Iterated monodromy groups

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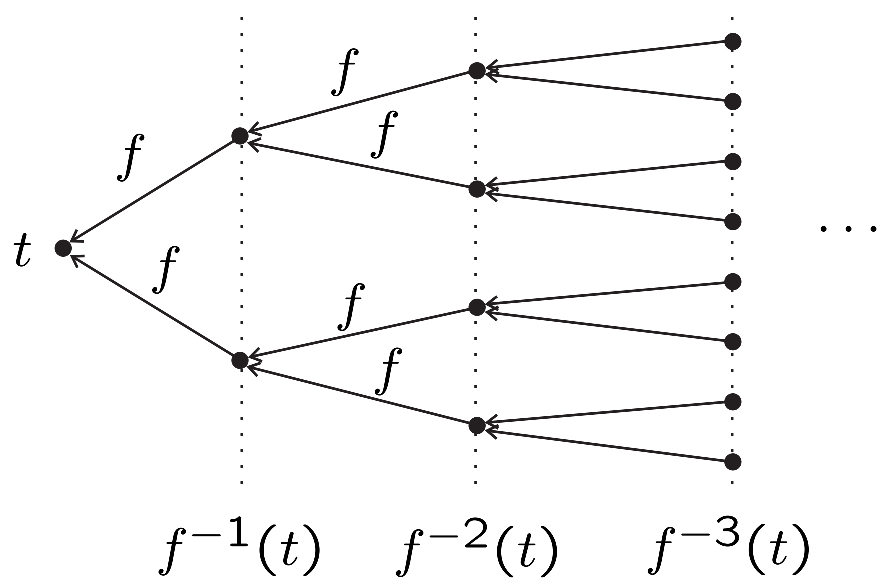
$\mathcal{M}$ — an arcwise connected and locally arcwise connected topological space (or an *orbispace*)

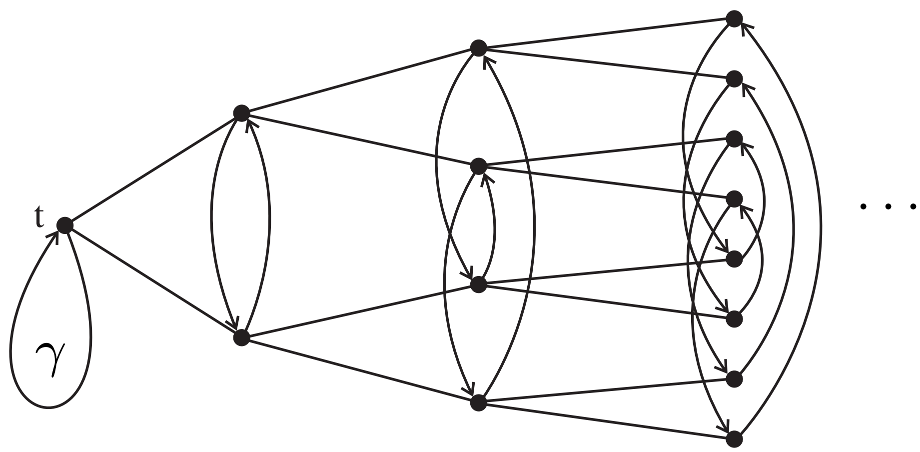
$\mathcal{M}_1$ — its open arcwise connected subset (resp. sub-orbispace).

$f : \mathcal{M}_1 \to \mathcal{M}$ — a d-fold covering map.

$t \in \mathcal{M}$ — a basepoint.



Preimage tree T_t.



Iterated monodromy action.

The action on T_t is an action by automorphisms of the rooted tree.

Definition. The quotient of the fundamental group $\pi_1(\mathcal{M}, t)$ by the kernel of the action is called the *iterated monodromy group* IMG(f) of the covering f.

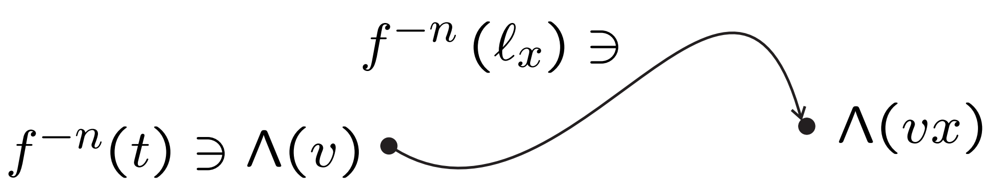
The group IMG(f) and its action on the tree T_t depends only on f.

Computation of IMG(f)

Choose an alphabet $X, |X| = d$, a bijection $\Lambda : X \to f^{-1}(t)$, and a path ℓ_x from t to $\Lambda(x)$ for every $x \in X$.

X^* is the rooted tree of finite words, where a word $v \in X^*$ is connected to vx, $\forall x \in X$.

Define the map $\Lambda : X^* \to T_t$ inductively by the rule:

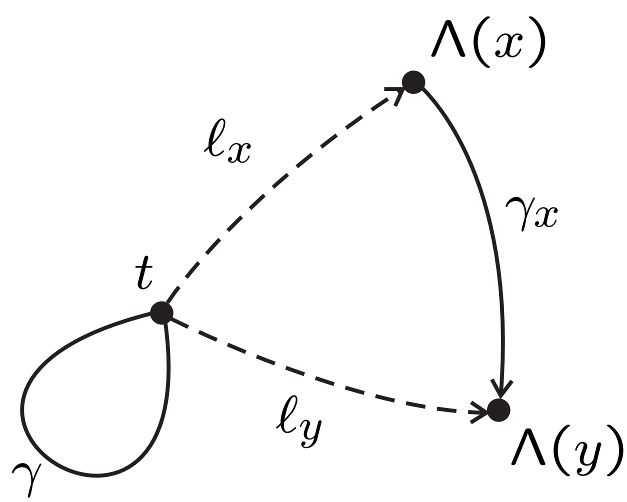


The map $\Lambda : X^* \to T_t$ is an isomorphism of rooted trees.

The isomorphism $\Lambda : X^* \to T_t$ conjugates the iterated monodromy action of $\pi_1(\mathcal{M}, t)$ on T_t to a self-similar action on X^* defined by the formula

$$(xv)^\gamma = y\left(v^{\ell_x \gamma_x \ell_y^{-1}}\right),$$

where $v \in X^*, x \in X$ and

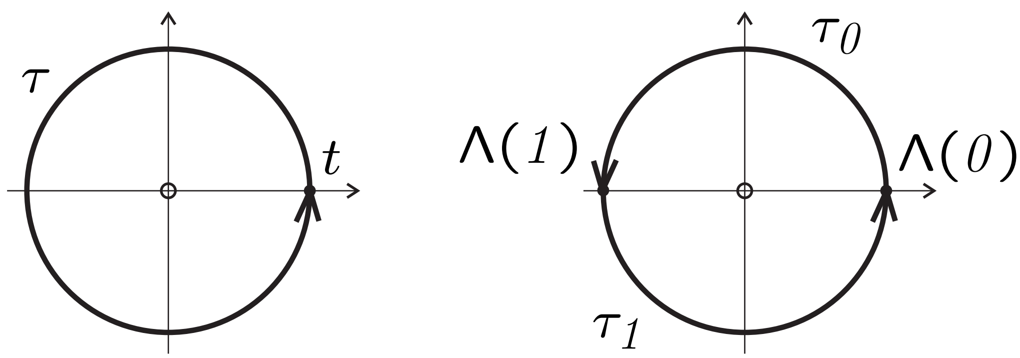


This is the *standard action* of IMG (f) on X^*.

Examples

1. $f(z) = z^2$ with $\mathcal{M} = \mathbb{C} \setminus \{0\}$ and $\mathcal{M}_1 = \mathcal{M}$.

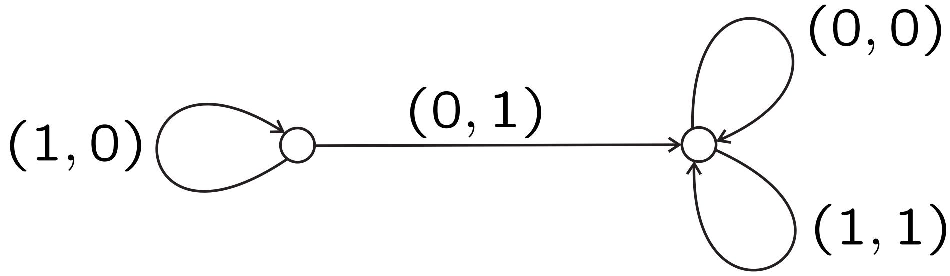
Let $X = \{0, 1\}$ and $t = 1$. The fundamental group of $\mathcal{M}$ is generated by a single loop τ around 0.



We take ℓ_0 is trivial at t, $\ell_1 = \tau_0$.

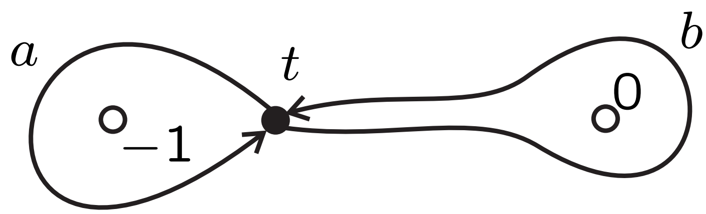
Then

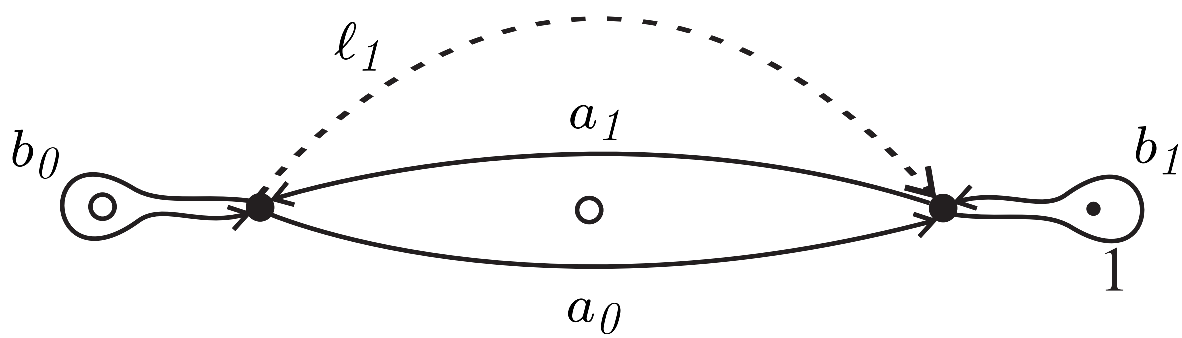
$$(0v)^\tau = 1v, \qquad (1v)^\tau = 0v^\tau.$$



1. $f(z) = z^2 - 1$ with $\mathcal{M} = \mathbb{C} \setminus \{0, -1\}$ and $\mathcal{M}_1 = \mathbb{C} \setminus \{0, -1, 1\}$.

Let $t = \frac{1-\sqrt{5}}{2}$. The fundamental group of $\mathcal{M}$ is generated by two loops a and b.





$$(0v)^a = 1v^b, \qquad (1v)^a = 0v$$
$$(0v)^b = 0v^a, \qquad (1v)^b = 1v.$$

Theorem. [R. Grigorchuk, A. Żuk] The group $G = \mathsf{IMG}\left(z^2 - 1\right)$

- is torsion free;
- has exponential growth;
- is just non-solvable;
- has solvable word and conjugacy problems;
- has no free subgroups.

Theorem. [L. Bartholdi]

$$G = \left\langle a, b \,\middle|\, \left[\left[a^{2^k}, b^{2^k}\right], b^{2^k}\right], \left[\left[b^{2^k}, a^{2^{k+1}}\right], a^{2^{k+1}}\right] \right\rangle.$$

Limit spaces

Let (G, X^*) be a self-similar *contracting* group action. Denote by $X^{-\omega}$ the space of all sequences

$$\ldots x_3 x_2 x_1, \qquad x_i \in X$$

with the product topology.

$\ldots x_2 x_1 \approx \ldots y_2 y_1$ if there exists a bounded sequence $g_n \in G$ such that

$$(x_n x_{n-1} \ldots x_1)^{g_n} = (y_n y_{n-1} \ldots y_1).$$

$X^{-\omega}/\approx$ is the *limit space* $\mathcal{J}_G$.

The shift $\ldots x_3 x_2 x_1 \mapsto \ldots x_4 x_3 x_2$ induces a continuous map $\mathsf{s} : \mathcal{J}_G \to \mathcal{J}_G$. We get the *limit dynamical system* $(\mathcal{J}_G, \mathsf{s})$.

The space $\mathcal{J}_G$ has a natural structure of an orbispace such that s is a $|X|$-fold covering by an open sub-orbispace of $\mathcal{J}_G$.

Theorem 1. Let $f : \mathcal{M}_1 \to \mathcal{M}$ be an expanding covering. Then the standard action of IMG(f) on X^* is contracting and the restriction of f onto its Julia set is topologically conjugate to the limit dynamical system $(\mathcal{J}_G, \mathsf{s})$.

An example. The Julia set of z^2 is the unit circle, and z^2 acts on it as a double covering. The automaton generating the adding machine action generates the equivalence relation:

$$\ldots 0001 x_n x_{n-1} \ldots x_1 \approx \ldots 1110 x_n x_{n-1} \ldots x_1$$
$$\ldots 0000 \approx 1111,$$

i.e., the limit space is the circle $\mathbb{R}/\mathbb{Z}$ and the map s is induced by $x \mapsto 2x$.

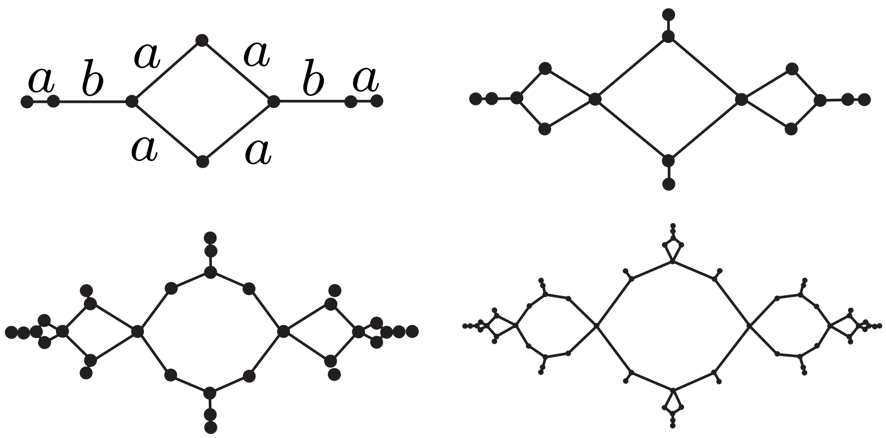
In general, if a rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is *sub-hyperbolic*, then f is expanding on a neighborhood of the Julia set. (Orbispaces have to be used in many cases!)

We get finite-to-one encodings of the Julia set by sequences. They are *finite presentations* of the respective dynamical systems.

For example, the Julia set of the polynomial $z^2 - 1$ is the quotient of the space $\{0, 1\}^{-\omega}$ by the equivalence relation

$$\ldots 0000001v \approx \ldots 0101011v \approx \ldots 1010100v$$
$$\ldots 0000 \approx \ldots 0101 \approx \ldots 1010$$

The Schreier graphs Γ_n of the action of the group IMG(f) on the sets X^n approximate the Julia set of f.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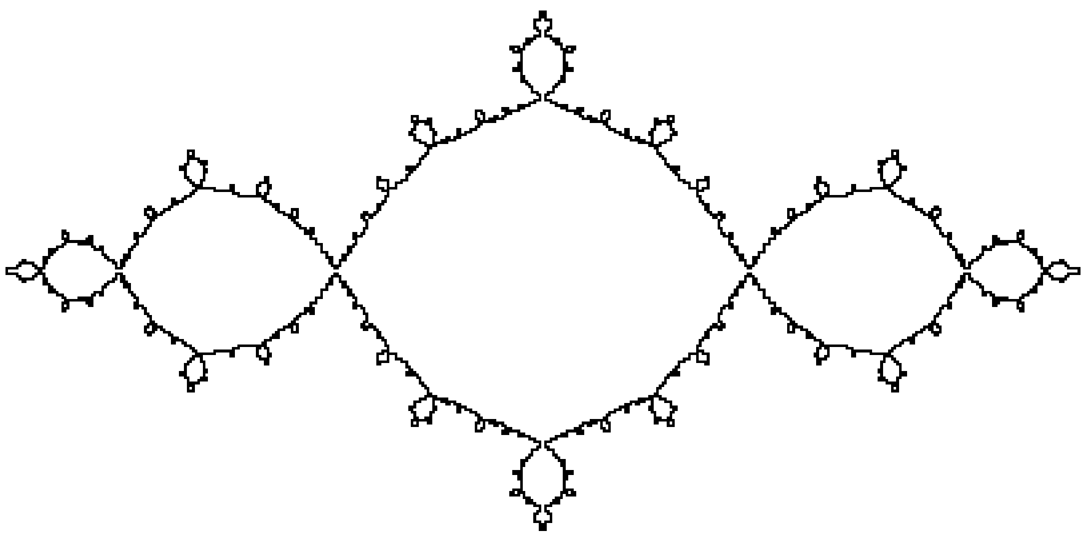
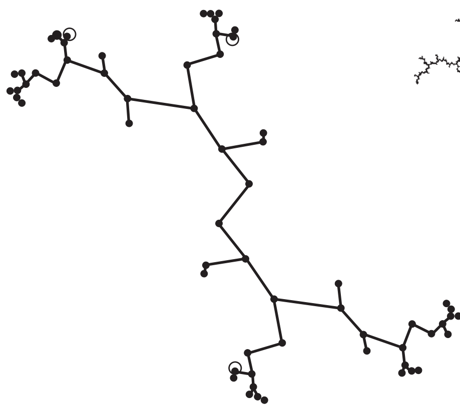
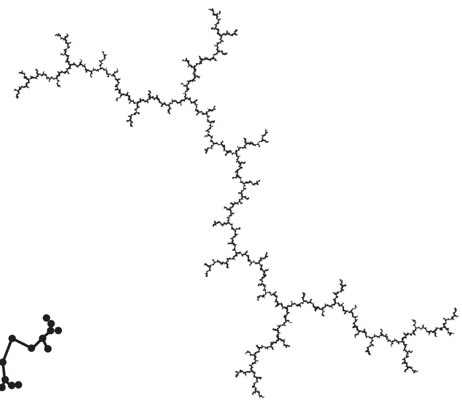
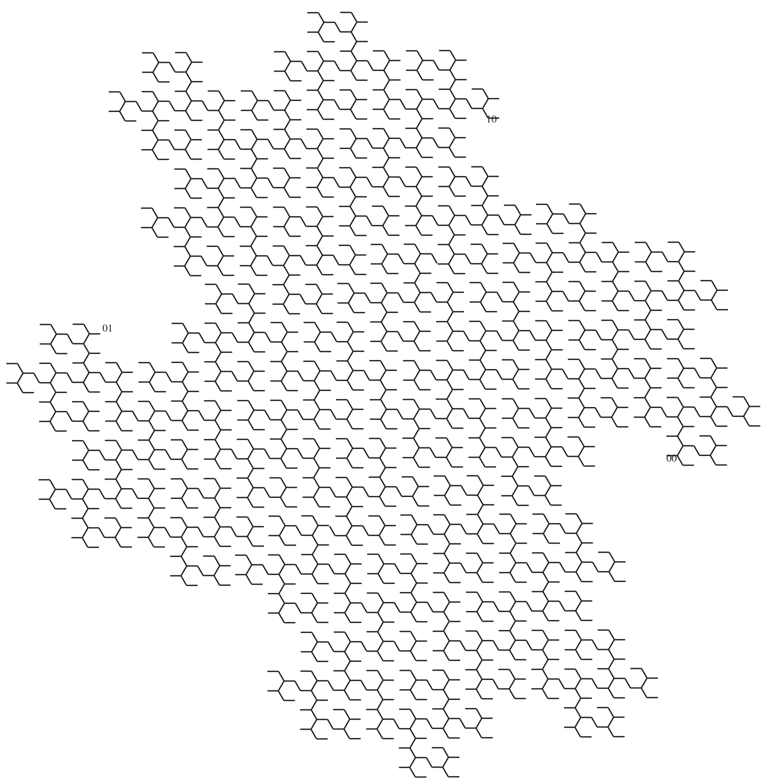


$z^2 + i$







The following is an analog of theorems by M. Shub, J. Franks and M. Gromov and can be proved using Theorem 1 and M. Gromov's theorem on groups of polynomial groups.

Theorem 2. Let $f : \mathcal{M} \to \mathcal{M}$ be an expanding finite self-covering of a compact orbifold. Then there exists a nilpotent simply-connected Lie group L, a proper affine action of $G = \pi_1(\mathcal{M})$ on L and an expanding automorphism $\phi : L \to L$ such that $\mathcal{M} = G\backslash L$, $\phi \cdot G \cdot \phi^{-1} < G$ and ϕ induces f.

$\mathsf{IMG}(f) = \pi_1(\mathcal{M})$ and is virtually nilpotent.
$(\mathcal{J}_{\mathsf{IMG}(f)}, \mathsf{s})$ is topologically conjugate to $(\mathcal{M}, f)$.

The encoding of the points of $\mathcal{M}$ by sequences generalize the numeration systems on $\mathbb{R}$: the sequence $\ldots x_2x_1$ corresponds to the point

$$\cdots \phi^{-2}(r_{x_3})\phi^{-1}(r_{x_2})r_{x_1} \in L$$

where $r_x \in L$ are "digits", indexed by $x \in X$.

Digit tile $\mathcal{T}$ is the set of elements

$$\cdots \phi^{-2}(r_{x_3})\phi^{-1}(r_{x_2})r_{x_1} \in L$$

for all possible sequences $\ldots x_2 x_1 \in X^{-\omega}$.

Some examples of digit tiles in $\mathbb{R}^2$: