10.7: Taylor and Maclaurin Series

- The Taylor series for $f(x)$ about $x = a$:
  \[
  f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots
  \]

- The Maclaurin series is the Taylor series about $x = 0$ (i.e. $a=0$):
  \[
  f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots
  \]

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \quad (1, 1)
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad (-\infty, \infty)
\]

\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad (-\infty, \infty)
\]

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad (-\infty, \infty)
\]

\[
\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad [-1, 1]
\]

Examples.

1. Given that function $f$ has power series expansion (i.e. Taylor series) centered at $a = \pi$. Find this expansion and its radius of convergence if it is given that
   \[
   f^{(n)}(\pi) = \frac{(-1)^n n!}{4^{2n+1}(2n+1)!}
   \]
2. Find the 20th derivative of $f(x) = e^{x^2}$ at $x = 0$.

3. Find Taylor series for $f(x) = e^{3x}$ centered at $x = 1/3$. What is the associated radius of convergence?
4. Find Taylor series for $f(x) = \frac{1}{x}$ centered at $x = 5$. What is the associated interval of convergence?
5. Find Maclaurin series for the following functions:

(a) \( f(x) = x^3 \sin x^5 \)

(b) \( f(x) = \sin^2 x \)

(c) \( x + 3x^2 + xe^{-x} \)

6. Express \( \int \frac{\sin (3x)}{x} \, dx \) as an infinite series.
7. Find the sum of the series:

(a) \[ \sum_{n=0}^{\infty} \frac{(-1)^n n^{2n}}{4^{2n}(2n)!} \]

(b) \[ \sum_{n=0}^{\infty} \frac{7^n}{n!} \]

(c) \[ \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n + 1} \]

8. Use series to approximate the integral \[ \int_{0.5}^{0.5} x^2 e^{-x^2} \, dx \] with error less than \(10^{-3}\).
10.9: Applications of Taylor Polynomials

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \]

\[ T_N(x) \]

\[ N \text{- th degree Taylor polynomial} \]

\[ R_N(x) \]

\[ \text{Remainder} \]

Examples.

9. Find the fourth-degree Taylor polynomial of \( f(x) = \frac{1}{2 + 6x} \) centered at \( a = 0 \).

10. Find the third-degree Taylor polynomial of \( f(x) = \sqrt{x} \) centered at \( a = 1 \).

11. Find the second degree Taylor Polynomial for \( f(x) = \ln x \) at \( a = 3 \).
11.1: Three-dimensional Coordinate System

- The distance between the points \( P(x_1, y_1, z_1) \) and \( Q(x_2, y_2, z_2) \) is

\[
|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.
\]

- Equation of a sphere \((x - a)^2 + (y - b)^2 + (z - c)^2 = r^2\) (completing the square)

Examples.

12. Graph the following regions:
   (a) \( x = 5 \) in \( \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3 \);

   (b) \( x^2 + y^2 - 1 = 0 \) in \( \mathbb{R}^2, \mathbb{R}^3 \).

13. Given the sphere \((x - 1)^2 + (y + 4)^2 + (z - 2)^2 = 16\).
   (a) What is the intersection of the sphere with the \( yz \)-plane.

   (b) Find the distance from the point \((1, -2, 3)\) to the center of the sphere.
14. What is the intersection of the surface $x^2 + y^2 = 49$ with the $xy$-plane.

15. Determine the radius and the center of the sphere given by the equation

$$x^2 + y^2 + z^2 + 2y + z - 1 = 0.$$