Self-similar groups and hyperbolic groupoids I

Volodymyr Nekrashevych

June 21, 2012,
Orleans

Self-similar groups

Definition

Let X be a finite alphabet. A *self-similar group* is a faithful action of a group G on the set X^* of finite words such that for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w)$$

for all $w \in X^*$.

Example: consider $X = \{0, 1\}$ and let a be defined by

$$a(0w) = 1w, \qquad a(1w) = 0a(w).$$

This is the rule of adding 1 to a dyadic integer. The transformation a (and the corresponding action of $\mathbb{Z}$) is called the *adding machine*.

Wreath recursions

It is easy to define a finitely generated self-similar group. Just choose for every generator g a permutation $\pi \in S_{\mathsf{X}}$ and elements $g_x \in G$ for all $x \in \mathsf{X}$, and then define

$$g(xw) = \pi(x) g_x(w)$$

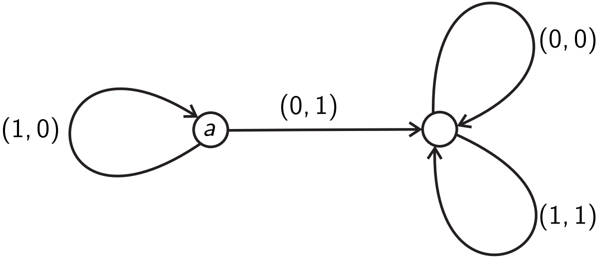
for all $w \in \mathsf{X}^*$. For example, take $G = \langle a, b \rangle$ and $\mathsf{X} = \{O, I, J\}$, where

$$a(Ow) = Iw, \quad a(Iw) = Oa(w), \quad a(Jw) = Jw$$

$$b(Ow) = Jw, \quad b(Iw) = Iw, \quad b(Jw) = Ob(w)$$

The rules can be compactly written as homomorphisms $G \longrightarrow G \wr S_{\mathsf{X}} : g \mapsto (g_x)\pi$, called wreath recursions.

If g_x are also generators, the group is generated by a *finite automaton*. For example, the adding machine action is described by the following graph.



To find $g(x_1x_2\ldots)$, find a directed path starting at g and labeled by $(x_1, y_1), (x_2, y_2), \ldots$. Then $g(x_1x_2\ldots) = y_1y_2\ldots$.

Bisets

Let (G, X) be a self-similar group. Identify letters $x \in \mathsf{X}$ with transformations

$$w \mapsto xw.$$

Then the equality $g(xw) = yg_x(w)$ can be written as an equality of compositions of transformations:

$$g \cdot x = y \cdot g_x.$$

Note that the set $\mathsf{X} \cdot G$ is invariant under post- and pre-compositions with elements of G:

$$g \cdot (x \cdot h) = y \cdot (g_x h), \qquad (x \cdot h) \cdot g = x \cdot (hg).$$

We get a *biset*: a set with commuting left and right actions of G.

Covering bisets

The right action of G on $\mathsf{X} \cdot G$ is *free* (i.e., $m \cdot g = m$ implies $g = 1$) and has $|\mathsf{X}|$ orbits labeled by the letters of X (since $x \cdot h \cdot g = x \cdot (hg)$).

Definition

A biset $\mathfrak{M}$ is a *covering biset* if the right action is free and has a finite number of orbits.

Let $\mathfrak{M}$ be a covering G-biset. Choose a right orbit transversal $\mathsf{Y} \subset \mathfrak{M}$, i.e., choose one element in each orbit. Then for every $g \in G$ and $x \in \mathsf{Y}$ the element $g \cdot x \in \mathfrak{M}$ can be uniquely written as

$$g \cdot x = y \cdot h$$

for some $y \in \mathsf{Y}$ and $h \in G$. We get the *associated action* on Y^* defined recurrently by the condition that

$$g(xw) = yh(w)$$

for all $w \in \mathsf{X}^*$ if $g \cdot x = y \cdot h$.

Consider the set $\mathfrak{M}$ of transformations of $\mathbb{R}$ of the form $x \mapsto (x+n)/2$ for $n \in \mathbb{Z}$. $\mathbb{Z}$ acts on $\mathbb{R}$ in the usual way. Then $\mathfrak{M}$ is invariant under pre- and post-compositions with elements of $\mathbb{Z}$:

$$x \mapsto (x+m+n)/2, \qquad x \mapsto (x+n)/2 + m = (x+n+2m)/2.$$

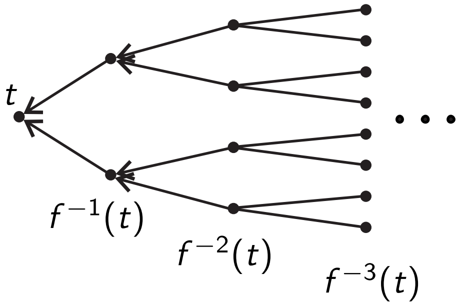
The action by post-compositions has two orbits (corresponding to parity of n).
If we choose the transversal $\{f_0 = x/2, f_1 = (x+1)/2\}$, then the generator $a : x \mapsto x+1$ satisfies (for the right action):

$$x \cdot (a \cdot f_0) = (x+1)/2 = x \cdot f_1, \quad x \cdot (a \cdot f_1) = (x+2)/2 = x/2 + 1 = x \cdot f_0 \cdot a.$$

We see that the action associated with the biset is the adding machine.

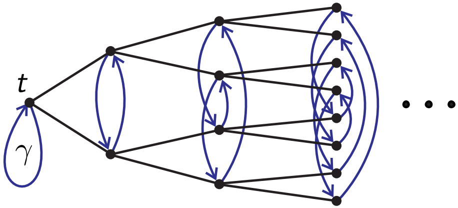
Iterated monodromy groups

Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a finite covering map, where $\mathcal{M}_1 \subset \mathcal{M}$. In general, one can consider a pair of maps $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$. Choose a basepoint $t \in \mathcal{M}$ and consider the tree of preimages $T = \bigsqcup_{n=0}^{\infty} f^{-n}(t)$:



Iterated monodromy group

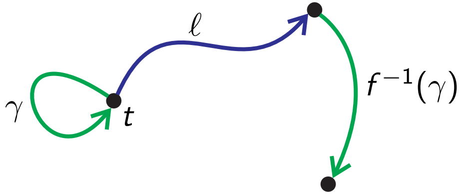
The fundamental group $\pi_1(\mathcal{M}, t)$ acts on T by the *monodromy actions* on each of the levels $f^{-n}(t)$.



The obtained action is the *iterated monodromy action*. The quotient of $\pi_1(\mathcal{M}, t)$ by the kernel of the iterated mondromy action is the *iterated monodromy group* $\mathrm{IMG}(f)$.

Associated biset

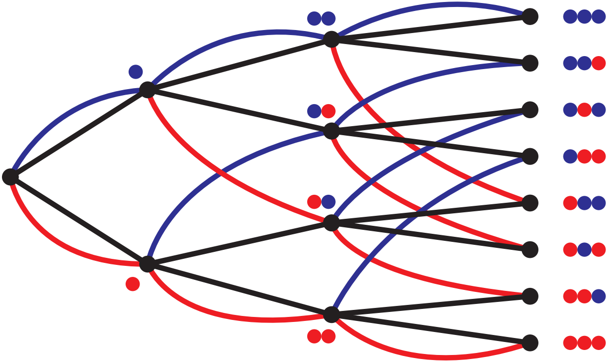
Let $\mathfrak{M}_f$ be the set of homotopy classes of paths in $\mathcal{M}$ from t to a point $z \in f^{-1}(t)$.
The fundamental group acts on $\mathfrak{M}_f$ by attaching loops to the beginning t and lifts of loops by f to the ends $z \in f^{-1}(t)$ of elements of $\mathfrak{M}_f$



It is easy to see that the two actions commute. The first action is free. The number of its orbits is equal to $|f^{-1}(t)|$, where the orbits are in a bijection with the ends of paths $\ell \in \mathfrak{M}$.

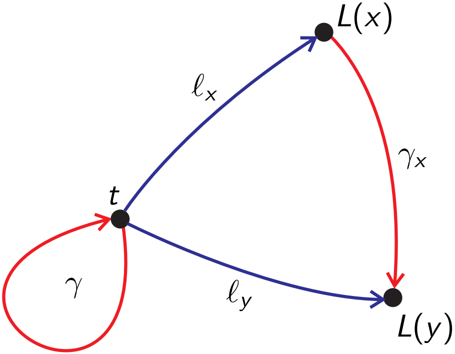
It is easy to check that the self-similar action of $\pi_1(\mathcal{M}, t)$ associated with the biset $\mathfrak{M}_f$ is conjugate to the iterated monodromy action.
Choosing a right orbit transversal, i.e., a collection of paths ℓ_x starting in t and ending in $L(x) \in f^{-1}(t)$ (where $x \in \mathsf{X}$, $|\mathsf{X}| = \deg f$, and $L : \mathsf{X} \longrightarrow f^{-1}(t)$ is a bijection) we get the *standard self-similar action* of the iterated monodromy group, and a labeling of the vertices of T by words over X.

The point labeled by xv is the end of the lift of the path ℓ_x by $f^{|v|}$ starting at the point labeled by v.

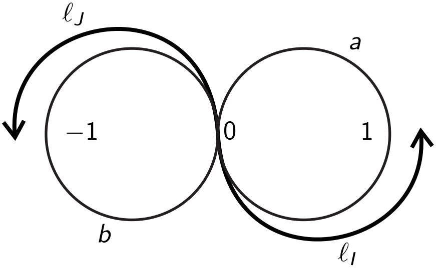


Let $\{\ell_x\}$ and $L : \mathsf{X} \longrightarrow f^{-1}(t)$, be as before. Let $\gamma \in \pi_1(\mathcal{M}, t)$, and let γ_x be the lift of γ starting at $L(x)$. Let $L(y)$ be the end of γ_x. Then the standard action is given by the rule

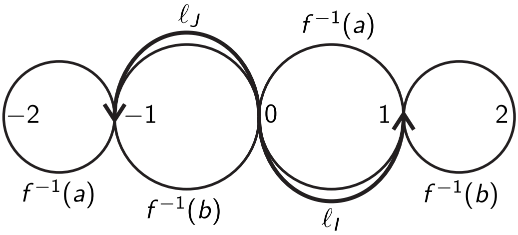
$$\gamma(xw) = y(\ell_y^{-1}\gamma_x\ell_x)(w).$$



The “interlaced adding machine” is the i.m.g. of $f(z) = -\frac{z^3}{2} + \frac{3z}{2}$ seen as the map $\mathbb{C} \setminus \{\pm 1, \pm 2\} \longrightarrow \mathbb{C} \setminus \{\pm 1\}$.



$\mathrm{IMG}(-z^3/2 + 3z/2)$ continued



$$a(Ow) = Iw, \quad a(Iw) = Oa(w), \quad a(Jw) = Jw$$

$$b(Ow) = Jw, \quad b(Iw) = Iw, \quad b(Jw) = Ob(w).$$

A multi-dimensional example

Consider the map F of $\mathbb{C}^2$:

$$(x, y) \mapsto \left(1 - \frac{y^2}{x^2}, 1 - \frac{1}{x^2}\right)$$

It can be naturally extended to the projective plane.

$$(x : y : z) \mapsto (x^2 - y^2 : x^2 - z^2 : x^2).$$

The set $\{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$ is the critical locus. The post-critical set is the union of the line at infinity with the lines $x = 0, x = 1, y = 0, y = 1, x = y$.
They are permuted as follows:

$$\{x = 0\} \mapsto \{z = 0\} \mapsto \{y = 1\} \mapsto \{x = y\} \mapsto \{x = 0\}$$

$$\{y = 0\} \mapsto \{x = 1\} \mapsto \{y = 0\}.$$

The iterated monodromy group of F (as computed by J. Belk and S. Koch) is generated by the transformations:

$$a(1v)=1b(v),\quad a(2v)=2v,\quad a(3v)=3v,\quad a(4v)=4b(v),$$

$$b(1v)=1c(v),\quad b(2v)=2c(v),\quad b(3v)=3v,\quad b(4v)=4v,$$

$$c(1v)=4d(v), c(2v)=3(ceb)^{-1}(v), c(3v)=2(fa)^{-1}(v), c(4v)=1v,$$

$$d(1v)=2v,\quad d(2v)=1a(v),\quad d(3v)=4v,\quad d(4v)=3a(v),$$

$$e(1v)=1f(v),\quad e(2v)=2v,\quad e(3v)=3f(v),\quad e(4v)=4v,$$

$$f(1v)=3b^{-1}(v),\quad f(2v)=4v,\quad f(3v)=1eb(v),\quad f(4v)=2e(v).$$

Bimodules

We have described a relation between two structures: pairs of maps $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$, where f is a covering map (and ι was an embedding), and G-bisets $\mathfrak{M}$. Both define *bimodules* over algebras.

In the case of a pair of maps $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}$, consider the algebra $A = C_0(\mathcal{M})$ of continuous functions on $\mathcal{M}$, and the space $\Phi = C_0(\mathcal{M}_1)$ of continuous functions on $\mathcal{M}_1$. Let Φ be a right module w.r.t.

$$\phi \cdot a(x) = \phi(x)a(f(x)), \quad x \in \mathcal{M}_1$$

with A-valued inner product

$$\langle \phi_1 | \phi_2 \rangle(x) = \sum_{z \in f^{-1}(x)} \overline{\phi_1(z)}\phi_2(z).$$

Define a structure of a left A-module on Φ by

$$(a \cdot \phi)(x) = a(\iota(x))\phi(x), \quad x \in \mathcal{M}_1.$$

In the case of a covering biset $\mathfrak{M}$ over a group G, consider the group ring $A = \mathbb{C}[G]$ and the linear space Φ spanned by $\mathfrak{M}$. Since G acts by commuting left and right actions on $\mathfrak{M}$, extending the action by linearity, we get a bimodule structure on Φ.

We also have an A-valued inner product on Φ given by the condition that if $x, y \in \mathfrak{M}$ are such that $y = x \cdot g$ for some $g \in G$, then $\langle x|y \rangle = g$; otherwise $\langle x|y \rangle = 0$.