1. Groups and their actions

Let G be a group. An *(left) action* of G on a set X is a map $G \times X \longrightarrow X : (g, x) \mapsto g \cdot x$ satisfying

(1) $1 \cdot x = x$;
(2) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

We say then that X is a (left) G-set. An *isomorphism* of left G-sets $\phi : X_1 \longrightarrow X_2$ is a bijection satisfying

$$\phi(g \cdot x) = g \cdot \phi(x)$$

for all $x \in X_1$.

The right actions are defined in a similar way, and are usually denoted $x \cdot g$ or x^g.

If X has some structure (e.g., topology, manifold, group, linear space), then we say that G preserves this structure if $x \mapsto g \cdot x$ is an automorphism of the structure (homeomorphism, diffeomorphism, automorphism, linearity). If G is a topological group, and X is a topological space, then the action is *continuous* if the map $G \times X \longrightarrow X$ is continuous.

An action defines a natural homomorphism from G to the symmetric group $\mathrm{Symm}\,(X)$ of all permutations $X \longrightarrow X$. Namely, for every $g \in G$ the corresponding element of $\mathrm{Symm}\,(X)$ is the permutation $x \mapsto g \cdot x$.

The *kernel of the action* is the kernel of the homomorphism $G \longrightarrow \mathrm{Symm}\,(X)$, i.e., the subgroup of elements $g \in G$ such that $g \cdot x = x$ for all $x \in X$. The action is called *faithful* if its kernel is trivial.

We say that $x, y \in X$ *belong to the same orbit* if there exists $g \in G$ such that $g \cdot x = y$. It is easy to see that this is an equivalence relation. The action is said to be *transitive* if there is only one orbit.

For $x \in X$ the *stabilizer* of x in G is the subgroup $G_x = \{g \in G \ : \ g \cdot x = x\}$

If X is a topological space, then there are different weaker notions of (topological transitivity). An action is called *topologically transitive* if there exists $x \in X$ such that the orbit of x is dense. It is called *minimal* if every orbit is dense.

PROPOSITION 1.1. *If X is a complete separable metric space, then a G-action is topologically transitive if and only if for any two non-empty open subsets $U, V \subset X$ there exists $g \in G$ such that $g \cdot U \cap V \neq \emptyset$.*

PROOF. If there exists a dense orbit $G \cdot x$, then for any non-empty open subsets U, V there exist $g_1, g_2 \in G$ such that $g_1 \cdot x \in U$ and $g_2 \cdot x \in V$. Then $g_2 g_1^{-1} \cdot U \cap V \ni g_2 \cdot x$.

In the other direction, suppose that for any two open non-empty subsets U, V there exists $g \in G$ such that $g \cdot U \cap V \neq \emptyset$. Let $\{U_i\}_{i=1}^{\infty}$ be a countable basis of topology of X (which exists, since X is a separable metric space). Denote $V_i = \bigcup_{g \in G} g \cdot U_i$. Then V_i is dense and open. By Baire's category theorem, $Y = \cap_{i=1}^{\infty} V_i$ is co-meager, and hence non-empty. Let $x \in Y$. Then for every open set W there exists $U_i \subset W$. We have $x \in V_i$, hence there exists $g \in G$ such that $g \cdot x \in U_i \subset W$. □

EXAMPLE 1.1. Consider the circle $\mathbb{R}/\mathbb{Z}$, and the map $R_\theta : x \mapsto x + \theta$. Consider the action of the infinite cyclic group $\mathbb{Z}$ generated by R_θ.

PROPOSITION 1.2. *For every $\alpha \in \mathbb{R}/\mathbb{Z}$ and every irrational $\theta \in \mathbb{R}$ the set $\alpha + n\theta \pmod 1$ is dense in $\mathbb{R}/\mathbb{Z}$.*

Proof. Denote by $\alpha \mod 1$ the number $t \in [0,1)$ such that $\alpha - t \in \mathbb{Z}$.

Since $n\theta \mod 1$ is infinite, for every N there exist $n_1 \neq n_2$ and $k \in \{0, 1, \ldots, N-1\}$ such that $(n_1\theta - n_2\theta) \mod 1 \in (k/N, (k+1)/N)$. Then $|(n_1 - n_2)\theta| \mod 1 < 1/N$. We have shown that for every $\epsilon > 0$ there exists $n \in \mathbb{Z}$ such that $n\theta \mod 1 < \epsilon$. Consider then the sequence $0, n\theta, 2n\theta, \ldots$ of points of $\mathbb{R}/\mathbb{Z}$. It divides the circle into arcs of length less than ϵ. It follows that for every $\alpha \in \mathbb{R}/\mathbb{Z}$ there exists k such that $|\alpha - nk\theta| \mod 1 < \epsilon$. Consequently, the orbit $\{n\theta\}$ of 0 is dense in $\mathbb{R}/\mathbb{Z}$. Rotating the circle by α we see that orbit of α is dense in $\mathbb{R}/\mathbb{Z}$. □

Example 1.2. Consider a finite alphabet A, and let $X = A^{\mathbb{Z}}$ be the topological space of all maps $w : \mathbb{Z} \longrightarrow A$ (with pointwise convergence, where A is discrete). Consider the shift map given by $s(w)(n) = w(n+1)$.

Proposition 1.3. *The shift is topologically transitive but not minimal.*

Proof. It is not minimal, since orbits of periodic sequences are finite. We can find a sequence w such that for any finite word v there exists i such that $w(i)w(i+1)\ldots w(i+k) = v$ for some i and k. Then the orbit of w is dense. One way to find such w is to list all possible finite words in a bi-infinite sequence (which is possible since the set of all finite words is countable). Another way is to write a sequence of independent random identically distributed letters (so that each letter has a non-zero probability). For a given word v and given number $i \in \mathbb{Z}$, probability that v is not equal to $w(i)w(i+1)\ldots w(i+|v|-1)$ is strictly less than one. It follows that probability that v is not a subword of a random infinite word w is zero. Consequently, with probability 1 a random infinite word $w \in A^{\mathbb{Z}}$ contains every finite subword, i.e., its orbit under the shift is dense. □