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Math 308 — Problem Set 7solutions
Issued: 10.29 Due: Training Homework Solutions

7s.1. The characteristic polynomial is $\lambda^2 - \lambda - 2$; therefore, the eigenvalues are 2 and -1. For $\lambda_1 = 2$ the matrix $A - \lambda_1 I$ is $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$, so that $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector.

For $\lambda_2 = -1$ the matrix $A = \lambda_2 I$ is $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$, so an eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

The corresponding solutions are $\mathbf{x}_1 = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Therefore the general solution is

$$\mathbf{x} = c_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

7s.2. The characteristic polynomial is $\lambda^2 + 3\lambda + 2$, hence the eigenvalues are -2 and -1. An eigenvector corresponding to the first eigenvalue is $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. An eigenvector corresponding to the second eigenvalue is $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore, the general solution is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ implies that

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Solving the system we get the solution

$$\mathbf{x}(t) = -2e^{-2t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 7e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Both components of the solution go to zero as $t \to \infty$.

7s.3. The characteristic polynomial is $\lambda^2 + 9$, hence the eigenvalues are $\pm 3i$. The first eigenvalue $\lambda_1 = 3i$ has corresponding eigenvector $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1-3i \end{pmatrix}$. It follows that

$$\mathbf{x}_1(t) = e^{3it}\begin{pmatrix} -2 \\ 1-3i \end{pmatrix}$$

is a solution of the equation.

Taking real and imaginary values we get two real solutions

$$\mathrm{Re}\mathbf{x}_1 = \begin{pmatrix} -2\cos(3t) \\ \cos(3t) + 3\sin(3t) \end{pmatrix}$$

and

$$\mathrm{Im}\mathbf{x}_1 = \begin{pmatrix} -2\sin(3t) \\ -3\cos(3t) + \sin(3t) \end{pmatrix}.$$

The general solution is the linear combination of these two solutions with coefficients c_1 and c_2.

7s.4. The eigenvalues of the matrix are $\lambda = -1 \pm 2i$.

An eigenvector corresponding to $-1 + 2i$ is $\begin{pmatrix} 2i \\ 1 \end{pmatrix}$. Therefore, a solution is

$$\mathbf{x}_1 = e^{(-1+2i)t}\begin{pmatrix} 2i \\ 1 \end{pmatrix}.$$

Taking real and imaginary parts we get fundamental solutions

$$e^{-t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix}, \qquad e^{-t}\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}.$$

The general solution is then

$$\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}.$$

The initial condition $\mathbf{x}(0) = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ implies

$$c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

Solving the system we get $c_1 = -3, c_2 = 2$. Both components of the solution decay to zero as $t \to \infty$.

7s.5. The characteristic polynomial is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, therefore the only eigenvalue is $\lambda = 1$. The matrix $A - I$ is $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$. An eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. A solution of the system of ODE's is then

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We need to look for a solution $\mathbf{w}$ of $(A - I)\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. A solution is $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore

$$\mathbf{x}_2 = te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is another solution. The general solution is then

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2.$$

The solution grows as $t \to \infty$.

7s.6. The roots of the characteristic equation are $\lambda = -4$ and 2. Therefore the general solution is

$$y(t) = c_1 e^{-4t} + c_2 e^{2t}.$$

7s.7. Let $y_2(t) = \sin(x^2)v(x)$. Substituting it into the differential equation we get the equation

$$v'' + \left(\frac{4x\cos(x^2)}{\sin(x^2)} - \frac{1}{x} \right) v' = 0.$$

Solving it with respect to $u = v'$ we get

$$v'(x) = \exp\left(\int \left(\frac{1}{x} - \frac{4x\cos(x^2)}{\sin(x^2)}\right) dx\right) = \frac{cx}{(\sin(x^2))^2}$$

(use substitution $s = \sin(x^2)$ to compute integral of the second summand), therefore,

$$v(x) = c_1 \frac{\cos(x^2)}{\sin(x^2)} + c_2$$

(use substitution $s = x^2$ to compute the integral). We get

$$y_2(x) = c_1 \cos(x^2) + c_2 \sin(x^2).$$

We can take then $y_2(x) = \cos(x^2)$.

7s.8. Let $z = \ln x$. Then our equation reduces to

$$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = 0.$$

The characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$. The roots are $\lambda = -1$ and -2. Therefore the general solution is

$$y(z) = c_1 e^{-2z} + c_2 e^{-z} = c_1 x^{-2} + c_2 x^{-1}.$$

7s.9. The spring dconstant is $k = .9/.05 = 19.6\text{N/m}$. The mass $m = 0.1\text{kg}$. Therefore, the equation of motion is $0.1y'' + 19.6y = 0$, or $y'' + 196y = 0$. We get $\omega_0 = 14$ and the general solution is $y = A\cos(14t) + B\sin(14t)$. The initial condition $y(0) = 0\text{m}$ and $y'(0) = 10\text{cm/sec}$ implies $A = 0$ and $B = 5/7$. Therefore, the solution is $y(t) = \frac{5}{7}\sin(14t)$.

7s.10. The characteristic equation for the homogeneous problem is $2\lambda^2 + 3\lambda + 1 = 0$, which has roots $-1, -1/2$. Therefore, the general solution of the homogeneous problem is $y_h(t) = c_1 e^{-t} + c_2 e^{-t/2}$. We look for a solution $Y_1(t)$ of the form $A + Bt + Ct^2$ of the equation $2y'' + 3y' + y = t^2$ and then look for a solution $Y_2(t)$ of the form $D\cos t + E\sin t$ of the equation $2y'' + 3y' + y = 3\sin t$. For Y_1 we get conditions $A + 3B + 4C = 0$, $B + 6C = 0$ and $C = 1$. Solving these equations we get $A = 14, B = -6$ and $C = 1$. For Y_2 we find $D = -9/10$ and $E = -3/10$. Therefore, the general solution of the equation is

$$y(t) = c_1 e^{-t} + c_2 e^{-t/2} + 14 - 6t + t^2 - 0.9\cos t - 0.3\sin t.$$