Combinatorial models of expanding maps and Julia sets

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May 18, 2011
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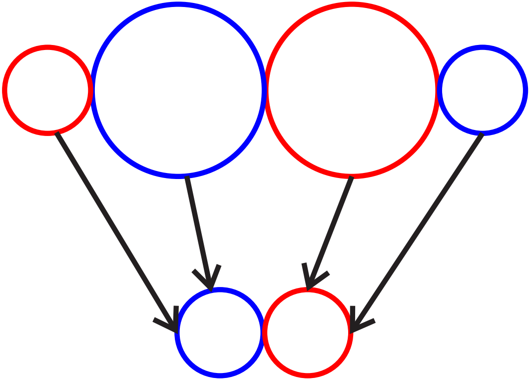
Definition

A *topological correspondence (topological automaton)* $\mathcal{F}$ is a quadruple $(\mathcal{M}, \mathcal{M}_1, f, \iota)$, where $\mathcal{M}$ and $\mathcal{M}_1$ are topological spaces (orbispaces), $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a finite covering map and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a continuous map.

Example: $-\frac{z^3}{2} + \frac{3z}{2}$



Transducers

Definition

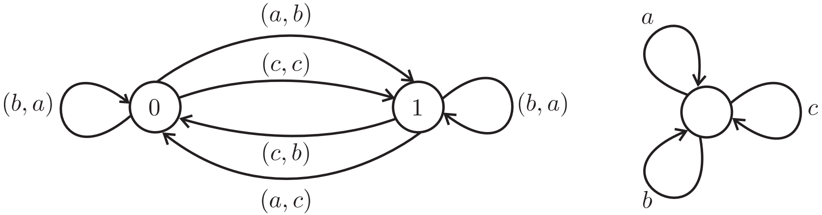
An *automaton* over an alphabet X is a triple (Q, τ, π), where Q is a set (of *internal states*) and τ and π are maps

$$\tau : Q \times \mathsf{X} \longrightarrow \mathsf{X}, \quad \pi : Q \times \mathsf{X} \longrightarrow Q,$$

called the *output* and *transition*. The automaton is called *invertible* if for every $q_0 \in Q$ the map $x \mapsto \tau(q_0, x)$ is a permutation. The automaton is *finite* if the set Q is finite.

Dual Moore diagram

Let $\mathcal{M}$ be the graph with one vertex and $|Q|$ arrows e_q, $q \in Q$. Let $\mathcal{M}_1$ be the graph with the set of vertices X where for every $x \in \mathsf{X}$ and $q \in Q$ we have an arrow $e_{q,x}$ from x to $\tau(q,x)$. Define $f(e_{q,x}) = e_q$ and $\iota(e_{q,x}) = e_{\pi(q,x)}$. If the automaton is invertible, then f is a covering.



Iterating

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & \ddots & \downarrow f_4 & & \downarrow f_3 & & \downarrow f_2 \\
\ldots & \xrightarrow{\iota_4} & \mathcal{M}_4 & \xrightarrow{\iota_3} & \mathcal{M}_3 & \xrightarrow{\iota_2} & \mathcal{M}_2 \\
 & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 \\
\ldots & \xrightarrow{\iota_3} & \mathcal{M}_3 & \xrightarrow{\iota_2} & \mathcal{M}_2 & \xrightarrow{\iota_1} & \mathcal{M}_1 \\
 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f \\
\ldots & \xrightarrow{\iota_2} & \mathcal{M}_2 & \xrightarrow{\iota_1} & \mathcal{M}_1 & \xrightarrow{\iota} & \mathcal{M}
\end{array}
$$

We get three inverse limits $\lim_f \mathcal{F}$, $\lim_\iota \mathcal{F}$ and $\lim_{f,\iota} \mathcal{F}$ with self-maps ι_∞, f_∞ and Δ.

Iterated monodromy groups

Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a topological correspondence. Identify $\pi_1(\mathcal{M}_1)$ with a subgroup of finite index in $\pi_1(\mathcal{M})$. Then $\iota_* : \pi_1(\mathcal{M}_1) \longrightarrow \pi_1(\mathcal{M})$ is the *virtual endomorphism* of $\pi_1(\mathcal{M})$ associated with the correspondence. Denote

$$N_{\iota_*} = \bigcap_{n \geq 1, g \in \pi_1(\mathcal{M})} g^{-1} \cdot \mathrm{Dom}\, \iota_*^n \cdot g$$

The *iterated monodromy group* of $\mathcal{F}$ is

$$\mathrm{IMG}\,(\mathcal{F}) = \pi_1(\mathcal{M})/N_{\iota_*}$$

together with the (conjugacy class of) the virtual endomorphism induced by ι_*. Two topological correspondences are *combinatorially equivalent* if they have the same iterated monodromy groups.

Contracting correspondences

Definition

Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a topological correspondence such that $\mathcal{M}$ is a compact path connected and locally path connected (orbi)space. $\mathcal{F}$ is *contracting* if there exists a length structure on $\mathcal{M}$ and $\lambda < 1$ such that for every rectifiable path γ in $\mathcal{M}_1$

$$\mathrm{length}(\iota(\gamma)) \leq \lambda \cdot \mathrm{length}(\gamma),$$

where length of γ is computed with respect the lift of the length structure by f.

Rigidity Theorem

Theorem

Let $\mathcal{F} = (\mathcal{M}, \mathcal{M}_1, f, \iota)$ be a contracting topological correspondence with locally simply connected $\mathcal{M}$. Then the system $(\lim_\iota \mathcal{F}, f_\infty)$ depends, up to a topological conjugacy, on $(\mathrm{IMG}(\mathcal{F}), \iota_*)$ only.

If $\mathcal{F}$ is a correspondence associated with an expanding partial self-covering $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$, then $\mathcal{F}$ is contracting, and the limit $(\lim_\iota \mathcal{F}, f_\infty)$ is restriction of f onto the attractor $\bigcap_{n\geq 0} \mathcal{M}_n$ of backward iterations of f (the "Julia set" of f). Constructing another combinatorially equivalent contracting topological correspondence $\mathcal{F}$, we get approximations of the Julia set.

A multi-dimensional example

Consider the following map $\mathbb{C}^n \longrightarrow \mathbb{C}^n$:

$$F(x_1, x_2, \ldots, x_n) = \left(1 - \frac{1}{x_n^2}, 1 - \frac{x_1^2}{x_n^2}, \ldots, 1 - \frac{x_{n-1}^2}{x_n^2}\right).$$

It can be extended to an endomorphism of $\mathbb{CP}^n$:

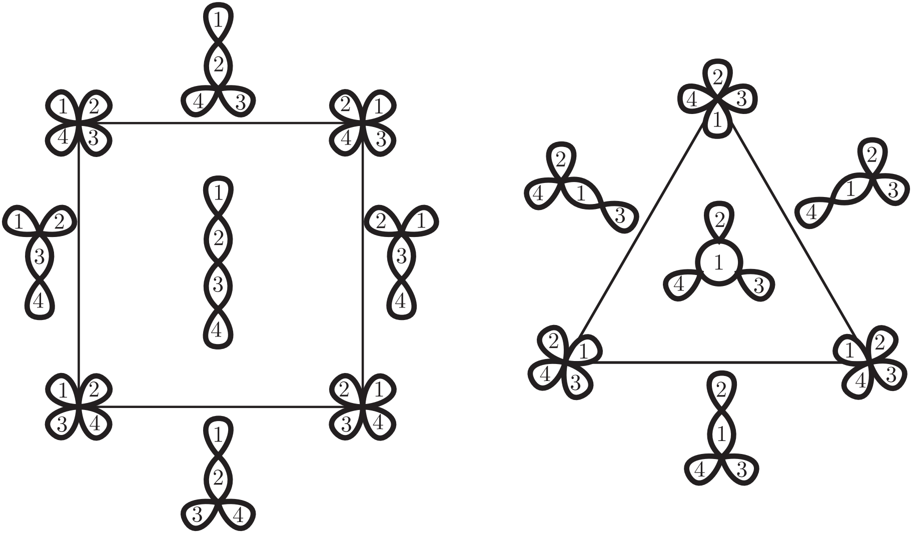
$$[x_1 : x_2 : \cdots : x_n : x_{n+1}] \mapsto [x_n^2 - x_{n+1}^2 : x_n^2 - x_1^2 : \cdots : x_n^2 - x_{n-1}^2 : x_n^2].$$

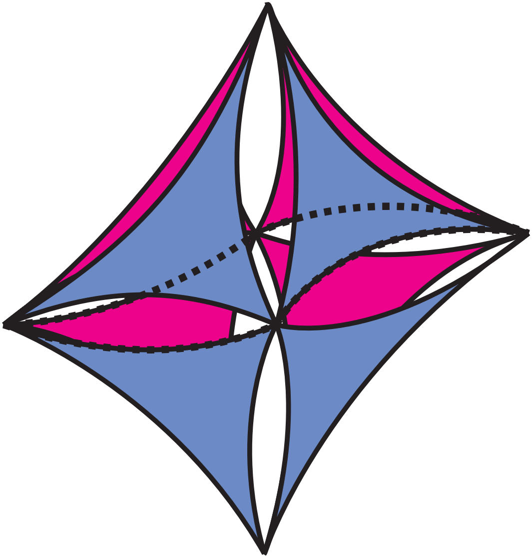
The union of the forward orbits of the set of critical points is the union P of the hyperplanes $x_i = 0$, $x_i = x_j$. We get a partial self-covering $F : \mathbb{CP}^n \setminus F^{-1}(P) \longrightarrow \mathbb{CP}^n \setminus P$.

A model of F

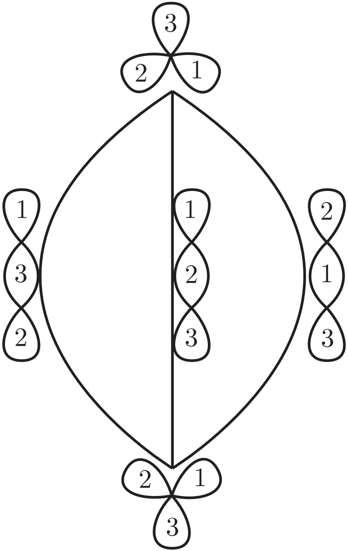
A *cactus diagram* is an oriented two-dimensional contractible cellular complex Γ consisting of $n+2$ discs D_i, $i = 0, 1, \ldots, n, n+1$, such that any two disc are either disjoint or have only one common point. A *planar cactus diagram* is a cactus diagram Γ together with an isotopy class of an orientation preserving embedding $\Delta : \Gamma \longrightarrow \mathbb{R}^2$ (i.e., cyclic orders of the discs adjacent to every given disc). A *metric cactus diagram* is a cactus diagram together with a metric on the one-skeleton, such that perimeter of the disc D_k is $\sqrt[n+2]{2^{-k}}$.

Let $\mathcal{M}$ be the space of all such metric planar cactus diagrams. It is an affine polyhedral complex. The cells are in a bijective correspondence with planar cactus diagrams.





For every planar metric cactus diagram Γ consider a diagram Γ_1 such that there exists a degree two branched covering map $\Gamma_1 \longrightarrow \Gamma$ with the critical point inside the disc D_0. Denote one of the preimages of D_i by D'_{i-1} (mod $n+2$). Let $\mathcal{M}_1$ be the configuration space of such labeled planar metric cactus diagrams Γ_1. We have a natural covering map $f : \Gamma_1 \mapsto \Gamma$.
For $\Gamma_1 \in \mathcal{M}_1$ contract the non-labeled discs, rename D'_i by D_i and divide all the distances by $\sqrt[n+2]{2}$. You get a point of $\mathcal{M}$. This gives the map $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$.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2 1
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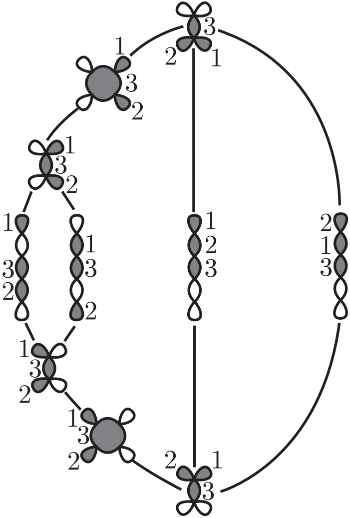
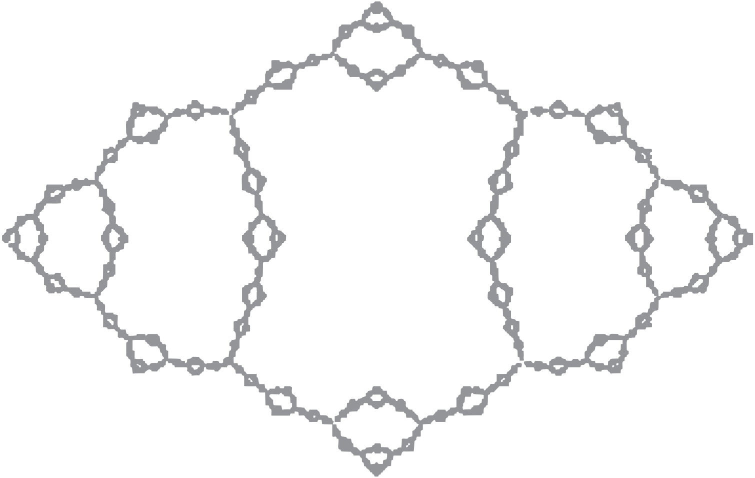
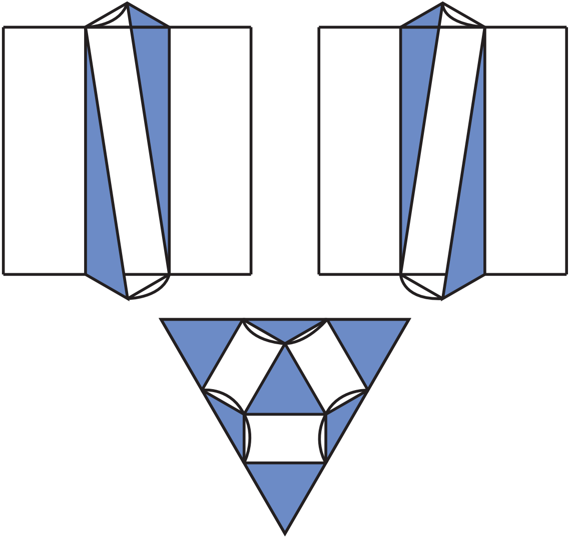
3



Julia set of $1 - \frac{1}{x^2}$





General approach

Let $\phi : G_1 \longrightarrow G$ be a surjective virtual endomorphism of a finitely generated group. If $\mathcal{X}$ is space with a co-compact proper G-action by isometries and $F : \mathcal{X} \longrightarrow \mathcal{X}$ is such that

$$F(x \cdot g) = F(x) \cdot \phi(g).$$

Then $(\mathcal{M} = \mathcal{X}/G, \mathcal{M}_1 = \mathcal{X}/G_1, f, \iota)$, where f, ι are induced by identity and F, is a topological correspondence with the associated virtual endomorphism ϕ.

Let $S = S^{-1} \ni 1$ be a finite generating set of G. *Rips complex* $\Gamma(G, S^n)$ is the simplicial complex with set of vertices G where A is a simplex if $Ag^{-1} \subset S^n$ for all $g \in A$.

Let $\phi : G_1 \longrightarrow G$ be a contracting surjective virtual endomorphism. Choose a left coset representative system $\{g_1, g_2, \ldots, g_d\}$ of G/G_1. Define $F(g) = \phi(g_i^{-1}g)$. Then $F(x \cdot g) = F(x) \cdot \phi(g)$ for all $x \in G$ and $g \in \mathrm{Dom}\,\phi$.

Theorem

There exist n and m such that $F^m : \Gamma(G, S^n) \longrightarrow \Gamma(G, S^n)$ is simplicial and equivariantly homotopic to a contracting map.

Here equivariant homotopy means that $H(x \cdot g) = H(x) \cdot \phi(g)$ for all maps H along the homotopy.