Iterated Monodromy Groups
Lecture 1

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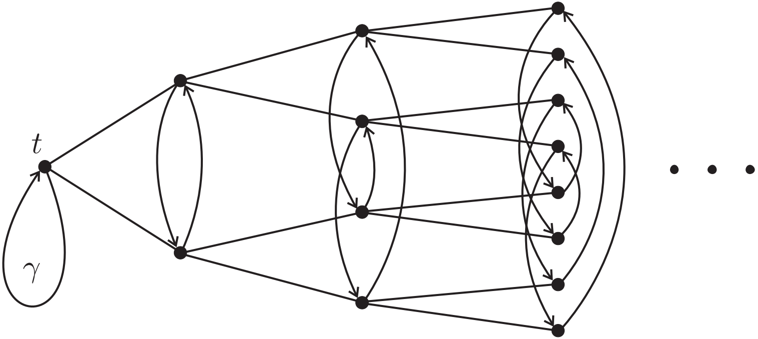
Bath

Outline

1. Main definition.
2. Computation of iterated monodromy groups.
3. Virtual endomorphisms and self-similar groups.
4. Limit spaces.
5. Examples and applications.

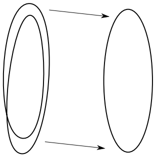
Definition

Let $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a covering of a space by a subset (a *partial self-covering*).

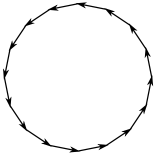


Double self-covering of the circle

Consider the map $p : x \mapsto 2x$ of the circle $\mathbb{R}/\mathbb{Z}$.



The fundamental group of the circle is generated by the loop γ equal to the image of $[0, 1]$ in $\mathbb{R}/\mathbb{Z}$.
The lifts of γ by p^n are the images of $\left[\frac{m}{2^n}, \frac{m+1}{2^n}\right]$, for $m = 0, \ldots, 2^n - 1$.



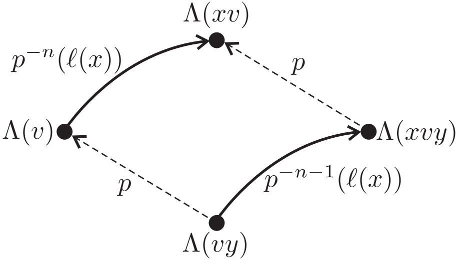
Encoding of the tree

Choose an alphabet X, $|\mathsf{X}| = \deg p$, a bijection $\Lambda : \mathsf{X} \to p^{-1}(t)$, and a path $\ell(x)$ from t to $\Lambda(x)$ for every $x \in \mathsf{X}$.

Define the map $\Lambda : \mathsf{X}^* \to T$ inductively by the rule:

$$\Lambda(xv) \text{ is the end of the } p^{|v|}\text{-lift of } \ell(x) \text{ starting at } \Lambda(v).$$

The map $\Lambda : \mathsf{X}^* \to T$ is an isomorphism of rooted trees, where v is connected to vy in X^*.



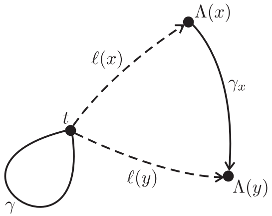
Recurrent formula

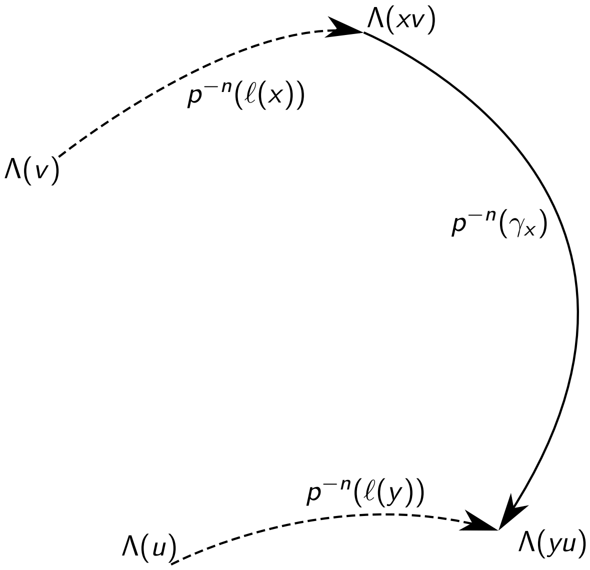
Let us identify the trees T and X^* using the isomorphism Λ. Then the iterated monodromy group acts on the tree X^*. Let γ be an element of the fundamental group $\pi_1(\mathcal{M}, t)$.

Proposition

For $x \in \mathsf{X}$, let γ_x be the lift of γ by p starting at $\Lambda(x)$. Let $y \in \mathsf{X}$ be such that $\Lambda(y)$ is the end of γ_x. Then for every $v \in \mathsf{X}^*$ we have

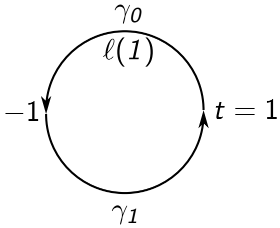
$$\gamma(xv) = y\left(\ell(x)\gamma_x\ell(y)^{-1}\right)(v).$$




$\Lambda(xv)$
$p^{-n}(\ell(x))$
$\Lambda(v)$
$p^{-n}(\gamma_x)$
$p^{-n}(\ell(y))$
$\Lambda(u)$
$\Lambda(yu)$

Example: z^2

$p : z \mapsto z^2$ induces a double self-covering of $\mathbb{C} \setminus \{0\}$ (homotopically equivalent to the 2-fold self-covering of the circle).
Chose the basepoint $t = 1$. $p^{-1}(1) = \{1, -1\}$. Let $\ell(\mathit{0})$ be trivial, and let $\ell(\mathit{1})$ be the unit upper half-circle. Let γ be the unit circle based at t with the positive orientation.



We get $\gamma(\mathit{0}v) = \mathit{1}v$, $\gamma(\mathit{1}v) = \mathit{0}\gamma(v)$. This is known as the *adding machine*.

Example: $-\frac{z^3}{2} + \frac{3z}{2}$

A rational function $f(z) \in \mathbb{C}(z)$ is *post-critically finite* if orbit of every critical point of f is finite. The union P_f of the orbits of critical values is the *post-critical set* of f.

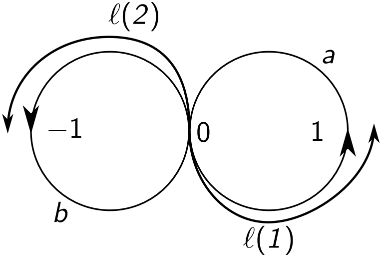
If f is post-critically finite, then it is a partial self-covering of $\widehat{\mathbb{C}} \setminus P_f$.

Consider $f(z) = -\frac{z^3}{2} + \frac{3z}{2}$. It has three critical points $\infty, 1, -1$, which are fixed under f.

Hence it is post-critically finite and is a covering of $\mathbb{C} \setminus \{\pm 1\}$ by the subset $\mathbb{C} \setminus f^{-1}(\{\pm 1\}) = \mathbb{C} \setminus \{\pm 1, \pm 2\}$.

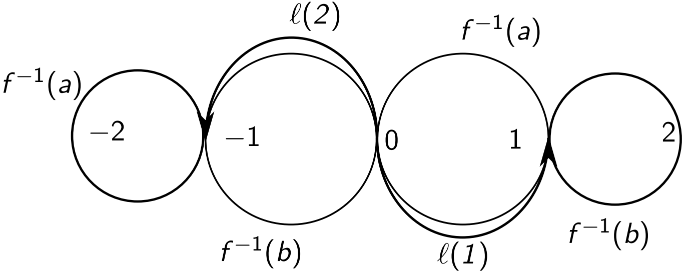
Example: $-\frac{z^3}{2} + \frac{3z}{2}$

Let $t = 0$. It has three preimages $0, \pm\sqrt{3}$. Choose the following connecting paths and generators of $\pi_1(\mathbb{C} \setminus \{\pm 1\}, 0)$ ($\ell(0)$ is trivial):



Example: $-\frac{z^3}{2} + \frac{3z}{2}$

The generators a and b are lifted to the following paths:



$a(0v) = 1v, \quad a(1v) = 0a(v), \quad a(2v) = 2v,$
$b(0v) = 2v, \quad b(1v) = 1v, \quad b(2v) = 0b(v).$

A multi-dimensional example

Consider the map F of $\mathbb{C}^2$:

$$(x, y) \mapsto \left(1 - \frac{y^2}{x^2}, 1 - \frac{1}{x^2}\right)$$

It can be naturally extended to the projective plane.

$$(x : y : z) \mapsto (x^2 - y^2 : x^2 - z^2 : x^2).$$

The set $\{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$ is the critical locus. The post-critical set is the union of the line at infinity with the lines $x = 0, x = 1, y = 0, y = 1, x = y$.

They are permuted as follows:

$$\{x = 0\} \mapsto \{z = 0\} \mapsto \{y = 1\} \mapsto \{x = y\} \mapsto \{x = 0\}$$

$$\{y = 0\} \mapsto \{x = 1\} \mapsto \{y = 0\}.$$

The iterated monodromy group of F (as computed by J. Belk and S. Koch) is generated by the transformations:

$$a(1v)=1b(v),\quad a(2v)=2v,\quad a(3v)=3v,\quad a(4v)=4b(v),$$

$$b(1v)=1c(v),\quad b(2v)=2c(v),\quad b(3v)=3v,\quad b(4v)=4v,$$

$$c(1v)=4d(v), c(2v)=3(ceb)^{-1}(v), c(3v)=2(fa)^{-1}(v), c(4v)=1v,$$

$$d(1v)=2v,\quad d(2v)=1a(v),\quad d(3v)=4v,\quad d(4v)=3a(v),$$

$$e(1v)=1f(v),\quad e(2v)=2v,\quad e(3v)=3f(v),\quad e(4v)=4v,$$

$$f(1v)=3b^{-1}(v),\quad f(2v)=4v,\quad f(3v)=1eb(v),\quad f(4v)=2e(v).$$

We have seen that for every $g \in \mathrm{IMG}(p)$ and for every $x \in \mathsf{X}$ there exists $y \in \mathsf{X}$ and $g_x \in \mathrm{IMG}(p)$ such that

$$g(xv) = yg_x(v)$$

for all $v \in \mathsf{X}^*$.
Groups satisfying this condition are called *self-similar*.
The map $\pi_g : x \mapsto y$ is a permutation (describing the action of g on the first level of the tree. Hence we get a map

$$g \mapsto \pi_g(g_1, g_2, \ldots, g_d),$$

from $\mathrm{IMG}(p)$ to $S_d \wr \mathrm{IMG}(p)$, where $\mathsf{X} = \{1, 2, \ldots, d\}$. It is easy to check that this map is a homomorphism.

Definition

A *wreath recursion* on a group G is a homomorphism

$$\Phi : G \longrightarrow S_d \wr G.$$

The wreath defining $\mathrm{IMG}\,(p)$ depends on the choice of the bijection of X with $p^{-1}(t)$ and on the choice of the connecting paths $\ell(x)$. Different choices produce wreath recursions, which differ from each other by application of an inner automorphism of $S_d \wr G$.

We say that $\Phi_1, \Phi_2 : G \longrightarrow S_d \wr G$ are *equivalent* if there exists an inner automorphism τ of $S_d \wr G$ such that $\Phi_2 = \tau \circ \Phi_1$.

Every wreath recursion defines an action on the tree $\{1, 2, \ldots, d\}^*$. If $\Phi(g) = \pi(g_1, g_2, \ldots, g_d)$ then

$$g(iv) = \pi(i)g_i(v)$$

for all $v \in \{1, 2, \ldots, d\}^*$ and $x \in \{1, 2, \ldots, d\}$.

These recurrent rules uniquely define the *action of G associated with* Φ.

The *associated faithful self-similar group* is the quotient of G by the kernel of the action. Equivalent wreath recursions define conjugate self-similar groups.

If G is generated by a finite set $\{g_1, g_2, \ldots, g_k\}$, then the wreath recursion is determined by its values on the generators:

$$\begin{aligned}
\Phi(g_1) &= \pi_1(g_{11}, g_{12}, \ldots, g_{1d}),\\
\Phi(g_2) &= \pi_2(g_{21}, g_{22}, \ldots, g_{2d}),\\
\vdots \quad & \quad \vdots\\
\Phi(g_k) &= \pi_k(g_{k1}, g_{k2}, \ldots g_{kd}).
\end{aligned}$$

If we write g_{ij} as groups words in $g_1, \ldots, g_k$, we get a finite description of the associated self-similar group. (As a wreath recursion over the free group.)