Self-similar groups and hyperbolic groupoids II

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Contracting groups

Let (G, X) be a self-similar group. For every $g \in G$ and $v \in \mathsf{X}^*$ there exists $g|_v \in G$ such that

$$g(vw) = g(v)g|_v(w)$$

for all $w \in \mathsf{X}^*$. We call $g|_v$ the *section* of g at v.

We say that (G, X) is *contracting* if there exists a finite subset $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all $v \in \mathsf{X}^*$ such that $|v| \geq n$.

Proposition

If $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is an expanding covering of a Riemannian manifold, then its iterated monodromy group is contracting.

Section $\gamma|_{x_1x_2\ldots x_n}$ is the loop

$$\ell_{y_n}^{-1}\ell_{y_ny_{n-1}}^{-1}\ldots\ell_{y_n\ldots y_1}^{-1}\gamma_{x_1\ldots x_n}\ell_{x_n\ldots x_1}\ldots\ell_{x_nx_{n-1}}\ell_{x_n},$$

where $\ell_{a_n\ldots a_k}$ is the lift of ℓ_{a_n} by f^{n-k} starting at the end of $\ell_{a_{n-1}\ldots a_k}$, and $\gamma_{x_1\ldots x_n}$ is the lift of γ by f^n starting at the end of $\ell_{x_1\ldots x_n}$. Since the length of lifts of paths by f^n exponentially decrease, we get a uniform bound on the length of γ_v for all v big enough.

Some properties of contracting groups

- Contracting groups have solvable word problem in polynomial time.
- Contracting groups have no free subgroups (amenable?).
- Many (all except for virtually nilpotent?) are not finitely presented.

Limit dynamical system

Let (G, X) be a contracting group. Consider the space $\mathsf{X}^{-\omega} = \{\ldots x_2 x_1\}$ of left-infinite sequences over X. Two sequences $\ldots x_2 x_1, \ldots y_2 y_1$ are equivalent w.r.t. G if there exists a sequence $g_n \in G$ assuming a finite set of values such that

$$g_n(x_n \ldots x_1) = y_n \ldots y_1.$$

The quotient of $\mathsf{X}^{-\omega}$ by this equivalence relation is the *limit space* $\mathcal{J}_G$ of (G, X). The equivalence relation is invariant under the shift

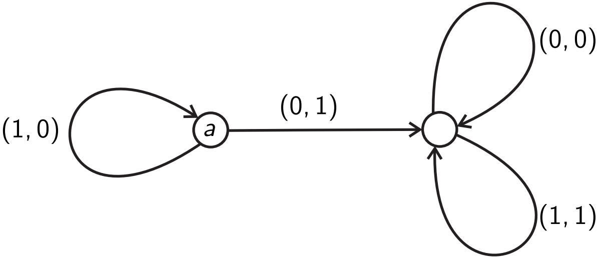
$$\ldots x_2 x_1 \mapsto \ldots x_3 x_2.$$

Hence, the shift induces a continuous map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$.

Another definition

Let Γ be the diagram of the automaton generating (G, X). Paths going along the arrows describe the action of the generators. Let $R \subset \mathsf{X}^{-\omega} \times \mathsf{X}^{-\omega}$ be the pairs of infinite words read along left-infinite paths (paths going against the arrows). Then the equivalence relation generated by R is the equivalence relation defining $\mathcal{J}_G$.

Adding machine action



We get identifications $\ldots 001x_n \ldots x_1 \sim \ldots 110x_n \ldots x_1$, and all their shifts, hence $\mathcal{J}_{\mathbb{Z}} = \mathbb{R}/\mathbb{Z}$.

Main theorem

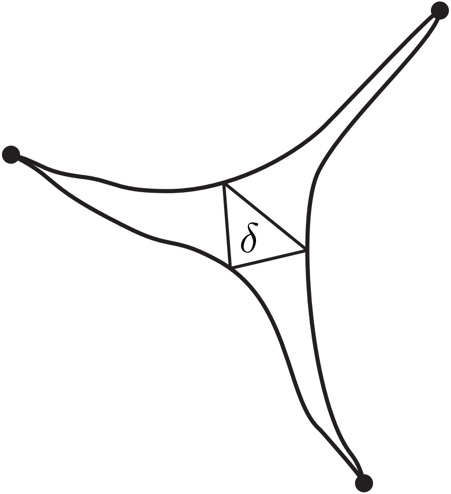
Theorem

Suppose that $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ *is expanding. Then* $G = \mathrm{IMG}(f)$ *is contracting and* $(\mathcal{J}_G, \mathsf{s})$ *is topologically conjugate to the action of* f *on the set of accumulation points of* $\bigcup_{n\geq 0} f^{-n}(t)$.

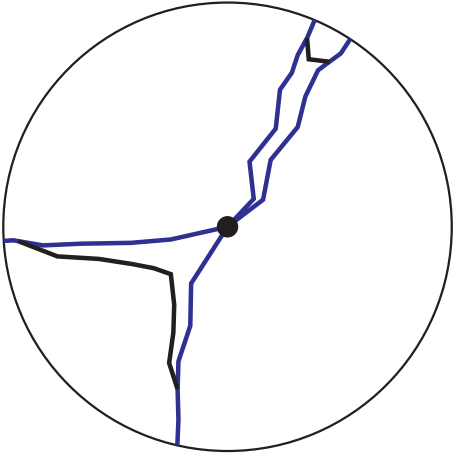
Hence, the contructions of the iterated monodromy group and the limit dynamical system are inverse to each other in hyperbolic cases (expanding maps and contracting self-similar groups).

Hyperbolic graphs

A geodesic metric space (e.g., a graph) is δ-hyperbolic if every geodesic triangle is δ-thin.



Hyperbolic boundary



Limit spaces as Gromov boundaries

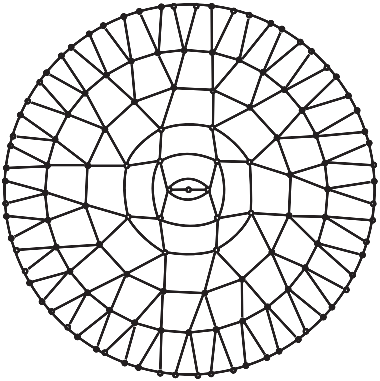
Consider the graph $\Gamma(G, \mathsf{X})$ with the set of vertices X^* in which two vertices are connected by an edge in two cases:

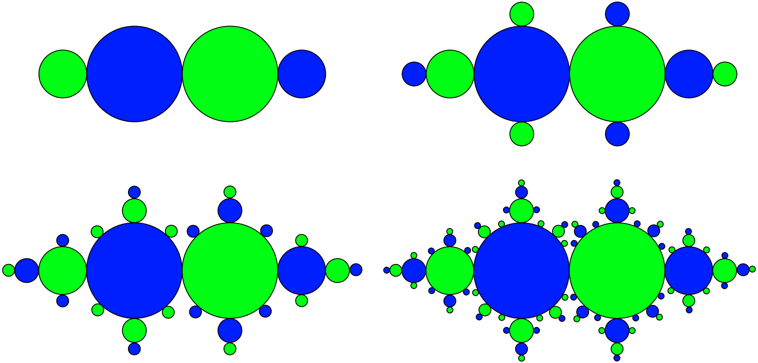
- they are of the form v and xv;
- they are of the form v and $s(v)$, where s belongs to a fixed finite generating set of G.

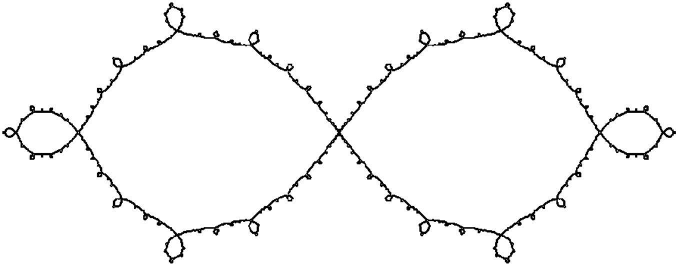
Theorem

If (G, X) is contracting, then $\Gamma(G, \mathsf{X})$ is Gromov hyperbolic. Its boundary is homeomorphic to $\mathcal{J}_G$. The shift $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ is induced by the map $vx \mapsto v$ on $\Gamma(G, \mathsf{X}) \setminus \{\emptyset\}$.

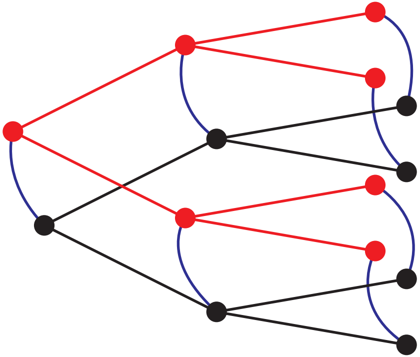
The graph $\Gamma(G, \mathsf{X})$ consists of the tree of edges (v, xv) and of the union of *Schreier* graphs describing the action of the generating set on the levels X^n.







Let $f : \mathcal{M} \longrightarrow \mathcal{M}$ be a self-covering. Let γ be a path in $\mathcal{M}$ from t_1 to t_2. The lifts of γ by iterations of f define an isomorphism H_γ between the trees $T_{t_i} = \bigsqcup_{n=0}^{\infty} f^{-n}(t_i)$.



If f is expanding, then the length of lifts approaches 0, and hence $d(v, H_\gamma(v)) \to 0$ in $\mathcal{M}$ as $v \to \infty$ in T_{t_1}.

If γ is a path in $\mathcal{M}$ between two vertices of T_t, then it defines an isomorphism between the corresponding sub-trees of T_t:
These are transformations of the form $v_1 w \mapsto v_2 g(w)$ for fixed $v_1, v_2 \in \mathsf{X}^*$ and $g \in \mathrm{IMG}(f)$. They form an inverse semigroup generated by G and the transformations $T_x : w \mapsto xw$.

Cuntz-Pimsner algebras of self-similar groups

Let (G, X) be a self-similar group. Its *Cuntz-Pimsner algebra* $\mathcal{O}_G$ is the universal C^* generated by G (as unitaries) and isometries S_x for $x \in \mathsf{X}$ satisfying

$$S_x^* S_x = 1, \qquad \sum_{x \in \mathsf{X}} S_x S_x^* = 1$$

and

$$g \cdot S_x = S_y \cdot h$$

whenever $g \cdot x = y \cdot h$. The algebra generated by S_x, $x \in \mathsf{X}$ is the *Cuntz algebra* $\mathcal{O}_{|\mathsf{X}|}$.

Let $H = L^2(\mathsf{X}^\omega)$ for the uniform Bernoulli measure on X^ω. G acts by measure-preserving transformations on X^ω, and we get the associated unitary representation of G on H. The transformations $T_x : w \mapsto xw$ satisfy $\mu(T_x(A)) = \frac{1}{|\mathsf{X}|}\mu(A)$, hence the operator S_x given by

$$S_x(f)(yw) = \begin{cases} \sqrt{|\mathsf{X}|} f(w) & \text{if } y = x \\ 0 & \text{othewise} \end{cases}$$

is an isometry. It is easy to check that G and S_x satisfying the defining relations of $\mathcal{O}_G$, hence they define a representation of $\mathcal{O}_G$.

A contracting group (G, X) is *regular* if for every $g \in G$ and $w \in \mathsf{X}^\omega$ either $g(w) \neq w$, or g acts trivially on a neighborhood of w. Equivalently, (G, X) is regular iff $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ is a covering map. For example, $\mathrm{IMG}(f)$ is regular if f is a hyperbolic rational function.

Theorem

If (G, X) is a regular contracting group, then $\mathcal{O}_G$ is defined by a finite set of relations, simple, purely infinite, nuclear, and satisfies the UCT.

In particular, $\mathcal{O}_G$ is a Kirchberg algebra, and is classifiable by its K-theory. If $f(z) = z^2 + c$ is a hyperbolic quadratic polynomial, then $\mathcal{O}_{\mathrm{IMG}(f)}$ is generated by the Cuntz algebra $\mathcal{O}_2 = \langle S_0, S_1 \rangle$ and a unitary a satisfying

$$a = S_1 S_0^* + S_0(1 - S_v S_v^* + S_v a S_v^*) S_1^*.$$

for a product $S_v = S_{x_1} S_{x_2} \cdots S_{x_n}$, where $x_1 x_2 \ldots x_n$ is related to the *kneading sequence* of the polynomial.

K-theory of hyperbolic rational functions

Theorem

Let $G = \mathrm{IMG}(f)$, where f is a hyperbolic rational function. Then

$$K_0(\mathcal{O}_{\mathrm{IMG}(f)}) \cong \mathbb{Z}/(d-1)\mathbb{Z} \oplus \mathbb{Z}^{c-1},$$

$$K_1(\mathcal{O}_{\mathrm{IMG}(f)}) \cong \mathbb{Z}/l\mathbb{Z} \oplus \mathbb{Z}^{c-1},$$

where $d = \deg f$, c is the number of attracting cycles, l is the g.c.d. of their lengths.

In particular, all algebras $\mathcal{O}_{\mathrm{IMG}(f)}$ are pairwise isomorphic for all hyperbolic quadratic polynomials $f(z) = z^2 + c$. The Cuntz-Pimsner algebras of the bimodules over $C_0(J_f)$ have the same K-theory.

The Cuntz-Pimsner algebra $\mathcal{O}_G$ have a natural action of the circle:

$$\Gamma_z(S_x) = zS_x, \qquad \Gamma_z(g) = g.$$

Theorem

Let (G_i, X) be regular contracting groups. The dynamical systems $(\mathcal{O}_{G_i}, \Gamma_z)$, $i = 1, 2$, are isomorphic iff the limit dynamical systems $(\mathcal{J}_{G_i}, \mathsf{s})$ are topologically conjugate.