10.5: Power Series

- For a given power series $\sum_{n=0}^{\infty} c_n (x - a)^n$ there are only 3 possibilities:

  1. There is $R > 0$ such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$. We call such $R$ the radius of convergence.
  2. The series converges only for $x = a$ (then $R = 0$).
  3. The series converges for all $x$ (then $R = \infty$).

- We find the radius of convergence using the **Ratio Test**.

- An interval of convergence is the interval of all $x$’s for which the power series converges.

- You must check the endpoints $x = a \pm R$ individually to determine whether or not they are in the interval of convergence.

1. For the following series find the radius and interval of convergence.

   (a) $\sum_{n=0}^{\infty} \frac{n^4 x^n}{7^n}$

   (b) $\sum_{n=0}^{\infty} \frac{8^n (x + 4)^{3n}}{n^3 + 1}$

   (c) $\sum_{n=1}^{\infty} \frac{(-9)^n (5x - 3)^n}{n}$

   (d) $\sum_{n=1}^{\infty} \frac{(n + 1)! (x - 1)^{n+1}}{4^{n+1}}$

   (e) $\sum_{n=0}^{\infty} \frac{(-6)^n x^n}{(3n + 1)!}$

2. Assume that it is known that the series $\sum_{n=0}^{\infty} c_n (x - 3)^n$ converges when $x = 5$ and diverges when $x = -2$. What can be said about the convergence or divergence of the following series:

   (a) $\sum_{n=0}^{\infty} c_n (-7)^n$

   (b) $\sum_{n=0}^{\infty} c_n 5^n$

   (c) $\sum_{n=0}^{\infty} c_n (-3)^n$

   (d) $\sum_{n=0}^{\infty} c_n 3^n$
10.6: Representation of Functions as Power Series

- Geometric Series Formula:
  \[ \frac{1}{1-x} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^{n}, \quad -1 < x < 1. \]

- Term-by-term Differentiation and Integration of power series:
  If \( \sum_{n=0}^{\infty} c_n(x-a)^n \) has radius of convergence \( R > 0 \), then \( f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \) is differentiable (and therefore continuous) on the interval \((a - R, a + R)\) and
  \[
  f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}
  \]
  \[
  \int f(x) \, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}
  \]
  The radii of convergence of the power series for \( f'(x) \) and \( \int f(x) \, dx \) are both \( R \).

3. Find a power series representation for the following functions and determine the interval of convergence.

   (a) \( f(x) = \frac{4}{1+x} \)
   (b) \( f(x) = \frac{4}{2+4x} \)
   (c) \( f(x) = \frac{-9}{9-x^4} \)
   (d) \( f(x) = \ln(3x+5) \)
   (e) \( f(x) = x^5 \ln(3x+5) \)
   (f) \( f(x) = \frac{x^4}{(1-4x)^2} \)
   (g) \( f(x) = \arctan(16x^4) \)

4. Express the integral \( \int_{-0.5}^{0} \frac{dx}{1-x^7} \) as a power series.