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Fall 2006

Groups and Holomorphic Dynamics — Problem Set 1
Basics of Holomorphic Dynamics

1.1. Cross-ratio. a) Show that the group of biholomorphic automorphisms of $\hat{\mathbb{C}}$ is generated by translations $z \mapsto z + a$ and the inversion $z \mapsto 1/z$.

b) Given four different points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ show that the cross-ratio

$$\chi(z_1, z_2, z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_2 - z_1)(z_4 - z_3)}$$

is invariant under the automorphisms of $\hat{\mathbb{C}}$.

1.2. Multiplier at infinity. Suppose that ∞ is a fixed point of a rational function f. Show that the multiplier of f at ∞ is equal to $\lim_{z\to\infty} 1/f'(z)$.

1.3. Degree 1 maps. Prove that the Julia set of a linear function $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ is either empty or consists of a single repelling or parabolic point.

1.4. Parts of the Mandelbrot set. Describe the set of values of c such that $f(z) = z^2+c$ has an attracting fixed point. The same question for an attracting cycle of length 2.

1.5. When a quadratic polynomial $z^2 + c$ has two repelling fixed points?

1.6. Let R be a rational function and let C be a circle such that $R(C) \subset C$. Prove that the Julia set of R is either C or is a totally disconnected subset of C.

1.7. Dense forward orbits. Find a number $z \in T = \{z \ : \ |z| = 1\} \subset \mathbb{C}$ such that z^{2^n} is dense in T.

1.8. Cantor set. Show that the Julia set of $z^2 - 6$ is a Cantor set. Give a more precise description.

1.9. Newton's method I. Let f be a polynomial and let $N(z) = z - \frac{f(z)}{f'(z)}$. Show that the finite fixed points of N are attracting with multiplier $1 - 1/m$, where m is the multiplicity of the point as a root of f, hence that the fixed points of N are exactly the roots of f (if we do not take into account the repelling fixed point ∞).

1.10. Newton's method II. Study the Newton's method for $f(z) = z^2 + 1$: which initial points will converge to which roots.

1.11. Pre-periodic Fatou components. Prove that if the Fatou set of a rational function has a pre-periodic component, then it has infinitely many components.

1.12. A sub-hyperbolic function. Show that the rational function $f(z) = \left(1 - \frac{2}{z}\right)^{2n}$ is sub-hyperbolic and that its Julia set is the whole sphere $\hat{\mathbb{C}}$. Show that for $n = 1$ the associated orbifold is Euclidean. Give an interpretation of f in the spirit of Lattès' examples.