4. Expanding dynamical systems

4.1. Metric definition.

DEFINITION 4.1. Let $\mathcal{X}$ be a compact metric space. A map $f : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be *expanding* if there exist $\epsilon > 0$ and $L > 1$ such that $d(f(x), f(y)) \geq Ld(x, y)$ for all $x, y \in \mathcal{X}$ such that $d(x, y) < \epsilon$.

If $\mathcal{X}$ is a compact metrizable space, then a continuous map $f : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be *(topologically) expanding* if there exists a metric such that f is expanding with respect to it.

LEMMA 4.1. *Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be such that f^n is topologically expanding. Then f is also topologically expanding.*

PROOF. Suppose that f^n is expanding with respect to a metric d. Let ϵ and L be as in Definition 4.1.

Consider the metric

$$d_n(x, y) = \sum_{k=0}^{n-1} L^{-k/n} d(f^k(x), f^k(y)).$$

Then

$$d_n(f(x), f(y)) = \sum_{k=1}^{n} L^{-(k-1)/n} d(f^k(x), f^k(y)) =$$

$$L^{-(n-1)/n} d(f^n(x), f^n(y)) + L^{1/n} \sum_{k=1}^{n-1} L^{-k/n} d(f^k(x), f^k(y)) \geq$$

$$L^{-1+1/n} Ld(x, y) + L^{1/n} \sum_{k=1}^{n-1} L^{-k/n} d(f^k(x), f^k(y)) = L^{1/n} d_n(x, y).$$

It follows that f is also expanding. □

EXAMPLE 4.1. Consider the circle $\mathbb{R}/\mathbb{Z}$, and for an integer k, $|k| > 1$, consider the self-covering $f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$ given by $f(x) = kx \pmod 1$. It is obviously expanding.

EXAMPLE 4.2. An endomorphism $f : M \longrightarrow M$ of a Riemannian manifold is called expanding if there exist $C > 0$ and $L > 1$ such that $\|Df^n\vec{v}\| \geq CL^n\|\vec{v}\|$ for every tangent vector $\vec{v}$. Note that, by Lemma 4.1, every expanding endomorphism of a compact Riemannian manifold is an expanding self-covering.

4.2. Hyperbolic rational functions.

4.2.1. *Some classical theorems of complex analysis.*

THEOREM 4.2 (Uniformization Theorem). *Any simply connected Riemann surface (i.e., a one dimensional smooth complex manifold) is conformally isomorphic to exactly one of the following surfaces.*

1. *The Riemann sphere $\widehat{\mathbb{C}}$.*
2. *The (Euclidean) plane $\mathbb{C}$.*
3. *the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, or, equivalently, the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$.*

THEOREM 4.3 (Schwarz Lemma). *If $f : \mathbb{D} \longrightarrow \mathbb{D}$ is holomorphic and $f(0) = 0$, then $|f'(0)| \le 1$. If $|f'(0)| = 1$, then f is a rotation $z \mapsto cz$ about 0 (with $|c| = 1$). If $|f'(z)| < 1$, then $|f(z)| < |z|$ for all $z \neq 0$.*

As a corollary we get

THEOREM 4.4 (Liouville Theorem). *If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic and bounded, then it is constant.*

The automorphism groups (i.e., groups of bi-holomorphic automorphisms) of the simply connected Riemannian surfaces are as follows:

(1) $Aut(\widehat{\mathbb{C}})$ is the group of all *Möbius transformations* $z \mapsto \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ such that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, and is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$.
(2) $Aut(\mathbb{C})$ is the group of all affine transformations $z \mapsto az + b$ for $a, b \in \mathbb{C}$, $a \neq 0$.
(3) $Aut(\mathbb{H})$ is the group of all transformations $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ are such that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$, and is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. Every automorphism of $\mathbb{D}$ is of the form $z \mapsto e^{i\theta} \frac{z-a}{1-\overline{a}z}$, where $\theta \in \mathbb{R}$, $|a| < 1$.

If S is a connected Riemann surface, then its universal covering $\widetilde{S}$ is one of the simply connected surfaces $\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{D}$, and the fundamental group $\pi_1(S)$ acts on $\widetilde{S}$ by conformal automorphisms. We say that S is *Euclidean* or *hyperbolic*, if $\widetilde{S}$ is isomorphic to $\mathbb{C}$ or $\mathbb{D}$, respectively.

Note that the action of $\pi_1(S)$ on $\widetilde{S}$ is fixed point free. Since every non-identical Möbius transformation has a fixed point, the only surface with universal covering $\widehat{\mathbb{C}}$ is the sphere $\widehat{\mathbb{C}}$ itself.

Any transformation $z \mapsto az + b$ for $a \neq 1$ has a fixed point, hence in the Euclidean case the fundamental group acts on the universal covering $\mathbb{C}$ by translations. It is easy to see that this implies that a Euclidean surface is isomorphic either to the cylinder $\mathbb{C}/\mathbb{Z}$, or to a torus $\mathbb{C}/\Lambda$, where Λ is the subgroup of the additive group of $\mathbb{C}$ generated by two non-zero complex numbers a, b such that $a/b \notin \mathbb{R}$. All the other Riemann surfaces are hyperbolic.

It is a direct corollary of the Liouville theorem that every holomorphic map from $\mathbb{C}$ to a hyperbolic surface is constant (since we can lift it to the universal covering). In particular, every holomorphic map $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\mathbb{C} \setminus f(\mathbb{C})$ has more than one point is constant (*Picard's Theorem*).

THEOREM 4.5 (Poincaré metric). *There exists, up to multiplication by a constant, unique Riemannian metric on $\mathbb{D}$ invariant under every conformal automorphism of $\mathbb{D}$. It is given by $ds = \frac{2|dz|}{1-|z|^2}$ (the coefficient 2 is in order to have Gaussian curvature equal to -1). Every orientation preserving isometry of $\mathbb{D}$ is a conformal automorphism.*

Every hyperbolic surface S has then a unique Poincaré metric coming from the Poincaré metric on the universal covering $\widetilde{S} \cong \mathbb{D}$ of S (since the fundamental group $\pi_1(S)$ acts on $\widetilde{S}$ by conformal automorphisms).

The following is a corollary of Schwarz Lemma.

THEOREM 4.6 (Pick Theorem). *Let $f : S \longrightarrow S'$ be a holomorphic map between hyperbolic surfaces. Then exactly one of the following cases is taking place.*

(1) *f is a conformal isomorphism and an isometry with respect to the Poincaré metrics.*
(2) *f is a covering map and is a local isometry.*
(3) *f is strictly contracting, i.e., for every compact set $K \subset S$ there is a constant $c_K < 1$ such that $d(f(x), f(y)) \leq c_k d(x, y)$ for all $x, y \in K$.*

4.2.2. *Julia set.*

Definition 4.2 (Compact-open topology). Let $\mathcal{X}$ be a locally compact space, and let $\mathcal{Y}$. *Compact open topology* on the space $Map(\mathcal{X}, \mathcal{Y})$ of continuous maps $\mathcal{X} \longrightarrow \mathcal{Y}$ is given by the bases of neighborhoods of a map $f : \mathcal{X} \longrightarrow \mathcal{Y}$ consisting of sets

$$N_{K,\epsilon}(f) = \{g \in Map(\mathcal{X}, \mathcal{Y}) \ : \ d(f(x), g(x)) < \epsilon \text{ for all } x \in K\}$$

where $K \subset \mathcal{X}$ is compact and $\epsilon > 0$.

In fact, the compact-open topology does not depend on the metric on $\mathcal{Y}$. Convergence in the compact-open topology is called *uniform convergence on compact subsets*.

Definition 4.3. A set $\mathcal{F}$ of holomorphic functions from a Riemann surface S to a compact Rieman surface T is called a *normal family* if its closure is compact in $Map(S, T)$. In the case when T is not compact, we replace T by its one-point compactification.

Thus, a family $\mathcal{F} \subset Hol(S, T)$ is normal if every sequence f_n of elements of $\mathcal{F}$ has either a subsequence f_{n_k} convergent uniformly on compact subsets, or a subsequence f_{n_k} converging to infinity uniformly on compact subsets (i.e., such that for all compact $K_1 \subset S$ and $K_2 \subset S$ the intersection $f_{n_k}(K_1) \cap K_2$ is empty for all k big enough).

Every holomorphic map $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is a rational function, i.e., a ratio of two polynomials.

Definition 4.4. Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a rational function. The *Fatou set* of f is the set of points $z \in \widehat{\mathbb{C}}$ such that there exists a neighborhood U of z such that $f^{\circ n} : U \longrightarrow \widehat{\mathbb{C}}$, for $n \geq 0$, is normal. The complement of the *Fatou set* is called the *Julia set*.

Some examples (some taken from a very good introductory book by J. Milnor *Dynamics in One Complex Variable*, Annals of Mathematics Studies, No 160, Princeton University Press 2006):

The Julia set J is totally invariant, i.e., $f(J) = J = f^{-1}(J)$. It is always non-empty (unless f is a Möbius transformation) and compact. It can be equal to the whole sphere.

4.2.3. *Hyperbolic rational functions.*

Definition 4.5. A rational function f is *hyperbolic* if it is expanding on its Julia set.

A *post-critical set* of f is the set of all points of the form $f^n(c)$, where c is a critical point of f, and $n \geq 1$.

Theorem 4.7. *Let f be a rational function of degree ≥ 2. Then the following conditions are equivalent.*

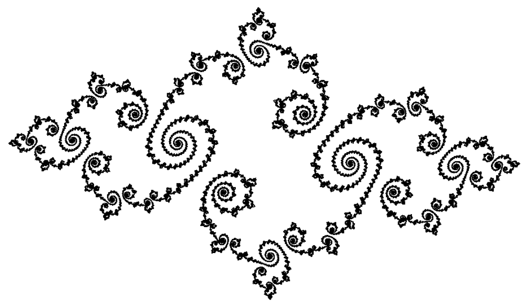

FIGURE 10. A Cantor set $z^2 - 0.765 + 0.12i$

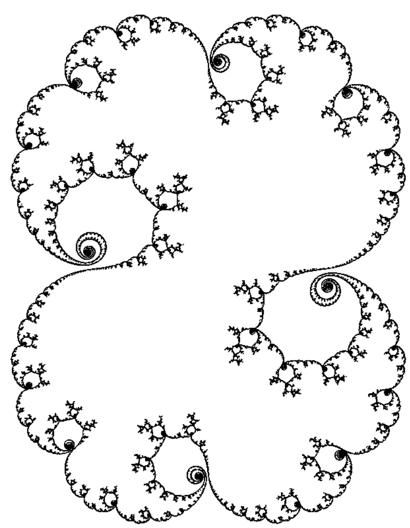

FIGURE 11. A simple closed curve $z^2 + (0.99 + 0.14i)z$

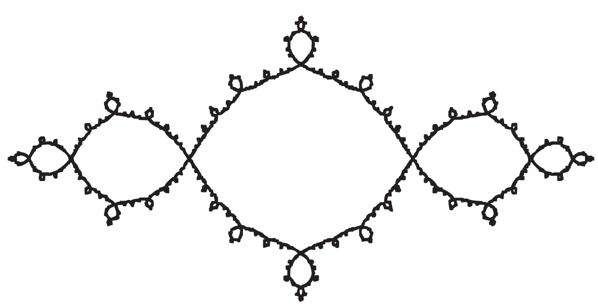

FIGURE 12. "Basilica" $z^2 - 1$

(1) *f is hyperbolic.*
(2) *Closure of the post-critical set of f is disjoint from its Julia set.*
(3) *Orbit of every critical point converges to an attracting cycle.*

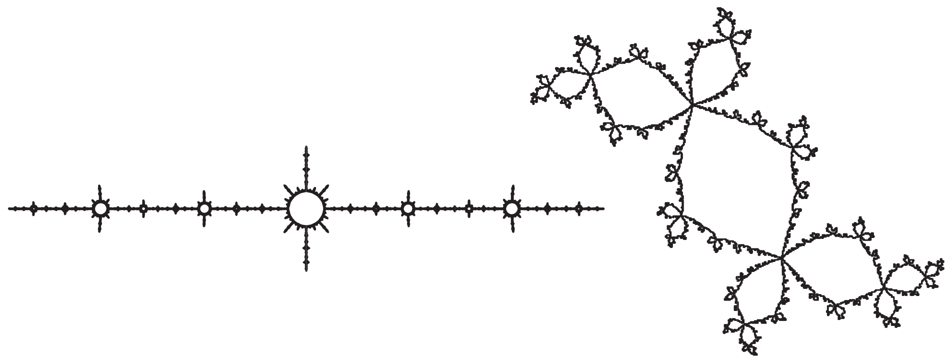

FIGURE 13. "Rabbit and Airplane"

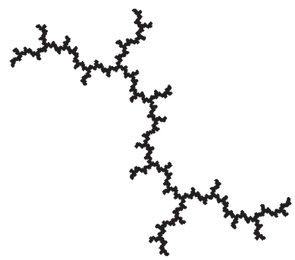

FIGURE 14. $z^2 + i$

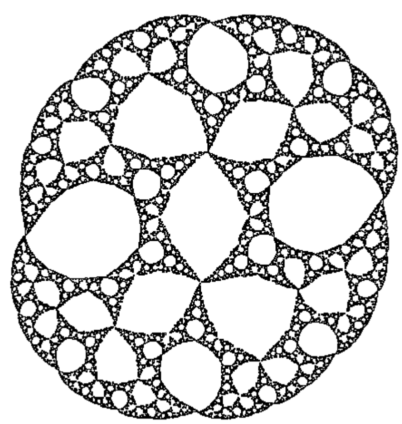

FIGURE 15. $((1 + i\sqrt{3})/2 + z^2)/(1 - z^2)$

Sketch of the proof. It is easy to see that (3) implies (2), since basins of attraction belong to the Fatou set. Let us show only that (2) implies (1) in the case when $\overline{P}$ has more than two points. Then $\mathcal{X} = \widehat{\mathbb{C}} \setminus \overline{P}$ is a hyperbolic surface containing the Julia set. Note that $f(\overline{P}) \subset \overline{P}$, hence $f^{-1}(\mathcal{X}) \subset \mathcal{X}$. The map $f : f^{-1}(\mathcal{X}) \longrightarrow \mathcal{X}$ is a covering. Consider the Poincaré metrics on $\mathcal{X}$ and $f^{-1}(\mathcal{X})$. The map f is a local

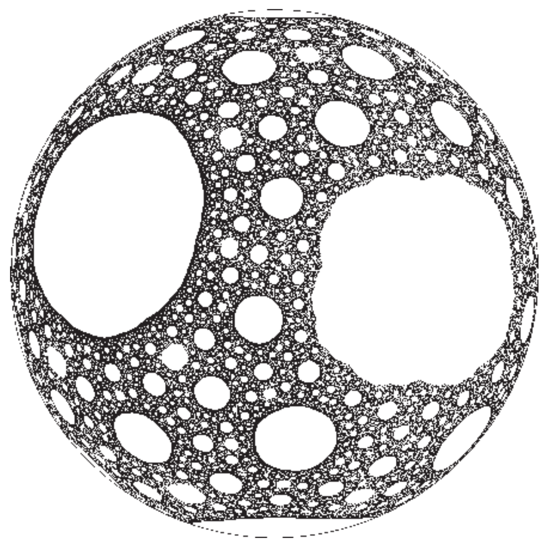

FIGURE 16. A Sierpiński carpet

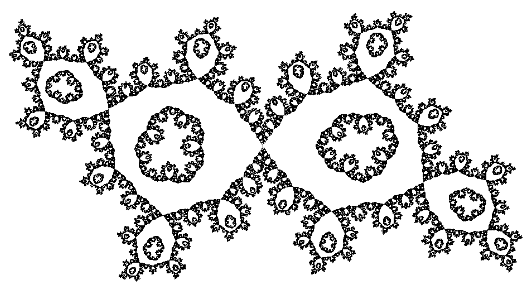

FIGURE 17. Herman rings

isometry with respect to these metrics. The inclusion map $Id : f^{-1}(\mathcal{X}) \longrightarrow \mathcal{X}$ is not a covering map, hence it is strictly contracting, see Theorem 4.6. It follows that if we consider the restriction of the Poincaré metric of $\mathcal{X}$ onto the subset $f^{-1}(\mathcal{X})$, then the map $f : f^{-1}(\mathcal{X}) \longrightarrow \mathcal{X}$ is expanding. Since the Julia set is compact and contained in $f^{-1}(\mathcal{X})$, the map f will be strictly expanding on the Julia set.

If closure of the post-critical set has less than three points, then they belong to attracting cycles, and we can take $\mathcal{X}$ equal to $\widehat{\mathbb{C}}$ minus a small neighborhood of $\overline{P}$, and repeat the proof.

Let us show that (1) implies (2). Let W be a neighborhood of the Julia set J such that f is expanding on W. Taking an ϵ-neighborhood of J in W, we get an open neighborhood U of J such that f is expanding on U, and $f^{-1}(U) \subset U$. Then $f^{-n}(U) \subset U$ for all $n \geq 1$. The set U does not contain critical points of f, since f is not one-to-one, hence not expanding, on any neighborhood of a critical point. If c is critical, and $f^n(c) \in U$, then $c \in f^{-n}(U) \subset U$, which is a contradiction. Consequently, U does not contain any post-critical points. This implies that intersection of U with the closure of the post-critical set is empty.

The fact that (2) implies (3) follows from classification of components of the Fatou set.

4.3. Problems. A *cycle* of a rational function $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is a sequence $x_0, x_1, \ldots, x_{n-1}$ such that $f(x_i) = x_{i+1}$ for all $i = 0, \ldots, n-2$, and $f(x_{n-1}) = x_0$. Its *multiplier* is the derivative $\frac{d}{dz} f^n(z)\big|_{z=x_i} = f'(x_0) f'(x_1) \cdots f'(x_{n-1})$. The cycle is called *attracting* if its multiplier is less than one in absolute value. It is *superattracting* if the multiplier is zero.

PROBLEM 4.1. Find the set of values of $c \in \mathbb{C}$ such that $z^2 + c$ has an attracting fixed point (i.e., a cycle of length 1).

PROBLEM 4.2. Find the set of values of $c \in \mathbb{C}$ such that $z^2 + c$ has an attracting cycle of length 2.

PROBLEM 4.3. Consider the Tchebyshev polynomials $T_d(x) = \cos(d \arccos x)$. Describe the Julia sets of T_d for $d \geq 1$.

PROBLEM 4.4. Let $\mathbb{C}/\mathbb{Z}[i]$ be the torus, and let $A : \mathbb{C}/\mathbb{Z}[i] \longrightarrow \mathbb{C}/\mathbb{Z}[i]$ be the map given by $A(z) = (1+i)z$. Find the Julia set of A. Using the fact that any holomorphic map $f : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda$ on a torus is induced by a linear map on $\mathbb{C}$, describe all possible Julia sets of holomorphic maps on the torus.

PROBLEM 4.5. Consider the group G of all maps of the form $z \mapsto (-1)^k z + a + ib$, where $k \in \{0, 1\}$, and $a, b \in \mathbb{Z}$. Show that $\mathbb{C}/G$ is homeomorphic to a sphere. Consider the map $A(z) = (1+i)z$. Show that it induces a well defined map on the sphere $\mathbb{C}/G$. Since the group G and the map A act by holomorphic maps, there is a well defined structure of a complex manifold on $\mathbb{C}/G$, and A induces a holomorphic map on $\mathbb{C}/G$, hence is can be realized by a rational function. What is the Julia set of this rational function?

4.4. Topological description of expanding maps.

DEFINITION 4.6. Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be a continuous map, where $\mathcal{X}$ is compact Hausdorff. An *expansion entourage* U is a compact neighborhood of the diagonal $\{(x, x) : x \in \mathcal{X}\}$ in $\mathcal{X} \times \mathcal{X}$ such that if $(f^n(x), f^n(y)) \in U$ for all $n \geq 0$, then $x = y$.

If U is an expansion entourage, then $U^\top = \{(x, y) : (y, x) \in U\}$ is also an expansion entourage. Then $U^\top \cap U$ is a symmetric expansion entourage. It follows that we may assume without loss of generality that expansion entourages are symmetric.

Suppose that U is an expansion entourage for a map $f : \mathcal{X} \longrightarrow \mathcal{X}$. Denote then by $U_n \subset \mathcal{X} \times \mathcal{X}$ the set of pairs of points (x, y) such that $(f^k(x), f^k(y)) \in U$ for all $k = 0, 1, \ldots, n$. In particular, $U_0 = U$. Denote $U_{-1} = \mathcal{X} \times \mathcal{X}$. By the definition of an expansion neighborhood, $\bigcap_{n \geq 0} U_n$ is equal to the diagonal. It is easy to see that U_n are compact neighborhoods of the diagonal.

LEMMA 4.8. *For every neighborhood V of the diagonal, there exists n such $U_n \subset V$.*

PROOF. It is enough to prove the lemma for open neighborhoods. Then $\mathcal{X} \times \mathcal{X} \setminus V$ is compact. By definition of the expansion entourage, open sets $\mathcal{X} \times \mathcal{X} \setminus U_n$, for $n \geq 0$, cover $\mathcal{X} \times \mathcal{X} \setminus V$. Since $\mathcal{X} \times \mathcal{X} \setminus U_n \supseteq \mathcal{X} \times \mathcal{X} \setminus U_{n+1}$, it follows that there exists n such that $\mathcal{X} \times \mathcal{X} \setminus U_n \supseteq \mathcal{X} \times \mathcal{X} \setminus V$. $\square$

For subsets A, B of $\mathcal{X} \times \mathcal{X}$, denote by $A \circ B$ the set of pairs (x, y) such that there exists z such that $(x, z) \in A$ and $(z, y) \in B$. Note that $A \circ B$ is the image of the closed subset $D = \{((x_1, y_1), (x_2, y_2)) \ : \ y_1 = x_2\}$ of $A \times B$ under the map $((x_1, y_1), (x_2, y_2)) \mapsto (x_1, y_2)$. If A and B are compact, then D is a closed subset of a compact space $A \times B$, hence D is compact, which implies that $A \circ B$ is compact.

LEMMA 4.9. *There exists* $\Delta \in \mathbb{N}$ *such that* $U_{n+\Delta} \circ U_{n+\Delta} \subset U_n$ *for all* $n \geq 1$.

PROOF. Suppose that there is no Δ such that $U_\Delta \circ U_\Delta \subset Int(U)$. Denote $B_k = (\mathcal{X}^2 \setminus Int(U)) \cap (U_k \circ U_k)$, for $k \geq 0$. Then the sets B_k are closed non-empty, and $B_{k+1} \subset B_k$. It follows from compactness of $\mathcal{X}^2$ that the intersection $\bigcap_{k \geq 1} B_k$ is non-empty. Let (x, y) be such that $(x, y) \in B_k$ for all k. Let $Z_k \subset \mathcal{X}$ be the set of points z such that $(x, z) \in U_k$ and $(z, y) \in U_k$. Since U_k is closed, the set Z_k is closed. It is non-empty, by the choice of (x, y). We also have $Z_{k+1} \subset Z_k$. It follows that the intersection of all Z_k is non-empty. Let $z_0 \in \bigcap_{k \geq 1} Z_k$. Then $(x, z_0) \in U_k$ for all k, hence $x = z_0$, and $(z_0, y) \in U_k$ for all k, hence $z_0 = y$, which implies $x = y$, which is a contradiction.

We have shown that there exists Δ such that $U_\Delta \circ U_\Delta \subset U$. If $(x, y) \in U_{\Delta+n} \circ U_{\Delta+n}$, then there exists z such that $(x, z) \in U_{\Delta+n}$ and $(z, y) \in U_{\Delta+n}$. Then $(f^i(x), f^i(z)) \in U_{\Delta+n-i} \subseteq U_\Delta$ and $(f^i(z), f^i(y)) \in U_{\Delta+n-i} \subseteq U_\Delta$ for all $i = 0, 1, \ldots, n$. It follows that $(f^i(x), f^i(y)) \in U_\Delta \circ U_\Delta \subset U$, hence $(x, y) \in U_n$. We have shown that $U_{n+\Delta} \circ U_{n+\Delta} \subset U_n$ for all $n \geq 0$. $\square$

DEFINITION 4.7. Denote, for $(x, y) \in \mathcal{X}^2$, by $\ell(x, y)$ the maximal value of n such that $(x, y) \in U_n$, and ∞ if $x = y$.

Lemma 4.9 is reformulated then as follows.

PROPOSITION 4.10. *There exists* $\Delta > 0$ *such that* $\ell(x, y) \geq \min(\ell(x, z), \ell(z, y)) - \Delta$ *for all* $x, y, z \in \mathcal{X}$.

THEOREM 4.11. *For all* $\alpha > 0$ *small enough there exists a metric* d *on* $\mathcal{X}$ *and* $C > 1$ *such that*

$$C^{-1} e^{-\alpha \ell(x,y)} \leq d(x, y) \leq C e^{-\alpha \ell(x,y)}$$

for all $x, y \in \mathcal{X}$. *We say then that* d *is a metric of* exponent α *associated with the entourage* U.

PROOF. One can use the following *Frink's metrization lemma*, see Lemma 12 on page 185 in J. L. Kelley, *General Topology*.

LEMMA 4.12. *Let* E_n, $n \geq 0$, *be a sequence of neighborhoods of the diagonal in* $\mathcal{X} \times \mathcal{X}$ *such that*

$$E_{n+1} \circ E_{n+1} \circ E_{n+1} \subset E_n$$

for every n. *Then there is a non-negative continuous real valued function* $d : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ *such that* d *satisfies the triangle inequality and*

$$E_n \subset \{(x, y) \ : \ d(x, y) < 2^{-n}\} \subset E_{n-1}$$

for every $n \geq 1$.

Define $E_n = U_{2\Delta n}$. Then $E_{n+1} \circ E_{n+1} \circ E_{n+1} = U_{2\Delta(n+1)} \circ U_{2\Delta(n+1)} \circ U_{2\Delta(n+1)} \subset U_{2\Delta(n+1)-\Delta} \circ U_{2\Delta(n+1)} \subset U_{2\Delta(n+1)-\Delta} \circ U_{2\Delta(n+1)-\Delta} \subset U_{2\Delta(n+1)-2\Delta} = U_{2\Delta n} = E_n$.

Let d be the function given by the Frink's metrization lemma. Since U_n are symmetric, $d(x,y) = d(y,x)$. Since $\bigcap_{n\ge 1} U_n$ is equal to the diagonal, $d(x,y) = 0$ implies that $x = y$, and hence d is a metric.

It satisfies

$$U_{2\Delta n} \subset \{(x,y) \ : \ d(x,y) < 2^{-n}\} \subset U_{2\Delta(n-1)}.$$

Consequently,

$$d(x,y) \le 2^{-\lfloor \ell(x,y)/2\Delta \rfloor} < 2^{-\ell(x,y)/2\Delta+1}$$

and

$$d(x,y) \ge 2^{-\lfloor \ell(x,y)/2\Delta \rfloor - 1} \ge 2^{-\ell(x,y)/2\Delta-2}.$$

It follows that for $\alpha = 2^{-1/2\Delta}$ we have

$$\frac{1}{4} \cdot \alpha^{\ell(x,y)} \le d(x,y) \le 2 \cdot \alpha^{\ell(x,y)}.$$

Which finishes the proof of the theorem. $\square$

Let us give an independent and more explicit construction of the metric d satisfying the conditions of Theorem 4.11.

Consider, for every $n \in \mathbb{N}$ the graph Γ_n with the set of vertices $\mathcal{X}$ in which two points x, y are connected by an edge if $(x,y) \in U_n$, i.e., if $\ell(x,y) \ge n$. Let d_n be the combinatorial distance between the vertices of Γ_n.

Lemma 4.13. *There exists $\alpha > 0$ and $C > 0$ such that*

$$d_n(x,y) \ge Ce^{\alpha(n-\ell(x,y))}$$

for all $x, y \in \mathcal{X}$ and $n \in \mathbb{N}$.

Proof. Let Δ be as in Proposition 4.10, and let us prove the lemma for $\alpha = \frac{\ln 2}{\Delta}$. If x_0, x_1, x_2 is a path in Γ_n, then $\ell(x_0, x_2) \ge n - \Delta$, hence x_0, x_2 is a path in $\Gamma_{n-\Delta}$. It follows that $d_{n-\Delta}(x,y) \le \frac{1}{2}(d_n(x,y)+1)$, or

$$d_{n+\Delta}(x,y) \ge 2d_n(x,y) - 1.$$

If $\ell(x,y) = m$, then $d_{m+1}(x,y) \ge 2$, and hence

$$d_{m+1+t\Delta}(x,y) \ge 2^t + 1.$$

It follows that for every n and $t = \lfloor \frac{n-\ell(x,y)-1}{\Delta} \rfloor > \frac{n-\ell(x,y)-1}{\Delta} - 1$ we have

$$d_n(x,y) > 2^t > 2^{(n-\ell(x,y)-1-\Delta)/\Delta} = Ce^{\alpha(n-\ell(x,y))},$$

where $C = 2^{(-1-\Delta)/\Delta}$ and $\alpha = \frac{\ln 2}{\Delta}$. $\square$

We say that $\alpha > 0$ is a *lower exponent* if there exists $C > 0$ such that α and C satisfy the conditions of Lemma 4.13. If α is a lower exponent, then all numbers in the interval $(0, \alpha)$ are lower exponents. Hence, the set of lower exponents is either an interval $(0, c)$ (including the case $c = +\infty$) or an interval $(0, c]$. The number c is called the *critical lower exponent*.

It is easy to see that if α is such that there exists a metric of exponent α, then α is a lower exponent.

Let α be a lower exponent, and let $\beta \in (0, \alpha)$. Let us show that there exists a metric d of exponent β. Denote by $d_\beta(x,y)$ the infimum of the number $\sum_{i=1}^m e^{-\beta\ell(x_i, x_{i+1})}$ over all sequences $x_0 = x, x_1, x_2, \ldots, x_m = y$. The function

d_β obviously satisfies the triangle inequality, $d_\beta(x,y) = d_\beta(y,x)$, and $d_\beta(x,y) \le e^{-\beta\ell(x,y)}$ for all $x,y \in \mathcal{X}$. It remains to show that there exists $C > 0$ such that $Ce^{-\beta\ell(x,y)} \le d_\beta(x,y)$. In other words, C is such that

$$Ce^{-\beta\ell(x,y)} \le \sum_{i=1}^{m} e^{-\beta\ell(x_i,x_{i+1})} \tag{1}$$

for all sequences $x_0, x_1, \ldots, x_m$ such that $x = x_0$ and $y = x_m$.

Let $C_0 \in (0,1)$ be such that $d_n(x,y) \ge C_0 e^{-\alpha(n-\ell(x,y))}$ for all $x,y \in \mathcal{X}$ and $n \in \mathbb{N}$. Let us prove inequality (1) for $C = \exp\left(\frac{\beta(\ln C_0 - 2\alpha\Delta)}{\alpha-\beta}\right)$.

Lemma 4.14. *Let $x_0, x_1, \ldots, x_m$ be a sequence such that $\ell(x_i, x_{i+1}) \ge n$ for all $i = 0, 1, \ldots, m-1$. Let $n_0 \le n$. Then there exists a sub-sequence $y_0 = x_0, y_1, \ldots, y_{t-1}, y_t = x_m$ of the sequence x_i such that*

$$n_0 - 2\Delta \le \ell(y_i, y_{i+1}) < n_0$$

for all $i = 0, 1, \ldots, t-1$.

Proof. Let us construct the subsequence y_i by the following algorithm. Define $y_0 = x_0$. Suppose we have defined $y_i = x_r$ for $r < m$. Let s be the largest index such that $s > r$ and $\ell(x_r, x_s) \ge n_0$. Note that since $\ell(x_r, x_{r+1}) \ge n \ge n_0$, such s exists.

If $s < m$, then $\ell(x_r, x_{s+1}) < n_0$, and

$$\ell(x_r, x_{s+1}) \ge \min\{\ell(x_r, x_s), \ell(x_s, x_{s+1})\} - \Delta \ge \min\{n_0, \ell(x_s, x_{s+1})\} - \Delta = n_0 - \Delta.$$

Define then $y_{i+1} = x_{s+1}$. We have

$$n_0 - k \le \ell(y_i, y_{i+1}) < n_0.$$

If $s+1 = m$, we stop and get our sequence $y_0, \ldots, y_t$, for $t = i+1$.

If $s = m$, then $\ell(x_r, x_m) = \ell(y_i, x_m) \ge n_0$, and

$$\ell(y_{i-1}, x_m) \ge \min\{\ell(y_{i-1}, y_i), \ell(y_i, x_m)\} - \Delta \ge \min\{n_0 - \Delta, n_0\} - k = n_0 - 2\Delta$$

and

$$\ell(y_{i-1}, x_m) < n_0,$$

since y_i was defined and was not equal to x_m. Then we redefine $y_i = x_m$ and stop the algorithm.

In all the other cases we repeat the procedure. It is easy to see that at the end we get a sequence y_i satisfying the conditions of the lemma. $\square$

Let $x_0 = x, x_1, \ldots, x_m = y$ be an arbitrary sequence of points of X. Let n_0 be the minimal value of $\ell(x_i, x_{i+1})$. Let $y_0 = x, y_1, \ldots, y_t = y$ be a sub-sequence of the sequence x_i satisfying conditions of Lemma 4.14.

Suppose at first that

$$n_0 < \ell(x,y) + \frac{2\alpha\Delta - \ln C_0}{\alpha - \beta}.$$

Remember that $n_0 = \ell(x_i, x_{i+1})$ for some i, hence

$$\sum_{i=1}^{m} e^{-\beta\ell(x_{i-1},x_i)} \ge e^{-\beta n_0} > \exp\left(-\beta\ell(x,y) - \frac{\beta(2\alpha\Delta - \ln C_0)}{\alpha-\beta}\right) = Ce^{-\beta\ell(x,y)},$$

and the statement is proved.

Suppose now that $n_0 \geq \ell(x,y) + \frac{2\alpha k - \ln C_0}{\alpha-\beta}$, which is equivalent to

$$(\alpha-\beta)n_0 - (\alpha-\beta)\ell(x,y) - 2\alpha\Delta + \ln C_0 \geq 0. \tag{2}$$

If $t = 1$, then $n_0 - 2\Delta \leq \ell(x,y) < n_0$, hence

$$n_0 \leq \ell(x,y) + 2\Delta = \ell(x,y) + \frac{2\alpha\Delta - 2\beta\Delta}{\alpha-\beta} < \ell(x,y) + \frac{2\alpha\Delta - \ln C_0}{\alpha-\beta},$$

since $\ln C_0 < 0 < 2\beta\Delta$. But this contradicts our assumption.

Therefore $t > 1$, and the inductive assumption implies

$$\sum_{i=1}^{m} e^{-\beta\ell(x_{i-1},x_i)} \geq \sum_{i=0}^{t-1} Ce^{-\beta\ell(y_i,y_{i+1})} > tCe^{-\beta n_0}.$$

We have $t \geq d_{n_0-2\Delta}(x,y) \geq C_0 e^{\alpha(n_0-2\Delta-\ell(x,y))}$, hence

$$\sum_{i=1}^{m} e^{-\beta\ell(x_{i-1},x_i)} \geq C_0Ce^{-\beta n_0 + \alpha n_0 - 2\alpha\Delta - \alpha\ell(x,y)} =$$

$$C\exp\left(\ln C_0 - \beta n_0 + \alpha n_0 - 2\alpha\Delta - \alpha\ell(x,y)\right) =$$

$$C\exp\left(-\beta\ell(x,y) + (\alpha-\beta)n_0 - (\alpha-\beta)\ell(x,y) - 2\alpha\Delta + \ln C_0\right) \geq Ce^{-\beta\ell(x,y)},$$

by (2). Which finishes the proof. □

We get the following characterization of topologically expanding maps.

Theorem 4.15. *Let $\mathcal{X}$ be a compact Hausdorff space. A continuous map $f : \mathcal{X} \longrightarrow \mathcal{X}$ is topologically expanding if and only if there exists an expansion entourage $U \subset \mathcal{X} \times \mathcal{X}$.*

Proof. Suppose that $f : \mathcal{X} \longrightarrow \mathcal{X}$ is topologically expanding. Then there exists a metric d and numbers $\epsilon > 0$ and $L > 1$ such that $d(f(x), f(y)) > Ld(x,y)$ for all $(x,y) \in \mathcal{X}^2$ such that $d(x,y) \leq \epsilon$. Then the set $\{(x,y) : d(x,y) \leq \epsilon\}$ is an expansion entourage.

Suppose now that U is an expansion entourage. Suppose that d is a metric associated with U. Let $C > 1$ and $\alpha > 0$ be such that

$$C^{-1}e^{-\alpha\ell(x,y)} \leq d(x,y) \leq Ce^{-\alpha\ell(x,y)}$$

for all $x, y \in \mathcal{X}$. Let k be a positive integer, and suppose that $\ell(x,y) \geq k$. Then $\ell(f^k(x), f^k(y)) = \ell(x,y) - k$, and

$$\frac{d(f^k(x), f^k(y))}{d(x,y)} \leq C^2e^{-\alpha k}.$$

It follows that for any integer k greater than $\frac{\ln C^2}{\alpha}$ we have $d(f^k(x), f^k(y)) \leq Ld(x,y)$, where $L = C^2e^{-\alpha k} < 1$, for all $(x,y) \in U_k$. If $\epsilon < C^{-1}e^{-\alpha k}$, then $\ell(x,y) \geq k$ for all $x, y \in \mathcal{X}$ such that $d(x,y) \leq \epsilon$. It follows that f^k is expanding, hence f is also expanding, by Lemma 4.1. □

Proposition 4.16. *Let U and V be expansion entourages for a map $f : \mathcal{X} \longrightarrow \mathcal{X}$. Then the sets of lower exponents for U and V coincide. If d_U and d_V are metrics associated with U and V of exponent α, then there exists $C > 1$ such that $C^{-1}d_U(x,y) \leq d_V(x,y) \leq Cd_U(x,y)$ (the metrics are* bi-Lipschitz equivalent*).*

PROOF. Let ℓ_U and ℓ_V be defined by U and V, respectively. By Lemma 4.8, there exists k such that $U_k \subset V$, hence $U_{n+k} \subset V_n$ for all $n \in \mathbb{N}$. It follows that $\ell_V(x,y) \geq \ell_U(x,y) + k$. The same arguments show that $\ell_U(x,y) \geq \ell_V(x,y) + k$ for some k, i.e., that $|\ell_U(x,y) - \ell_V(x,y)|$ is uniformly bounded. The statements of the proposition easily follow from this fact. $\square$

We see that for any expanding map $f : \mathcal{X} \longrightarrow \mathcal{X}$ the *critical lower exponent* α is well defined (i.e., depends only on the topological dynamical system $(\mathcal{X}, f)$), and for every $\beta \in (0, \alpha)$ the corresponding metric of exponent β is uniquely defined, up to a bi-Lipschitz equivalence. Note that if d is a metric of exponent β, then any metric bi-Lipschitz equivalent to d is also a metric of exponent β.

PROBLEM 4.6. Find the critical lower exponents of the one-sided shift $s : \{0,1\}^{\mathbb{N}} \longrightarrow \{0,1\}^{\mathbb{N}}$ and the angle doubling map $x \mapsto 2x : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$.