1. Use the transformation \( x = \sqrt{2} u - \sqrt{2/3} v, \ y = \sqrt{2} u + \sqrt{2/3} v \) to evaluate the integral \( \iint_{R} (x^2 - xy + y^2) \, dA \), where \( R \) is the region bounded by the ellipse \( x^2 - xy + y^2 = 2 \).

Jacobian of the transformation is \( \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| = \left| \begin{array}{cc} \sqrt{2} & \sqrt{2/3} \\ -\sqrt{2/3} & \sqrt{2} \end{array} \right| = \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}} \).

The function \( x^2 - xy + y^2 \) in new variables becomes
\[
(\sqrt{2} u - \sqrt{2/3} v)^2 - (\sqrt{2} u - \sqrt{2/3} v)(\sqrt{2/3} v + \sqrt{2/3} u) + (\sqrt{2} u + \sqrt{2/3} v)^2 = (2u^2 - 4\sqrt{3} uv + \frac{2}{3}v^2) - (2u^2 - \frac{2}{3}v^2) + (2u^2 + 4\sqrt{3} uv + \frac{2}{3}v^2) = 2u^2 + 2v^2.
\]

Therefore, the curve \( x^2 - xy + y^2 = 2 \) becomes \( 2u^2 + 2v^2 = 2 \), which is equivalent to \( u^2 + v^2 = 1 \).

The integral is equal then to
\[
\int_{u^2 + v^2 \leq 1} (2u^2 + 2v^2) \cdot \frac{4}{\sqrt{3}} \, du \, dv.
\]

Passing to polar coordinates, we get that it is equal to
\[
\frac{8}{\sqrt{3}} \int_{0}^{2\pi} \int_{0}^{1} r^2 \cdot r \, dr \, d\theta = \frac{8}{\sqrt{3}} \cdot 2\pi \cdot \frac{1}{4} = \frac{4\pi}{\sqrt{3}}.
\]

2. Find a function \( f \) such that \( \nabla f = F \) for \( F = y \, \mathbf{i} + (x + z) \, \mathbf{j} + y \, \mathbf{k} \). Use it to find the integral \( \int_{C} F \cdot d\mathbf{r} \) along the curve \( C : r(t) = te^t \, \mathbf{i} + (1 + t) \, \mathbf{j} + t \, \mathbf{k}, \ 0 \leq t \leq 1 \).

We have to find a function \( f(x,y,z) \) such that \( \frac{\partial f}{\partial x} = y, \ \frac{\partial f}{\partial y} = x + z, \ \frac{\partial f}{\partial z} = y \). It follows from the first equality, that \( f \) is of the form \( xy + g(y,z) \). Substituting this into the second equality, we get that \( x + \frac{\partial g(y,z)}{\partial y} = x + z \). It follows that \( g(y,z) \) is of the form \( yz + h(z) \), hence \( f \) is of the form \( xy + yz + h(z) \). Substituting this into the third equation, we get that \( y + h'(z) \) is equal to \( y \), which implies that \( h(z) \) is a constant. It follows that we can take \( f(x,y,z) = xy + yz = y(x + z) \).

Beginning of the curve \( C \) is the point \( (0, e^0, 1 + 0, 0) = (0, 1, 0) \), its end is \( (e, 2, 1) \). It follows that the integral \( \int_{C} F \cdot d\mathbf{r} \) is equal to \( f(e, 2, 1) - f(0, 1, 0) = 2(e + 1) - 1(0 + 0) = 2(e + 1) \).

3. Find the area of the region bounded by the curve with parametric equation \( (x(t), y(t)) = (\cos^3 t \, i + \sin^3 t \, j), 0 \leq t \leq 2\pi \), using Green’s Theorem, by evaluating one of the integrals \( \int_{C} -y \, dx, \ \int_{C} x \, dy, \) or \( \frac{1}{2} \int_{C} (-y \, dx + x \, dy) \).

Let us evaluate the third integral (one can also use the first two).
\[
\frac{1}{2} \int_{0}^{2\pi} (-\sin^3 t \cdot (\cos^3 t)' + \cos^3 t(\sin^3 t)') \, dt = \\
\frac{1}{2} \int_{0}^{2\pi} (3\sin^3 t \cos^3 t \sin t + 3\cos^3 t \sin^2 t \cos^2 t) \, dt = \\
\frac{3}{2} \int_{0}^{2\pi} (\sin^4 t \cos^2 t + \cos^4 t \sin^2 t) \, dt = \\
3 \int_{0}^{2\pi} \sin^2 t \cos^2 t (\sin^2 t + \cos^2 t) \, dt = 3 \int_{0}^{2\pi} \sin^2 t \cos^2 t \, dt = \\
\frac{3}{8} \int_{0}^{2\pi} \sin^2 2t \, dt = \frac{3}{8} \int_{0}^{2\pi} \frac{1 - \cos 4t}{2} \, dt = \frac{3\pi}{8}.
\]
4. Find the area of the surface given by the parametric equation \( \mathbf{r}(u, v) = uv \mathbf{i} + (u + v) \mathbf{j} + (u - v) \mathbf{k} \) for \( u^2 + v^2 \leq 1 \).

Let us find the area element \( dS = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv \). We have

\[
\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 0 & 0 \\
1 & 1 & -1
\end{vmatrix} = (-2, u + v, v - u).
\]

Its length is \( \sqrt{4 + (u + v)^2 + (v - u)^2} = \sqrt{4 + 2u^2 + 2v^2} \). Hence, the area is equal to

\[
\iint_{u^2+v^2\leq 1} \sqrt{4 + 2(u^2 + v^2)} \, du \, dv.
\]

Let us pass to polar coordinates:

\[
\int_0^{2\pi} \int_0^1 \sqrt{4 + 2r^2} \cdot r \, dr \, d\theta = 2\pi \int_0^1 \sqrt{4 + 2r^2} \cdot r \, dr.
\]

Make a substitution \( t = 4 + 2r^2 \). Then \( dt = 4r \, dr \), so that the area is equal to

\[
\frac{2\pi}{4} \int_4^6 \sqrt{t} \, dt = \frac{\pi}{2} \left( \frac{2}{3} t^{3/2} \right) \bigg|_{t=4}^{t=6} = \frac{\pi}{3} (6^{3/2} - 4^{3/2}).
\]

5. Evaluate \( \iint_S \mathbf{F} \cdot d\mathbf{S} \) for \( \mathbf{F} = -y \mathbf{j} + z \mathbf{k} \), where \( S \) is the part of the paraboloid \( z = x^2 + y^2 \) below the plane \( z = 1 \) with upward orientation.

The surface is the graph of the function \( g(x, y) = x^2 + y^2 \). It follows that \( d\mathbf{S} = (-2x, -2y, 1) \).

The integral is equal to

\[
\iint_{x^2+y^2\leq 1} (0, -y, x^2 + y^2) \cdot (-2x, -2y, 1) \, dx \, dy = \iint_{x^2+y^2\leq 1} 2y^2 + x^2 + y^2 \, dx \, dy.
\]

Let us pass to polar coordinates:

\[
\int_0^{2\pi} \int_0^1 (2r^2 \sin^2 \theta + r^2) \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (2\sin^2 \theta + 1) r^3 \, dr \, d\theta = \int_0^1 r^3 \, dr \int_0^{2\pi} (2\sin^2 \theta + 1) \, d\theta = \frac{1}{4} \int_0^{2\pi} (1 - \cos 2\theta + 1) \, d\theta = \frac{1}{4} \int_0^{2\pi} (2 - \cos 2\theta) \, d\theta = \frac{1}{4} \cdot 2 \cdot 2\pi = \pi.
\]

6. Use Stokes’ Theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F} = z^2 \mathbf{i} + y^2 \mathbf{j} + xy \mathbf{k} \), where \( C \) is the triangle with vertices \((1,0,0),(0,1,0),(0,0,2)\) oriented counterclockwise as viewed from above.

Curl of the vector field is

\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
z^2 & y^2 & xy
\end{vmatrix} = (x, 2z - y, 0).
\]

Let us use the plane of the triangle as our surface. Equation of the plane is \( x + y + z/2 = 1 \). We can write it as \( z = 2 - 2x - 2y \), and parametrize it by \((x, y)\). Then \( d\mathbf{S} = (2, 2, 1) \, dx \, dy \). It follows that the integral is equal to

\[
\iint_T (x, 2(2-2x-2y)-y, 0) \cdot (2, 2, 1) \, dx \, dy = \iint_T (2x + 8x - 8y - 2y) \, dx \, dy = \iint_T (8 - 6x - 10y) \, dx \, dy,
\]
where $T$ is the projection of the triangle onto the $xy$-plane, i.e., the triangle in the $xy$ plane with vertices $(1,0)$, $(0,1)$, and $(0,0)$. It follows that the integral is equal to

$$
\int_0^1 \int_0^{1-y} 8 - 6x - 10y \, dx \, dy = \int_0^1 (8(1-y) - 3(1-y)^2 - 10y(1-y)) \, dy = \\
\int_0^1 (8 - 8y - 3 + 6y - 3y^2 - 10y + 10y^2) \, dy = \int_0^1 (7y^2 - 12y + 5) \, dy = \frac{7}{3} - \frac{12}{2} + 5 = \frac{4}{3}.
$$