Chapter 14. **Vector calculus.**
Section 14.5 **Curl and divergence.**

Curl.

If $\vec{F} = P\vec{\imath} + Q\vec{\jmath} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R exist, then the **curl** of $\vec{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Let $\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$ be the vector differential operator.

$$\nabla f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}$$

Then

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \text{curl } \vec{F}.$$

Example 1. Find curl $\vec{F}$ if $\vec{F}(x, y, z) = xe^y\vec{\imath} - ze^{-y}\vec{\jmath} + y\ln(z)\vec{k}$.

Theorem 1. If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \vec{0}$$

Theorem 2. If $\vec{F}$ is a vector field defined on all on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\vec{F} = \vec{0}$, then $\vec{F}$ is a conservative vector field.

Example 2. Determine whether or not the vector field $\vec{F} = zx\vec{\imath} + xy\vec{\jmath} + yx\vec{k}$ is conservative. If it is conservative, find a function f such that $\vec{F} = \nabla f$.

The curl vector is associated with rotation. If for a vector field $\vec{F}$ curl $\vec{F} = \vec{0}$, then the field $\vec{F}$ is **irrotational**.

Example 3. Let B be a rigid body rotation about the z-axis. The rotation can be described by the vector $\vec{w} = \omega\vec{k}$, where ω is the angular speed of B, that is the tangential speed of any point P in B divided by the distance d from the axis of rotation. Let $\vec{r} =< x, y, z >$ be the position vector of P.

1. Show that velocity field of B is given by $\vec{v} = \vec{w} \times \vec{r}$

2. Show that $\vec{v} = -\omega y\vec{\imath} + \omega x\vec{\jmath}$

3. Show that curl $\vec{v} = 2\vec{w}$.

Divergence.

Definition. If $\vec{F} =< P, Q, R >$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z exist, then the **divergence** of $\vec{F}$ is the function of tree variables defined by

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Example 4. Find the divergence of the vector field $\vec{F}(x, y, z) = xe^y\vec{\imath} - ze^{-y}\vec{\jmath} + y\ln(z)\vec{k}$.

Theorem 3. If $\vec{F} = P\vec{\imath} + Q\vec{\jmath} + Q\vec{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order derivatives, then

$$\text{div curl } \vec{F} = 0$$

Divergence is a vector operator that measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar. If div $\vec{F} = 0$, then $\vec{F}$ is said to be **incompressible**.

Laplace operator:

$$\text{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$$

Properties of the curl and divergence.

If f is a scalar field and $\vec{F}$, $\vec{G}$ are vector fields, then $f\vec{F}$, $\vec{F} \cdot \vec{G}$, and $\vec{F} \times \vec{G}$ are vector fields defined by

$$(f\vec{F})(x, y, z) = f(x, y, z)\vec{F}(x, y, z)$$
$$(\vec{F} \cdot \vec{G})(x, y, z) = \vec{F}(x, y, z) \cdot \vec{G}(x, y, z)$$
$$(\vec{F} \times \vec{G})(x, y, z) = \vec{F}(x, y, z) \times \vec{G}(x, y, z)$$

and

1. $\text{div}(\vec{F} + \vec{G}) = \text{div } \vec{F} + \text{div } \vec{G}$
2. $\text{curl}(\vec{F} + \vec{G}) = \text{curl } \vec{F} + \text{curl } \vec{G}$
3. $\text{div}(f\vec{F}) = f\text{div } \vec{F} + \vec{F} \cdot \nabla f$
4. $\text{curl}(f\vec{F}) = f\text{curl } \vec{F} + (\nabla f) \times \vec{F}$
5. $\text{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot \text{curl } \vec{F} - \vec{F} \cdot \text{curl } \vec{G}$
6. $\text{div}(\nabla f \times \nabla g) = 0$
7. $\text{curl curl}(\vec{F}) = \text{grad div } \vec{F} - \nabla^2\vec{F}$

8. $\nabla(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\nabla)\vec{G} + (\vec{G}\cdot\nabla)\vec{F} + \vec{F}\times \text{curl } \vec{G} + \vec{G}\times \text{curl } \vec{F}$

Vector forms of Green's Theorem.

Let $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$ be a vector field. We suppose that the plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P dx + Q dy$$

and

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Therefore $$(\text{curl } \vec{F})\cdot\vec{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\cdot\vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can rewrite the equation in Green's Theorem in the vector form

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (\text{curl } \vec{F})\cdot\vec{k}\, dA$$

If C is given by the vector equation $\vec{r}(t) =< x(t), y(t) >$, $a \le t \le b$, then the unit tangent vector

$$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|} = \frac{x'(t)}{|\vec{r'}(t)|}\vec{\imath} + \frac{y'(t)}{|\vec{r'}(t)|}\vec{\jmath}$$

Then the outward unit normal vector to C is given by

$$\vec{n}(t) = \frac{y'(t)}{|\vec{r'}(t)|}\vec{\imath} - \frac{x'(t)}{|\vec{r'}(t)|}\vec{\jmath}$$

Then

$$\oint_C \vec{F}\vec{n}\, ds = \int_a^b (\vec{F}\cdot\vec{n})(t)|\vec{r'}(t)|dt$$

$$= \int_C P dy - Q dx = \iint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

So,

$$\oint_C \vec{F}\cdot\vec{n} ds = \iint_D \text{div } \vec{F}(x,y) dA$$