Self-similar and branch groups III

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October 6, 2010,
Montreal

Rigidity

Recall that a level-transitive group $G \leq \mathrm{Aut}(T)$ is weakly branch if for every $v \in T$ there is $g \in G \setminus \{1\}$ acting trivially outside of T_v.

Theorem (Y. Lavrenyuk, V.N.)

Let $G_1, G_2 \leq \mathrm{Aut}(T)$ be weakly branch groups. Then every isomorphism $\phi : G_1 \longrightarrow G_2$ is induced by a measure preserving homeomorphism of ∂T.

An isomorphism $\phi : G_1 \longrightarrow G_2$ is *saturated* if there is a sequence of subgroups H_n of the nth level stabilizers in G_1 such that H_n and $\phi(H_n)$ stabilize the nth level, and act level-transitively on subtrees growing from the nth level.

Proposition

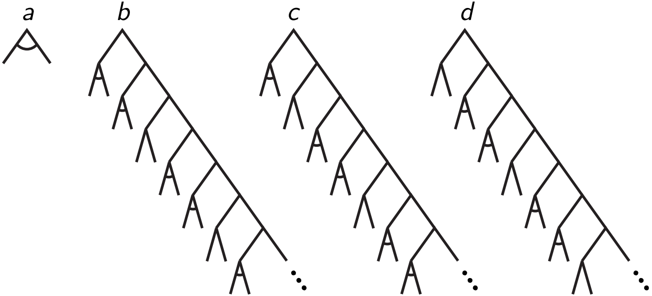
If $\phi : G_1 \longrightarrow G_2$ is a saturated isomorphism of weakly branch groups, then it is induced by an automorphism of T.

Grigorchuk groups

For $w = x_1 x_2 \ldots \in \{0, 1, 2\}$ and $w' = x_2 x_3 \ldots$ consider the group G_w acting on $\mathsf{X}^* = \{0, 1\}^*$ and generated by

$$a_w(0v) = 1v, \quad a_w(1v) = 0v,$$
$$b_w(1v) = 1b_{w'}(v), \quad c_w(1v) = 1c_{w'}(v), \quad d_w(1v) = 1d_{w'}(v)$$
$$b_w(0v) = \begin{cases} 0a_{w'}(v), & \text{if } x_1 \in \{0, 1\}, \\ 0v, & \text{otherwise}, \end{cases}$$
$$c_w(0v) = \begin{cases} 0a_{w'}(v), & \text{if } x_1 \in \{0, 2\}, \\ 0v, & \text{otherwise}, \end{cases}$$
$$d_w(0v) = \begin{cases} 0a_{w'}(v), & \text{if } x_1 \in \{1, 2\}, \\ 0v, & \text{otherwise}. \end{cases}$$

Grigorchuk groups



Grigorchuk groups

If w is not eventually constant, then G_w is branch. Using the sequence $H_n = (H_{n-1})^2$ we see that all isomorphisms are saturated.

Proposition

Two groups G_{w_1} and G_{w_2} such that w_1 and w_2 are not eventually constant are isomorphic if and only if they are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$.

The map $\mathrm{Aut}(\mathsf{X}^*) \longrightarrow \mathrm{Aut}(\mathsf{X}^*)/\mathrm{Aut}(\mathsf{X}^*)' \cong C_2^\infty$ maps conjugate subgroups of $\mathrm{Aut}(\mathsf{X}^*)$ to the same groups. Images of G_{w_1} and G_{w_2} are equal if and only if $G_{w_1} = G_{w_2}$, i.e., if w_1 is obtained from w_2 by application of a permutation $\pi \in \mathrm{Symm}(\{0, 1, 2\})$ to each coordinate.

Quadratic polynomials

Theorem (L. Bartholdi, V.N.)

The iterated monodromy groups of p.c.f. quadratic polynomials are generated by bounded automata, weakly branch, and saturated.

Theorem (V.N.)

Let G_1, G_2 be contracting weakly branch saturated groups generated by bounded automata. Let $(\mathcal{X}_1, s_1)$ and $(\mathcal{X}_2, s_2)$ be the respective limit dynamical systems. If G_1 and G_2 are isomorphic then there exists n_1, n_2 such that $(\mathcal{X}_1, s_1^{n_1})$ and $(\mathcal{X}_2, s_2^{n_2})$ are topologically conjugate.

Rabbit and Airplane

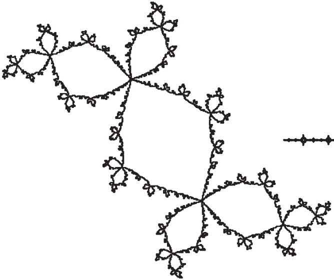
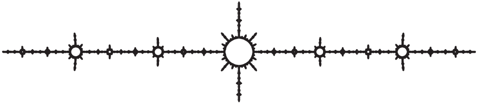
Consider the groups G_1 and G_2 generated by

$$\begin{aligned} a(0v) &= 1v, & a(1v) &= 0b(v), \\ b(0v) &= 0v, & b(1v) &= 1c(v), \\ c(0v) &= 0v, & c(1v) &= 1a(v), \end{aligned}$$

and

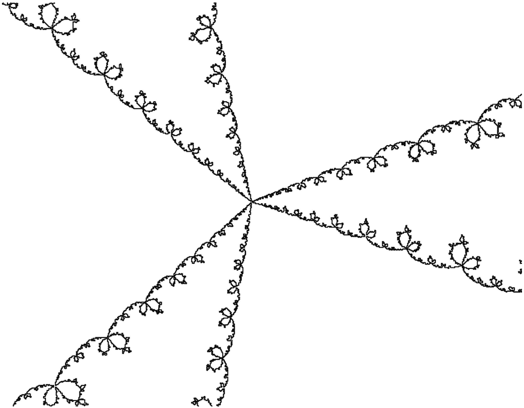
$$\begin{aligned} a(0v) &= 1v, & a(1v) &= 0b(v), \\ b(0v) &= 0v, & b(1v) &= 1c(v), \\ c(0v) &= 0a(v), & c(1v) &= 1v. \end{aligned}$$

Rabbit and Airplane

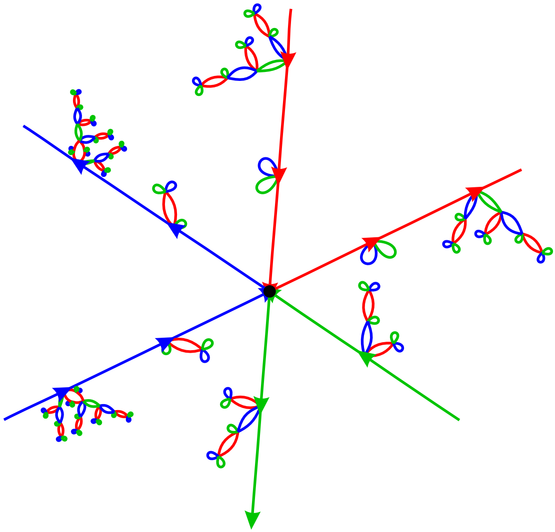
 

Consequently G_1 and G_2 are not isomorphic as abstract groups. What is growth of G_2? Does it have free sub-semigroups? For which quadratic polynomials f growth of $\mathrm{IMG}(f)$ is sub-exponential? Which contracting groups are (weakly) branch?

A free sub-semigroup of G_1



A free sub-semigroup of G_1



Functoriality of the limit dynamical system

Let G and H be self-similar groups over alphabets X and Y. Suppose that $\phi : G \longrightarrow H$ and $F : \mathsf{X} \longrightarrow \mathsf{Y}$ are such that

$$\phi(g)(F(x)) = F(g(x)), \qquad \phi(g|_x) = \phi(g)|_{F(x)}$$

for all $g \in G$ and $x \in \mathsf{X}$. Then the map $F(\ldots x_2x_1) = \ldots F(x_2)F(x_1)$ induces a continuous map from the limit space of G to the limit space of H commuting with the shift.

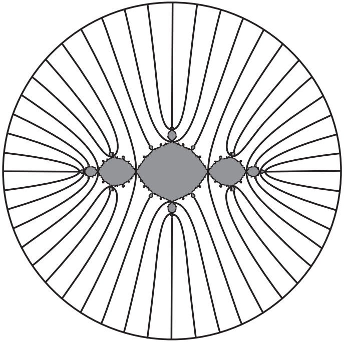
$\mathrm{IMG}(z^2 - 1)$ is generated by

$$a(0w) = 1w, \quad a(1w) = 0b(w), \quad b(0w) = 0w, \quad b(1w) = 1a^b(w)$$

then

$$ab(0w) = a(0w) = 1w, \quad ab(1w) = a(1a^b(w)) = 0ab(w),$$

hence the adding machine is contained in $\mathrm{IMG}(z^2 - 1)$. We get the *Carateodori loop*.



$\mathrm{IMG}(z^2 + i)$ is generated by

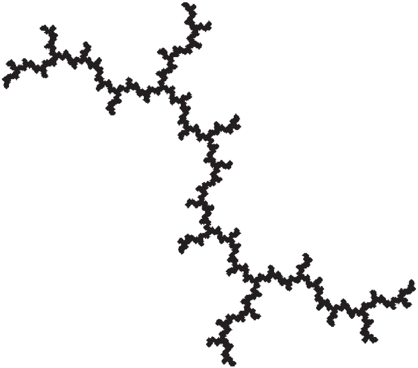
$$\alpha(0v) = 1v, \qquad \alpha(1v) = 0v,$$
$$\beta(0v) = 0\alpha(v), \quad \beta(1v) = 1\gamma(v),$$
$$\gamma(0v) = 0\beta(v), \quad \gamma(1v) = 1v$$

The limit space of the group $\langle \alpha, \beta, \gamma, a, b, c \rangle$

$$a(0v) = 1v, \qquad a(1v) = 0v,$$
$$b(0v) = 0a(v), \quad b(1v) = 1c(v),$$
$$c(0v) = 0b(v), \quad c(1v) = 1b(v)$$

is the same as of $\langle a, b, c \rangle$ and is the isosceles right triangle.

Julia set of $z^2 + i$



Filling a triangle