]
  \[
  = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots
  \]

- The Maclaurin series is the Taylor series about \( x = 0 \) (i.e. \( a=0 \)):
  \[
  f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots
  \]

- Known Maclaurin series and their intervals of convergence you must have memorized:

  \[
  \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \quad (1,1)
  \]

  \[
  e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad (-\infty, \infty)
  \]

  \[
  \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad (-\infty, \infty)
  \]

  \[
  \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad (-\infty, \infty)
  \]

  \[
  \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad [-1,1]
  \]

Examples.

1. Given that function \( f \) has power series expansion (i.e. Taylor series) centered at \( a = \pi \). Find this expansion and its radius of convergence if it is given that

   \[
   f^{(n)}(\pi) = \frac{(-1)^n n!}{4^{2n+1}(2n+1)!}
   \]

2. Find the 20th derivative of \( f(x) = e^{x^2} \) at \( x = 0 \).

3. Find Taylor series for \( f(x) = e^{3x} \) centered at \( x = 1/3 \). What is the associated radius of convergence?

4. Find Taylor series for \( f(x) = \frac{1}{x} \) centered at \( x = 5 \). What is the associated interval of convergence?

5. Find Maclaurin series for the following functions:

   (a) \( f(x) = x^3 \sin x^5 \)
(b) \( f(x) = \sin^2 x \)
(c) \( x + 3x^2 + xe^{-x} \)

6. Express \( \int \frac{\sin(3x)}{x} \, dx \) as an infinite series.

7. Find the sum of the series:
   (a) \( \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!} \)
   (b) \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \)
   (c) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n + 1} \)

8. Use series to approximate the integral \( \int_0^{0.5} x^2 e^{-x^2} \, dx \) with error less than \( 10^{-3} \).

10.9: Applications of Taylor Polynomials

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x - a)^n + \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

\[\text{T}_N(x)\]
\[\text{Remainder}\]

Taylor polynomial

Examples.

9. Find the fourth-degree Taylor polynomial of \( f(x) = \frac{1}{2 + 6x} \) centered at \( a = 0 \).

10. Find the third-degree Taylor polynomial of \( f(x) = \sqrt{x} \) centered at \( a = 1 \).

11. Find the second degree Taylor Polynomial for \( f(x) = \ln x \) at \( a = 3 \).

11.1: Three-dimensional Coordinate System

- The distance between the points \( P(x_1, y_1, z_1) \) and \( Q(x_2, y_2, z_2) \) is
  \[|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \]

- Equation of a sphere \( (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \) (completing the square)

Examples.

12. Graph the following regions:
   (a) \( x = 5 \) in \( \mathbb{R}^2, \mathbb{R}^3 \); (b) \( x^2 + y^2 - 1 = 0 \) in \( \mathbb{R}^2, \mathbb{R}^3 \).

13. Given the sphere \( (x - 1)^2 + (y + 4)^2 + (z - 2)^2 = 16 \).
   (a) What is the intersection of the sphere with the \( yz \)-plane.
   (b) Find the distance from the point \( (1, -2, 3) \) to the center of the sphere.

14. What is the intersection of the surface \( x^2 + y^2 = 49 \) with the \( xy \)-plane.

15. Determine the radius and the center of the sphere given by the equation
    \[ x^2 + y^2 + z^2 + 2y + z - 1 = 0. \]