2. Cayley and Schreier graphs

Let G be a group, and $H \leq G$ its subgroup. Then the set of left cosets $G/H = \{gH \ : \ g \in G\}$ is a left G-set with respect to the natural action $g \cdot hH = (gh)H$. Every transitive G-set is isomorphic to a G-set of the form G/H. More precisely:

Theorem 2.1. *Let X be a transitive G-set. Choose a basepoint $x_0 \in X$. Then the map $g \cdot x_0 \mapsto gG_{x_0}$ is a well defined isomorphism $X \mapsto G/G_{x_0}$ of G-sets.*

The proof is straightforward. As a corollary, we get that cardinality of a G-set X is equal to the index of a stabilizer G_x in G.

Definition 2.1. An (*oriented*) *graph* Γ is given by the set V of its *vertices*, the set E of its *edges*, and the maps $\alpha, \omega : E \longrightarrow V$. Here $\alpha(e)$ and $\omega(e)$ are the *beginning* and *end* of the edge e.

A *labeled graph* is a graph together with a map $\lambda : E \longrightarrow A$, where A is a *set of labels*. We consider sometimes the Cayley graph $\Gamma(G, S)$ as labeled graph with the natural labeling $\lambda(s, g) = s$.

Sometimes we also add an involution $i : E \longrightarrow E$ satisfying $\alpha(i(e)) = \omega(e)$ and $\omega(i(e)) = \alpha(e)$. In this case the graph is considered to be non-oriented.

Suppose that S is a finite generating set of G, and let X be a G-space. Then *graph of action* is the graph with the set of vertices $V = X$, set of edges $E = S \times X$, beginning and end maps

$$\alpha(s, x) = x, \qquad \omega(s, x) = s \cdot x,$$

and labeling $\lambda(s, x) = s$.

Note that the graph of the action completely determines the action of G on X.

Particular cases of graphs of actions have special names.

Definition 2.2. *(Left) Cayley graph* $\Gamma(G, S)$ of G is the graph of the action of G on itself by left multiplication.

For a subgroup $H \leq G$ the *Schreier graph* $\Gamma(G/H, S)$ is the graph of the action of G on the set of left cosets G/H.

Note that by Theorem 2.1, graph of every transitive action is isomorphic to the Schreier graph $\Gamma(G/G_x, S)$, where G_x is the stabilizer of a point $x \in X$. In view of this, graphs of actions are often also called Schreier graphs.

If X is a topological space, then the set of edges $S \times X$ is also a topological space (where S is discrete), and the structure maps α, ω, λ are continuous. Thus, the graph of action is a *topological graph*.

We have a natural map $p : \Gamma(G, S) \longrightarrow \Gamma(G/H, S)$ given by $p(g) = gH$. It is a morphism of graphs, and is a *covering*, i.e., for every vertex v of $\Gamma(G, S)$ the induced maps $p : \alpha^{-1}(v) \longrightarrow \alpha^{-1}(p(v))$ and $p : \omega^{-1}(v) \longrightarrow \omega^{-1}(p(v))$ are bijections.

Note also that G acts on $\Gamma(G, S)$ on the right: the map $g \mapsto gh$ is an automorphism of $\Gamma(G, S)$ for every $h \in G$.

The Schreier graph $\Gamma(G/H, S)$ is the quotient of $\Gamma(G, S)$ by the right action of H. If H is normal, then $\Gamma(G/H, S)$ coincides with the Cayley graph of G/H.

2.1. Example: a virtually abelian group. Let G be the group of isometries of $\mathbb{Z}^2$ with its natural action on $\mathbb{R}^2$. It is generated by translations and the group

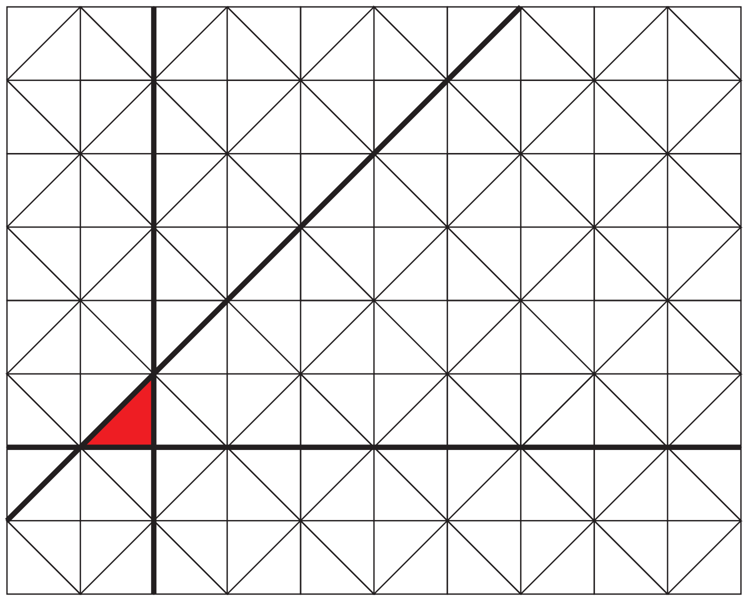


FIGURE 1. Fundamental domain of G

D_4 of isometries of the square with vertices $(\pm 1, \pm 1)$. One can check that it is also generated by the following three elements:

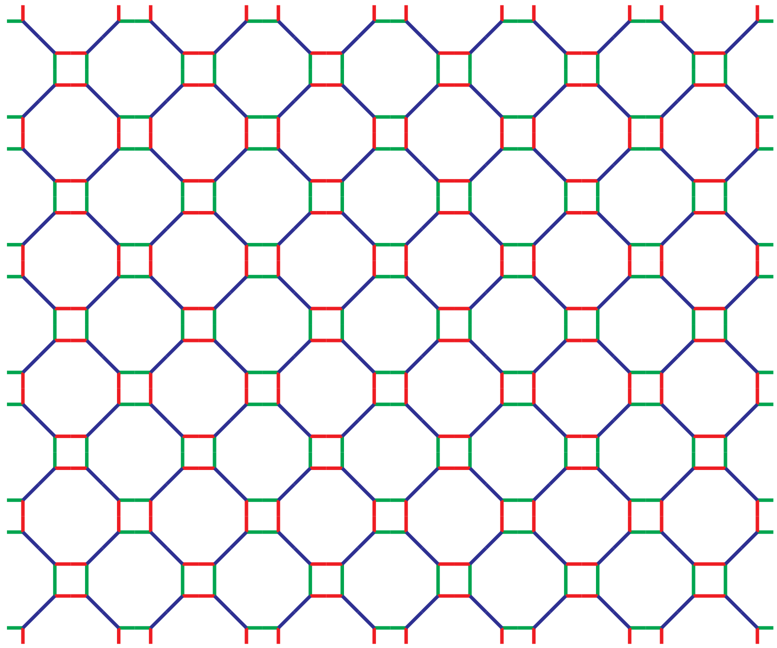
$$a : (x, y) \mapsto (x, -y), \quad b : (x, y) \mapsto (y, x), \quad c : (x, y) \mapsto (1 - x, y).$$

The elements a, b, and c are reflections with respect to the lines $x = 0$, $x = y$, and $x = 1/2$, respectively. These three lines form triangle Δ with vertices $(0, 0)$, $(1/2, 1/2)$, and $(1/2, 0)$. It is easy to see that Δ is a *fundamental domain* of the action of G on $\mathbb{R}^2$: images $g \cdot \Delta$ for $g \in G$ have disjoint interiors and tile the whole plane $\mathbb{R}^2$, see Figure 1. Moreover, the action of G on the orbit of Δ is free: only the trivial element of G leaves Δ invariant.

Let us describe the connected components of the graph of action of G on $\mathbb{R}^2$. Suppose that (x, y) belongs to the interior of Δ. Then the stabilizer of (x, y) in G is trivial, since $g \cdot (x, y) \notin \Delta$ for all $g \neq 1$.

Since the action of G on the orbit of Δ is free, the orbit of (x, y) is in a bijection with the triangles $g \cdot \Delta$. Namely, for every triangle there exists exactly one element of the orbit of (x, y) belonging to that triangle. A natural graph would be then the adjacency graph, in which two vertices are connected by an edge if and only if the corresponding triangles have a common side, see Figure 2.1. But this is not the graph of the action.

On the other hand, if (x', y') is a point of the orbit of (x, y) belonging to one of the three neighbors of Δ, then (x', y') is the image of (x, y) under the action of the corresponding generator $s \in \{a, b, c\}$. Applying an arbitrary element $g \in G$, we see that $g \cdot (x, y)$ and $gs \cdot (x, y)$ belong to neighboring triangles. It follows that the adjacency graph coincides with the *right* Cayley graph of G. But left and right Cayley graphs are isomorphic: the isomorphism is the map $g \mapsto g^{-1}$. We see that the graph of the action on the orbit of a point with trivial stabilizer (i.e., the Cayley graph) is isomorphic to the adjacency graph.



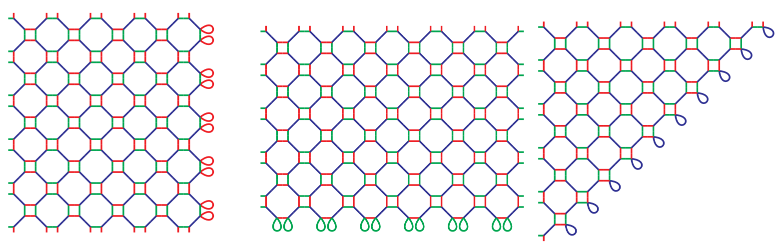


FIGURE 2. Graphs of action for points with $\mathbb{Z}/2\mathbb{Z}$ stabilizer

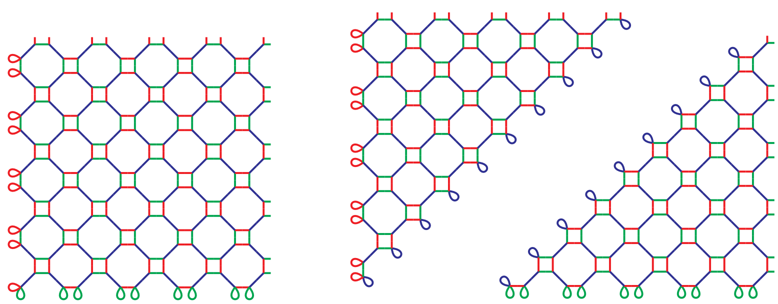


FIGURE 3. Graphs of action for points with dihedral stabilizer

If (x, y) belongs to the boundary of Δ, then the graph of the action is isomorphic to the quotient of the Cayley graph by the action of its stabilizer. The latter is either $\mathbb{Z}/2\mathbb{Z}$, or D_4, or D_2. See the corresponding graphs of action on Figures 2 and 3.

2.2. Graphs of minimal actions. Let G be a finitely generated group acting by homeomorphisms on a complete metric space $\mathcal{X}$. For $x \in \mathcal{X}$, denote by Γ_x the graph of the action of G on the orbit $G \cdot x$ of x (defined with respect to a fixed finite generating set S).

Definition 2.3. A point $x \in \mathcal{X}$ is called G-generic if for every $g \in G$ either $g \cdot x \neq x$, or x belongs to the interior of the set of fixed points of g.

Note that for every $g \in G$ the set of points that do not satisfy the conditions of the definition (i.e., are such that $g \cdot x = x$ but x does not belong to the interior of the set of fixed points of g) is a closed nowhere dense set. Consequently, by Baire's category theorem, the set of G-generic points is co-meager, in particular it is non-empty.

Proposition 2.2. *The set of G-generic points is G-invariant.*

Proof. Let x be a G-generic point, and let $h \in G$ be an arbitrary element. Then the stabilizer $G_{h\cdot x}$ of $h \cdot x$ is equal to hG_xh^{-1}. For every $g \in G_x$ there exists a neighborhood U of x such that g acts trivially on U. But then hgh^{-1} acts trivially on hU. It follows that every element of $G_{h\cdot x}$ acts trivially on a neighborhood of $h \cdot x$, i.e., that $h \cdot x$ is G-generic. □

Let Γ be a labeled graph. For a vertex v of Γ and $R > 0$ denote by $B_v(R)$ the ball of radius R with center in v, seen as a *rooted* labeled graph (with root v). In particular, it means that whenever we are talking about morphisms between balls, the morphisms are assumed to preserve the centers.

Definition 2.4. We say that a G-orbit $G \cdot x$ is *locally contained* in the orbit $G \cdot y$ if for every vertex $v \in \Gamma_x$ and every $R > 0$ there exists a vertex $u \in \Gamma_y$ such that the corresponding balls $B_v(R)$ and $B_u(R)$ are isomorphic.

Two orbits are said to be *locally isomorphic* if each is contained in the other one.

It is not hard to show that the definition does not depend on the choice of the generating set.

Proposition 2.3. *Suppose that the action of G on $\mathcal{X}$ is minimal. Then orbit of any G-generic point is locally contained in any other orbit. In particular, orbits of G-generic points are all locally isomorphic to each other.*

Proof. Let $u \in \mathcal{X}$. The condition that the ball $B_v(R)$ is isomorphic to $B_u(R)$ can be written as a system of equalities and inequalities of the form $g \cdot u = u$ and $g \cdot u \neq u$ for a finite set of elements $g \in G$. Namely, for all pairs $g_1, g_2 \in G$ of elements of length less than R we must have $g_1 \cdot u = g_2 \cdot u$ if and only if $g_1 \cdot v = g_1 \cdot v$.

If u is G-generic and $g \cdot u = u$, then there exists an open neighborhood U of u such that $g \cdot v = v$ for all $v \in U$. For every u and g such that $g \cdot u \neq u$ there exists a neighborhood U of u such that $g \cdot v \neq v$ for all $v \in U$. It follows that there exists a neighborhood U_R of u such that if $v \in U_R$, then $B_v(R)$ and $B_u(R)$ are isomorphic.

Since the action is minimal, for every $v \in \mathcal{X}$ there exists $g \in G$ such that $g \cdot v \in U_R$. □

Proposition 2.4. *Suppose that $\mathcal{X}$ is compact, and the action is minimal. Let $x \in \mathcal{X}$ be a G-generic point. Then for every $R > 0$ and $y \in \mathcal{X}$ there exists $\Delta > 0$ such that for every vertex $u \in \Gamma_y$ there exists a vertex $u' \in \Gamma_y$ on distance less than Δ from u such that $B_{u'}(R)$ is isomorphic to $B_x(R)$.*

In other words, the set of points u such that $B_u(R)$ and $B_v(R)$ are isomorphic is a *net* in the graph of actions, so that it essentially "looks the same" everywhere.

PROOF. Recall, that by the proof of Proposition 2.3, there exists a neighborhood U of x such that for every $v \in U$ the balls $B_x(R)$ and $B_v(R)$ are isomorphic.

Since the action is minimal, for every $y \in \mathcal{X}$ there exists $g_y \in G$ such that $g_y \cdot y \in U$. Then $g_y^{-1} \cdot U$ cover the space $\mathcal{X}$. Since $\mathcal{X}$ is compact, there exists a finite sub-cover, i.e., a finite set of group elements $g_1, g_2, \ldots, g_k$ such that $g_i \cdot U$ cover $\mathcal{X}$. Let Δ be the maximal length of the elements g_i. Then for every $y \in \mathcal{X}$ there exists g_i such that $B_{g_i \cdot y}(R)$ and $B_x(R)$ are isomorphic. The distance from y to $g_i \cdot y$ is not more than the length of g_i, i.e., not more than Δ. $\square$

2.3. Schreier graphs of the free group. Note that if $H < G$ and N is a normal subgroup of G such that $N \leq H$, then the Schreier graphs $\Gamma(G/H, S)$ and $\Gamma((G/N)/(H/N), S')$ are isomorphic, where S' is the image of the generating set S in G/N. It follows that every Schreier graph is a Schreier graph of the free group.

On the other hand, we have the following obvious observation.

PROPOSITION 2.5. *Let Γ be a graph whose edges are labeled by elements of a finite set S. Suppose that for every vertex v of Γ and for every $s \in S$ there exists exactly one arrow starting in v and labeled by s, and exactly one arrow ending in v and labeled by s. Then Γ is isomorphic to the graph of an action of the free group F_S generated by S on the set of vertices of Γ.*

Namely, for every $s \in S$ and a vertex v, define $s \cdot v$ as the end of the edge starting in v that is labeled by s. Then it follows from the conditions of Proposition 2.5 that $v \mapsto s \cdot v$ is a permutation of the set of vertices. We get hence an action of F_S generated by these permutations. It is obvious that Γ is the graph of the action. We say that this action is the action *defined by the graph* Γ.

For an arbitrary word $g = s_1 s_2 \ldots s_n \in F_S$, where $s_i \in S \cup S^{-1}$ the image $g \cdot v$ is found in the following way. Find the path (sequence of edges) $e_n, e_{n-1}, \ldots e_2, e_1$, where $v = \alpha(e_n)$, $\omega(e_i) = \alpha(e_{i-1})$ for all $i = 2, \ldots, n$, and $\lambda(e_i) = s_i$. Here we assume that every edge of Γ consists in two incarnations: e and $i(e)$, satisfying $\alpha(i(e)) = \omega(e)$, $\omega(i(e)) = \alpha(e)$, and $\lambda(i(e)) = \lambda(e)^{-1}$. Then $g \cdot v = \omega(e_1)$.

In other words, in order to find $g \cdot v$, one has to find a path starting at v on which one reads g, the end of the path is $g \cdot v$. (It is nicer for the right actions…)

The stabilizer of a vertex v consists of words that are read on paths starting and ending in v. It is easy to see that the stabilizer of v is naturally identified with the fundamental group $\pi_1(\Gamma, v)$. This implies the following.

PROPOSITION 2.6. *Let Γ be as in the previous proposition, and suppose that it is connected. Then Γ is isomorphic to the Schreier graph of F_S by a subgroup isomorphic to the fundamental group of Γ.*

We say that a graph Γ labeled by elements of a set S is *well labeled* if it satisfies the conditions of Proposition 2.5.

PROBLEM 2.1. Let Γ be an unlabeled oriented graph such that for every vertex v the number of incoming arrows and the number of outgoing arrows are both equal to some fixed number d. Prove that we can label edges of Γ by elements of a set S, $|S| = d$, so that the obtained labeled graph is well labeled. **Hint:** prove this at first for finite graphs, and then go to a limit.

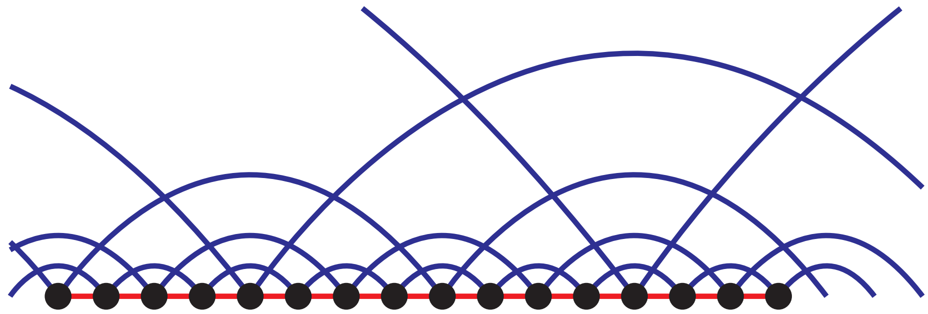


FIGURE 4. Graph Λ_w

One can use Schreier graphs to define groups. Suppose that Γ is a well labeled graph. Then it defines an action of the free group F_S, as above. The action may be non-faithful, so it is natural to take the quotient G of F_S by the kernel of the action. The graph Γ will be also a Schreier graph of G, by the remark at the beginning of the subsection. The group G is the group generated by the permutations $v \mapsto s \cdot v$.

EXAMPLE 2.1. Let $w = \ldots a_{-1}a_0a_1 \ldots \in S^{\mathbb{Z}}$ be an infinite sequence of elements of a finite set S. Suppose that $a_i \neq a_{i+1}$ for all $i \in \mathbb{Z}$. Consider the graph Γ_w with the set of vertices $\mathbb{Z}$, where for every i we have an arrow from i to $i+1$ and an arrow from $i+1$ to i both labeled by a_i, and $|S|-2$ loops at the vertex i labeled by $S \setminus \{a_i, a_{i-1}\}$. Then Γ_w is well labeled, hence it defines an action of F_S. Let G_w be the corresponding group, i.e., quotient of F_S by the kernel of the action.

PROBLEM 2.2. Identify G_w for a random w, i.e., find a group G such that G_w is isomorphic to G with probability 1, if the letters of w are found using the following Markov chain: a_i is chosen from the elements of the set $S \setminus \{a_{i-1}\}$ with equal probability.

2.4. "Long range" graph. Let $V = \mathbb{Z}$, $S = \{a, b\}$, and let $w = x_0x_1\ldots$, $x_i \in \{0, 1\}$, be an infinite sequence. Denote $w_k = x_0 + 2x_1 + 2^2x_2 + \cdots + 2^kx_k$, for $k = 0, 1, \ldots$. Arrows labeled by a start in n and end in $n+1$. Arrows labeled by b start in $w_k + 2^k(2n+1)$ and end in $w_k + 2^k(2n+3)$ for $k = 0, 1, 2, \ldots$, and $n \in \mathbb{Z}$. Additionally, if the sequence x_i is eventually constant, then there will be one vertex that is not connected to any other vertex by an arrow labeled by b. In this case we add a loop labeled by b to this vertex. Denote the obtained graph by Λ_w.

Another description of the graph Λ_w is as follows. We start with the Cayley graph of $\mathbb{Z}$ with respect to the generating set $\{1\}$. The edges of this graph are labeled by a. Then we connect every other vertex by b-labeled arrows, then among the remaining vertices we connect every other vertex, and so on, see Figure 4. The choice of vertices that are connected on each stage is done so that on the stage number k (starting with $k = 0$) we are NOT connecting the vertex w_k. At the end we either connect all vertices by b-labeled arrows, or there will remain one vertex. In the latter case we attach to it a loop.

PROBLEM 2.3. Prove that all graphs Λ_w, for w not eventually constant, are locally isomorphic (i.e., are locally contained in each other).

PROBLEM 2.4. The graph Λ_w is the graph of the action of a two-generated group. Taking quotient by the kernel we get a group generated by two permutations of $\mathbb{Z}$:

$$a : n \mapsto n+1, \qquad b : w_k + 2^k(2n+1) \mapsto w_k + 2^k(2n+3).$$

Prove that this group does not depend on w.

Note that a^2b^{-1} and $ab^{-1}a$ act non-trivially on disjoint sets, so that their commutator $[a^2b^{-1}, ab^{-1}a]$ is trivial.

Let us interpret the sequences w as dyadic numbers, so that $x_0x_1\ldots$ is interpreted as the sum

$$x_0 + 2x_1 + 2^2x_2 + \cdots .$$

Note that then eventually constant sequences are precisely the elements of $\mathbb{Z}$.

PROBLEM 2.5. Show that Γ_{w_1} and Γ_{w_2} are isomorphic as non-rooted trees if and only if $w_1 - w_2 \in \mathbb{Z}$.

2.5. Space of Schreier graphs. Let G be a countable group. The set 2^G of all subsets of G is naturally identified with the set of all functions $G \longrightarrow \{0,1\}$ (their characteristic functions), and hence has a natural direct product topology. Recall that a basis of topology consists of sets of the form

$$U_{C_0,C_1} = \{A \subset G \;:\; C_1 \subset A, C_0 \cap A = \emptyset\}$$

for finite sets C_0, C_1. In terms of maps $G \longrightarrow \{0,1\}$, the set U_{C_0,C_1} consists of functions $f : G \longrightarrow \{0,1\}$ such that $f(x) = 0$ for $x \in C_0$, and $f(x) = 1$ for $x \in C_1$.

It follows form Tykhonoff theorem that 2^G is compact. It is easy to see that U_{C_0,C_1} is closed (write its complement as union of the elements of the basis of topology!), hence we get a basis of topology consisting of closed and open (clopen) sets. By definition, this means that 2^G is *totally disconnected*. In fact, it is homeomorphic to the Cantor set.

PROBLEM 2.6. Prove that the subset of 2^G consisting of subgroups of G is closed. Prove that the set of all normal subgroups of G is also closed.

We get hence a natural topology on the set of all subgroups of G and on the set of all normal subgroups of G. The latter set is naturally identified with the set of all quotients of G (with a *marking* defined by the epimorphism from G to the quotient).

Both spaces (space of subgroups and the space of normal subgroups) are compact and totally disconnected. Note that this does not mean that they are homeomorphic to the Cantor set, since we they may have isolated points.

Let F_S be, as before, the free group generated by S. Then we get a bijection between the set of all subgroups of F_S and the set $\mathcal{G}_S$ of all *rooted* well labeled (by S) connected graphs Γ. Namely, given a subgroup $H \le F_S$, the corresponding graph is the Schreier graph $\Gamma(F_S/H, S)$ with the root $1 \cdot H$. Given a well labeled rooted graph $(\Gamma, v) \in \mathcal{G}_S$, the corresponding subgroup is $\pi_1(\Gamma, v)$, i.e., the subgroup consisting of words that are read on loops starting and ending in v.

There is a straightforward way of describing the topology on the space of subgroups of F_S using this identification.

Let $(\Gamma_1, v_1), (\Gamma_2, v_2) \in \mathcal{G}_S$. We say that distance between them is $1/(n+1)$, if n is the biggest number such that $B_{v_1}(n)$ and $B_{v_2}(n)$ are isomorphic.

Proposition 2.7. *The defined notion of distance is an ultrametric on $\mathcal{G}_S$ consistent with the topology on the space of subgroups of F_S.*

Proof. A basis of topology on the space of subgroups of F_S is the collection of the sets of the form

$$V_{A_0,A_1} = \{H \le F_S \ : \ A_0 \cap H = \emptyset, A_1 \subset H\}.$$

For every well labeled rooted graph (Γ, v) and every $R > 0$ the ball $B_v(R)$ can be completely described by a finite set of equations and inequalities of the form $g_1 \cdot v = g_2 \cdot v$ and $g_1 \cdot v \neq g_2 \cdot v$ for some $g_1, g_2 \in F_S$. These equations and inequalities are equivalent to $g_2^{-1}g_1 \cdot v = v$ and $g_2^{-1}g_1 \cdot v \neq v$. It follows that there exist finite sets $A_0, A_1 \subset F_S$ such that for every subgroup $L \in V_{A_0,A_1}$ the ball of radius R in the Schreier graph $\Gamma(F_S/L, S)$ with center in $1 \cdot L$ is isomorphic to $B_v(R)$.

Conversely, for every pair of finite sets $A_0, A_1 \subset F_S$ there exists R (equal to the maximum of lengths of elements of A_0 and A_1) such that for every subgroup $H \le F_S$, if we know the ball of radius R with center in $1 \cdot H$ in the Schreier graph $\Gamma(F_S/H, S)$, then we know whether H belongs to V_{A_0,A_1} or not. $\square$

Problem 2.7. Show that the only isolated points of $\mathcal{G}_S$ are finite graphs. This will show that the set of infinite graphs is homeomorphic to the Cantor set. Equivalently, the space of subgroups of infinite index in the free group F_S is homeomorphic to the Cantor set.

On the other hand, the set of normal subgroups of infinite index has isolated points (e.g., infinite finitely presented simple groups).

Given a rooted graph $(\Gamma, v) \in \mathcal{G}_S$ and an element $g \in F_S$, denote $g \cdot (\Gamma, v)$ the graph $(\Gamma, g \cdot v)$. It is easy to see that this defines an action of F_S on $\mathcal{G}_S$ by homeomorphisms.

The orbit of (Γ, v) is equal to the set of rooted trees (Γ, u), where u runs through the set of all vertices of Γ. Denote by $\overline{\Gamma}$ the closure of the orbit of (Γ, v). Note that the map $v \mapsto (\Gamma, v)$ from the set of vertices of Γ to $\mathcal{G}_S$ is not necessarily injective. For example, if Γ is a Cayley graph of a group, then $\overline{\Gamma}$ is a single point.

For every $(\Gamma, v) \in \mathcal{G}_S$, the set $\overline{\Gamma}$ is F_S-invariant.

Proposition 2.8. *Let K be the kernel of the action of F_S on the set of vertices of Γ. Then K acts trivially on $\overline{\Gamma}$. Consequently, the quotient F_S/K acts on $\overline{\Gamma}$.*

Proof. The group K acts trivially on the orbit of (Γ, v), which is a dense subset of $\overline{\Gamma}$. Since the action is continuous, it is trivial on $\overline{\Gamma}$. $\square$

Proposition 2.9. *Let $(\Gamma, v) \in \mathcal{G}_S$. The following conditions are equivalent.*

(1) *The action of F_S on $\overline{\Gamma}$ is minimal.*
(2) *For every $R > 0$ and every vertex u of Γ there exists $\Delta > 0$ such that for every vertex w of Γ there exists a vertex u' such that $d(w, u') < \Delta$ and the balls $B_u(R)$ and $B_{u'}(R)$ are isomorphic.*

Proof. Implication (1)$\Longrightarrow$(2) is proved in the same way as Proposition 2.4.

Suppose that (2) is satisfied. Let $(\Gamma_i, v_i) \in \overline{\Gamma}$ for $i = 1, 2$. It is enough to show that for every $R > 0$ there exists a vertex u of (Γ_2, v_2) such that the ball $B_u(R) \subset \Gamma_2$ is isomorphic to $B_{v_1}(R)$, since it will mean that the orbit of Γ_2 intersects every neighborhood of (Γ_1, v_1).

Ball $B_{v_1}(R)$ is isomorphic to some ball $B_v(R)$ of Γ. It follows from condition (2) that there exists $u \in \Gamma_2$ such that $B_u(R)$ is isomorphic to $B_v(R) \cong B_{v_1}(R)$. $\square$

2.6. "Long range" graphs. Let Λ_w for $w \in \mathbb{Z}_2$ be as in Subsection 2.4

PROBLEM 2.8. Show that the map $w \mapsto (\Lambda_w, 0)$ is continuous at every point $w \in \mathbb{Z}_2 \setminus \mathbb{Z}$ and discontinuous for $w \in \mathbb{Z}$.

PROBLEM 2.9. Show that we can redefine Λ_w for $w \in \mathbb{Z}$ in such a way that the map $w \mapsto (\Lambda_w, 0)$ is continuous and is a homeomorphism on its range.