5. Iterated monodromy groups

5.1. Expanding coverings. Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be an expanding *covering* map. Let $\epsilon > 0$, $L > 1$, and a metric d on $\mathcal{X}$ are such that $d(f(x), f(y)) \geq Ld(x, y)$ for all $x, y \in \mathcal{X}$ such that $d(x, y) \leq \epsilon$.

By definition, $f : \mathcal{X} \longrightarrow \mathcal{X}$ is a covering map if for every $x \in \mathcal{X}$ there exists an open neighborhood U of x that is *evenly covered*, i.e., such that $f^{-1}(U)$ can be decomposed into a disjoint union $f^{-1}(U) = U_1 \cup U_2 \cup \cdots \cup U_m$ such that $f : U_i \longrightarrow U$ is a homeomorphism for every i. The decomposition is finite, since $\mathcal{X}$ is compact (hence $f^{-1}(x)$ is compact for every $x \in \mathcal{X}$). Note that in general (if $\mathcal{X}$ is not locally connected) the decomposition is not unique. But we can use the fact that f is expanding to choose canonical decompositions for sets U of small diameter as follows.

Since $\mathcal{X}$ is compact, there exists a finite cover $\mathcal{U}$ of $\mathcal{X}$ by open evenly covered sets. Then, by Lebesgue's lemma, there exists $\delta_0 > 0$ such that for every set B of diameter less than δ_0 there exists $U \in \mathcal{U}$ such that $B \subset U$. It follows that every set of diameter less than δ_0 is evenly covered.

Consider decompositions of $f^{-1}(U)$, for $U \in \mathcal{U}$, into disjoint unions $U = U_1 \cup \cdots \cup U_m$ such that $f : U_i \longrightarrow U$ are homeomorphisms, and consider the corresponding inverse maps $f^{-1} : U \longrightarrow U_i$. By continuity of these inverse maps, there exists $\delta < \delta_0$ such that for every set A of diameter less than δ the set $f^{-1}(A)$ can be decomposed into a disjoint union of sets $A_1 \cup \cdots \cup A_m$ of sets of diameter less than ϵ. Then the diameters of A_i will be less than $L^{-1}\delta$. Note that then distance between any two different points of $f^{-1}(x)$ for $x \in \mathcal{X}$ is not less than ϵ. Consequently, for any $x_1 \in A_i$ and $x_2 \in A_j$ for $i \neq j$ we have $d(x_1, x_2) > \epsilon - 2L^{-1}\delta$. If δ is small enough, then $\epsilon - 2L^{-1}\delta > \delta$, and we get the following.

Lemma 5.1. *If δ is small enough, then for every set $A \subset \mathcal{X}$ of diameter less than δ the set $f^{-1}(A)$ is decomposed in a unique way into a disjoint union $f^{-1}(A) = A_1 \cup \cdots \cup A_m$ such that $f : A_i \longrightarrow A$ are homeomorphisms, sets A_i have diameters less than δ, and distance between any two points belonging to different sets A_i is greater than δ.*

We will call the sets A_i the *components* of $f^{-1}(A)$. For $n > 1$, the components of $f^{-n}(A)$ are defined inductively as components of $f^{-1}(A_i)$, where A_i is a component of $f^{-(n-1)}(A)$. Note that since components of $f^{-1}(A)$ are of diameter less than $L^{-1}\delta < \delta$, we have a unique decomposition of $f^{-n}(A)$ into components. If A is connected, then components of $f^{-n}(A)$ are its connected components.

Fix some $\delta > 0$ satisfying the conditions of Lemma 5.1. Let $U \subset \mathcal{X}$ be a set of diameter less than δ. Consider the rooted tree T_U whose nth level is the set of components of $f^{-n}(U)$, and in which a component A of $f^{-n}(U)$ is connected to the component $f(A)$ of $f^{-(n-1)}(U)$. The root is $f^{-0}(U) = \{U\}$.

Similarly, for every $x \in \mathcal{X}$, denote by T_x the tree whose levels are the sets $f^{-n}(x)$, in which a vertex $t \in f^{-n}(x)$ is connected to the vertex $f(t) \in f^{-(n-1)}(x)$. For every $x \in U$ the trees T_x and T_U are naturally isomorphic: the isomorphism maps a vertex $t \in f^{-n}(x)$ of T_x to the unique component of $f^{-n}(U)$ containing t.

The *boundary* ∂T_x of the tree T_x is the inverse limit of the sets $f^{-n}(x)$ with respect to the maps $f : f^{-n}(x) \longrightarrow f^{-(n-1)}(x)$. In other words, it is the space of all simple (i.e., without repetition) infinite paths in T_x starting at the root with the topology of coordinatewise convergence: two paths are close to each other if they coincide on a long initial segment. The boundary ∂T_U is defined in the same way.

Consider the natural extension $\hat{f} : \widehat{\mathcal{X}} \longrightarrow \widehat{\mathcal{X}}$ of $f : \mathcal{X} \longrightarrow \mathcal{X}$. Let $P : \widehat{\mathcal{X}} \longrightarrow \mathcal{X}$ be the natural projection map. For every set $U \subset \mathcal{X}$ of diameter less than δ the set $P^{-1}(U) \subset \mathcal{X}$ is naturally decomposed into the direct product $U \times \partial T_U$. Namely, every point $(x_0, x_1, \ldots) \in \widehat{\mathcal{X}}$ is a point of ∂T_{x_0}. Let $\xi \in \partial T_U$ be the image of this point under the natural isomorphism $T_{x_0} \longrightarrow T_U$. The point ξ is the unique sequense $(U, U_1, U_2, \ldots)$, where U_n is the component of $f^{-n}(U)$ containing x_n. Then it is easy to see that the map $(x_0, x_1, \ldots) \mapsto (x_0, \xi)$ is a homeomorphism $P^{-1}(U) \longrightarrow U \times \partial T_U$.

Suppose that $A, B \subset \mathcal{X}$ are sets of diameters less than δ such that $A \cap B \neq \emptyset$. Then for every component A_1 of $f^{-1}(A)$ there exists a unique component B_1 of $f^{-1}(B)$ such that $A_1 \cap B_1 \neq \emptyset$ (if there is another such component B_1', then for any points $x_1 \in B_1 \cap A_1$ and $x_2 \in B_1' \cap A_1$ we have $d(x_1, x_2) < \delta$, which contradicts our choice of $\mathcal{U}$). By induction, for every component A_n of $f^{-n}(A)$ there exists a unique component B_n of $f^{-n}(B)$ such that $A_n \cap B_n \neq \emptyset$. It follows that there exists a unique map $S_{A,B} : T_A \longrightarrow T_B$ such that $V \cap S_{A,B}(V) \neq \emptyset$ for all vertices V of T_A. It is easy to see that it is an isomorphism of rooted trees. It is equal to the composition of the natural homeomorphisms $T_A \longrightarrow T_x \longrightarrow T_B$ for any $x \in A \cap B$.

We will also denote by $S_{A,B}$ the induced homeomorphism $\partial T_A \longrightarrow \partial T_B$. It describes the gluing rule between the pieces $P^{-1}(A)$ and $P^{-1}(B)$ of $\widehat{\mathcal{X}}$ for the decompositions $P^{-1}(A) = A \times \partial T_A$ and $P^{-1}(B) = B \times \partial T_B$.

LEMMA 5.2. *If U_1, U_2, U_3 be subset of diameter less than δ such that $U_1 \cap U_2 \cap U_3 \neq \emptyset$, then $S_{U_2,U_3} \circ S_{U_1,U_2} = S_{U_1,U_3}$.*

PROOF. Choose a point $x \in U_1 \cap U_2 \cap U_3$. Then S_{U_i,U_j} is equal to the composition of the natural isomorphisms $T_{U_i} \longrightarrow T_x \longrightarrow T_{U_j}$. □

Let $\mathcal{U}$ be a finite set of subsets of $\mathcal{X}$ of diameter less than δ such that their union is whole $\mathcal{X}$. Recall that a *nerve* of the cover $\mathcal{U}$ is the simplicial complex with the set of vertices equal to $\mathcal{U}$ in which a subset $\mathcal{C} \subset \mathcal{U}$ is a simplex if and only if $\bigcap_{A \in \mathcal{C}} A$ is non-empty.

Let $\Gamma_{\mathcal{U}}$ be the nerve of the cover $\mathcal{U}$. For every edge (U_1, U_2) of $\Gamma_{\mathcal{U}}$ we have the isomorphism $S_{U_1,U_2} : T_{U_1} \longrightarrow T_{U_2}$. For every path $\gamma = (U_1, U_2, \ldots, U_n)$, we get isomorphisms $S_{U_i,U_{i+1}}$. Their composition is an isomorhism $S_\gamma : T_{U_1} \longrightarrow T_{U_n}$.

It follows from Lemma 5.2 that paths homotopic in $\Gamma_{\mathcal{U}}$ define equal isomorphisms. In particular, the map $\gamma \mapsto S_\gamma$ is a homormophism from $\pi_1(\Gamma_{\mathcal{U}}, V)$ to the automorphism group of T_V. Let us denote the image of $\pi_1(\Gamma_{\mathcal{U}}, V)$ under this homomorphism by $\mathrm{IMG}\,(f, \mathcal{U}, V)$. Note that every isomorphism S_{U_1,U_2} induces an isomorphism $\mathrm{IMG}\,(f, \mathcal{U}, U_1) \longrightarrow \mathrm{IMG}\,(f, \mathcal{U}, U_2)$ (by conjugation). It follows that if the nerve $\Gamma_{\mathcal{U}}$ is connected (e.g., if $\mathcal{X}$ is connected), then the group $\mathrm{IMG}\,(f, \mathcal{U}, V)$ does not depend on V. We call the group $\mathrm{IMG}\,(f, \mathcal{U}, V)$ the *iterated monodromy group.*

LEMMA 5.3. *Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are finite covers of $\mathcal{X}$ by sets of diameter less than δ such that for every element $A \in \mathcal{U}_1$ there exists $B \in \mathcal{U}_2$ such that $A \subset B$. Suppose that $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$ are such that $U_1 \subset U_2$. Then $S_{U_2,U_1}^{-1} \mathrm{IMG}\,(f, \mathcal{U}_1, U_1)\, S_{U_2,U_1} \leq \mathrm{IMG}\,(f, \mathcal{U}_2, U_2)$.*

Proof. Consider the nerve $\Gamma_{\mathcal{U}_1 \cup \mathcal{U}_2}$. Then $\Gamma_{\mathcal{U}_i}$ are sub-complexes of $\Gamma_{\mathcal{U}_1 \cup \mathcal{U}_2}$. Let $(U_1, A_1, A_2, \ldots, A_n, U_1)$ be a loop in $\Gamma_{\mathcal{U}_1}$. Let $B_i \in \mathcal{U}_2$ be such that $A_i \subset B_i$. Then $(U_2, B_1, B_2, \ldots, B_n, U_2)$ is a loop in $\Gamma_{\mathcal{U}_2}$. Note that $\{B_i, B_{i+1}, A_i, A_{i+1}\}$, $\{U_2, B_1, U_1, A_1\}$, and $\{U_2, B_n, U_1, A_n\}$ are simplices in $\Gamma_{\mathcal{U}_1 \cup \mathcal{U}_2}$. It follows that the loop $(U_2, U_1, A_1, A_2, \ldots, A_n, U_1, U_2)$ is homotopic in $\Gamma_{\mathcal{U}_1 \cup \mathcal{U}_2}$ to the loop $(U_2, B_1, B_2, \ldots, B_n, U_2)$, which implies that the automorphisms of T_{U_2} defined by them are equal. $\square$

Lemma 5.4. *Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are finite covers of $\mathcal{X}$ by sets of diameter less than δ such that every element of $\mathcal{U}_2$ is connected and is equal to a union of elements of $\mathcal{U}_1$. Then* $\mathrm{IMG}\,(f, \mathcal{U}_2, U_2) \le S_{U_2,U_1}^{-1} \mathrm{IMG}\,(f, \mathcal{U}_1, U_1)\, S_{U_2,U_1}$.

Proof. Let $\gamma = (B_0 = U_2, B_1, B_2, \ldots, B_{n-1}, B_n = U_2)$ be a loop in $\Gamma_{\mathcal{U}_2}$. Choose $x_i \in B_i \cap B_{i+1}$ for $i = 0, \ldots, n-1$. Since each B_i is connected and equal to a union of elements of $\mathcal{U}_1$, for every i there exists a path $A_{1,i}, A_{2,i}, \ldots, A_{k_i,i}$ in $\Gamma_{\mathcal{U}_1}$ such that $A_{j,i} \subset B_i$ and $x_{i-1} \in A_{1,i}$, $x_i \in A_{k_i,i}$. Replacing in the loop γ the vertex B_i by the path $(A_{1,i}, A_{2,i}, \ldots, A_{k_i,i})$, we will get a loop in $\Gamma_{\mathcal{U}_1}$ homotopic in $\Gamma_{\mathcal{U}_1 \cup \mathcal{U}_2}$ to γ. $\square$

As a direct corollary of Lemmas 5.3 and 5.4 we get the following.

Proposition 5.5. *Suppose that $\mathcal{X}$ is locally connected and connected, and let $\mathcal{U}$ be a finite cover of $\mathcal{X}$ by open connected sets of diameter less than δ. Then* $\mathrm{IMG}\,(f, \mathcal{U}, U)$ *does not depend on $\mathcal{U}$ and U.*

If $\mathrm{IMG}\,(f, \mathcal{U}, U)$ does not depend on U and $\mathcal{U}$, then we denote it $\mathrm{IMG}\,(f)$.

5.2. Iterated monodromy group for path-connected spaces. Suppose that $\mathcal{X}$ is path connected and locally path connected. Let γ be a path starting at $t_1 \in \mathcal{X}$ and ending in $t_2 \in \mathcal{X}$. By uniqueness of lifts of paths by covering maps, for every $z \in f^{-1}(t)$ there exists a unique path γ_z starting in z and such that $f(\gamma_z) = \gamma$. Similarly, for every vertex $z \in f^{-n}(t)$ of the nth level of the tree T_t there exists a unique path γ_z starting in z such that $f^n(\gamma_z) = \gamma$. Let $\gamma(z)$ be the end of γ_z. Then the map $z \mapsto \gamma(z)$ is an isomorphism $T_{t_1} \longrightarrow T_{t_2}$. Let us denote it by S_γ.

In particular, the map $\gamma \mapsto S_\gamma$ is a homomorphism from the fundamental group $\pi_1(\mathcal{X}, t)$ to the automorphism group of T_t. Its image is called the *iterated monodromy group* of $f : \mathcal{X} \longrightarrow \mathcal{X}$. It is easy to see that it does not depend (up to conjugacy of automorphism groups of rooted trees) on the choice of the basepoint t.

Proposition 5.6. *If the space $\mathcal{X}$ is path connected and locally path connected, then our two definitions of iterated monodromy groups (using covers and using paths) coincide.*

5.3. Modelling expanding coverings by graphs. Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be an expanding covering, let $\epsilon > 0$ and $L > 1$ be such that $d(f(x), f(y)) \ge L d(x, y)$ for all $x, y \in \mathcal{X}$, $d(x, y) < \epsilon$, and let δ be as in Lemma 5.1. We do not impose any connectivity conditions on $\mathcal{X}$ in this subsection.

Let $\mathcal{U}$ be a finite cover of $\mathcal{X}$ by subsets of diameter less than δ. Denote by $\mathcal{U}_n$ the set of components of $f^{-n}(U)$ for $U \in \mathcal{U}$. We also denote $\mathcal{U}_0 = \mathcal{U}$.

Denote by Γ_n the nerve of the cover $\mathcal{U}_n$.

The map f induces simplicial maps $f_n : \Gamma_{n+1} \longrightarrow \Gamma_n$ by the rule that $f_n(U) = f(U)$, where $f(U)$ is the image of U as a set under the map $f : \mathcal{X} \longrightarrow \mathcal{X}$, i.e., U is a component of $f^{-1}(f_n(U))$.

LEMMA 5.7. *The maps $f_n : \Gamma_{n+1} \longrightarrow \Gamma_n$ are coverings.*

PROOF. For $U \in \mathcal{U}_n$, denote by N_U the sub-complex of Γ_n equal to the union of simplices containing U.

It is enough to show that $f : N_U \longrightarrow N_{f(U)}$ is an isomorphism for every $U \in \mathcal{U}_{n+1}$. It is obviously a simplicial map.

Let us show that $f : N_U \longrightarrow N_{f(U)}$ is injective on the set of vertices adjacent to U. Suppose that it is not, then there exist elements $A, B, C \in \mathcal{U}_{n+1}$ such that $A \cap C$ and $B \cap C$ are non-empty, and $f(A) = f(B)$. But then there exist $x \in A$ and $y \in B$ such that $d(x, y) < \delta$, which contradicts the conditions of Lemma 5.1.

For every simplex $\Delta = \{f(U), A_1, A_2, \ldots, A_k\}$ of Γ_n containing $f(U)$ there exists a unique simplex $\Delta' = \{U, B_1, B_2, \ldots, B_k\} = \{U, S_{f(U),A_1}(U), S_{f(U),A_2}(U), \ldots, S_{f(U),A_k}(B_k)\}$ of Γ_{n+1} containing U such that $f(\Delta') = \Delta$. Consequently, $f : N_U \longrightarrow N_{f(U)}$ is an isomorphism. $\square$

DEFINITION 5.1. We say that $\mathcal{U}$ is *semi-Markovian* if for every $U \in \mathcal{U}_1$ there exists $U' \in \mathcal{U}$ such that $U \subset U'$.

LEMMA 5.8. *There exists a finite semi-Markovian cover.*

PROOF. Let $\mathcal{V}$ be a cover of $\mathcal{X}$ by sets of diameter less than δ_0. As before, we denote by $\mathcal{V}_n$ the set of components of $f^{-n}(A)$ for $A \in \mathcal{V}$. Define, for every $V \in \mathcal{V}$ the sets $V^{(n)}$ inductively by the rule that $V^{(0)} = V$, and $V^{(n+1)}$ is equal to the union of $V^{(n)}$ and all elements $W \in \mathcal{V}_{n+1}$ such that $W \cap V^{(n+1)} \neq \emptyset$. Define $V^{(\infty)} = \bigcup_{n \geq 1} V^{(n)}$, and let $\mathcal{V}^{(\infty)} = \{V^{(\infty)} : V \in \mathcal{V}\}$.

Diameter of $V^{(n)}$ is less than

$$2\delta_0(1 + L^{-1} + L^{-2} + \cdots + L^{-n}) < 2\delta_0/(1 - L^{-1}).$$

Consequently, diameter of $V^{(\infty)}$ is not more than $2\delta_0/(1 - L^{-1})$. Assume that $\delta_0 < (1 - L^{-1})\delta/2$. Then all elements of $\mathcal{V}^{(\infty)}$ have diameters less than δ.

It is easy to see that then $(\mathcal{V}_n)^{(\infty)} = (\mathcal{V}^{(\infty)})_n$, and that if $U \in \mathcal{V}_1$ and $V \in \mathcal{V}$ are such that $U \cap V \neq \emptyset$, then $U^{(\infty)}$ (as an element of $\mathcal{V}_1^{(\infty)}$) is contained in $V^{(\infty)}$, which implies that $\mathcal{V}^{(\infty)}$ is semi-Markovian. $\square$

Let $\mathcal{U}$ be a semi-Markovian cover. Choose for every $U \in \mathcal{U}_1$ an element $\iota(U) \in \mathcal{U}_0$ such that $U \subset \iota(U)$. It is easy to see that ι is a simplicial map (that it sends simplices to simplices).

Since U and $\iota(U)$ intersect, the map $S_{U,\iota(U)} : T_U \longrightarrow T_{\iota(U)}$ is defined. For every n it defines a bijection between the set of components of $f^{-n}(U)$ and the set of components of $f^{-n}(\iota(U))$. These sets are subsets of $\mathcal{U}_{n+1}$ and $\mathcal{U}_n$ respectively, and union of the maps $S_{U,\iota(U)}$ for $U \in \mathcal{U}_1$ is a map from $\mathcal{U}_{n+1}$ to $\mathcal{U}_n$, which we will denote ι_n.

Equivalently, $\iota_n(A)$ is the unique component of $f^{-1}(\iota_{n-1}(f(A)))$ containing A.

The map ι_n is uniquely defined by the condition that if A is a component of $f^{-n}(U)$ for $U \in \mathcal{U}_1$, then $\iota_n(A)$ is the unique component of $f^{-n}(\iota(U))$ such that

$\iota_n(A) \supset A$. It follows that $\iota_n : \Gamma_{n+1} \longrightarrow \Gamma_n$ is simplicial and that the diagram

$$\begin{array}{ccc} \Gamma_{n+2} & \xrightarrow{\iota_{n+1}} & \Gamma_{n+1} \\ \big\downarrow{\scriptstyle f_{n+1}} & & \big\downarrow{\scriptstyle f_n} \\ \Gamma_{n+1} & \xrightarrow{\iota_n} & \Gamma_n \end{array} \tag{3}$$

is commutative.

Let us show that the pair $f_0, \iota_0 : \Gamma_1 \longrightarrow \Gamma_0$ uniquely determines the sequence $f_n, \iota_n : \Gamma_{n+1} \longrightarrow \Gamma_n$.

Proposition 5.9. *Let $\widetilde{\Gamma}_n$ be the complex whose set of vertices is equal to the set of sequences $(v_1, v_2, \ldots, v_n)$ of vertices of Γ_1 such that $f_0(v_i) = \iota_0(v_{i-1})$. A set $\{(v_{1i}, v_{2i}, \ldots, v_{ni})\}_{i=1,\ldots,k}$ is a simplex of $\widetilde{\Gamma}_n$ if $\{v_{j1}, v_{j2}, \ldots, v_{jk}\}$ is a simplex for every $j = 1, \ldots, n$.*

Then there exists isomorphisms $\phi_n : \widetilde{\Gamma}_n \longrightarrow \Gamma_n$ such that

$$f_n(\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})) = \phi_n(v_2, v_3, \ldots, v_{n+1})$$

and

$$\iota_n(\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})) = \phi_n(v_1, v_2, \ldots, v_n)$$

for all $n \geq 1$.

Proof. Let us construct and prove properties of ϕ_n by induction. For $n = 1$ the graph $\widetilde{\Gamma}_1$ coincides with Γ_1, so set ϕ_1 to be equal to the identity map.

Suppose that ϕ_n is defined and satisfies the properties of the proposition. Let $(v_1, v_2, \ldots, v_{n+1})$ be an arbitrary vertex of $\widetilde{\Gamma}_{n+1}$.

If $n = 1$, then we have $v_2 \subset f(v_1)$, since $(v_1, v_2) \in \Gamma_1$. For $n > 1$ we have We have $\phi_n(v_2, v_3, \ldots, v_{n+1}) \subset f(\phi_n(v_1, v_2, \ldots, v_n))$, since $\phi_{n-1}(v_2, v_3, \ldots, v_n) = \iota_n(\phi_n(v_2, v_3, \ldots, v_{n+1}))$ and $f(\phi_n(v_1, v_2, \ldots, v_n)) = \phi_{n-1}(v_2, v_3, \ldots, v_n)$, by the inductive hypothesis.

Consequently, for $n = 1$ there exists a unique component of $f^{-1}(v_2)$ contained in v_1. We set $\phi_2((v_1, v_2))$ to be equal to this component. Similarly, for $n > 1$ there exists a unique component of $f^{-1}(\phi_n(v_2, v_3, \ldots, v_{n+1}))$ contained in $\phi_n(v_1, v_2, \ldots, v_n)$. We set $\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$ to be equal to it.

Formally, in both cases we defined ϕ_{n+1} by the rule

$$\phi_{n+1}(v_1, v_2, \ldots, v_{n+1}) = S_{f_{n-1}(\phi_n(v_1, v_2, \ldots, v_n)), \phi_n(v_2, v_3, \ldots, v_{n+1})}(\phi_n(v_1, v_2, \ldots, v_n)). \tag{4}$$

We get a map $\phi_{n+1} : \widetilde{\Gamma}_{n+1} \longrightarrow \Gamma_{n+1}$ (between sets of vertices). Let us show that it satisfies the conditions of the proposition and that it is an isomorphism of simplicial complexes.

It follows directly from the definition that $f_n(\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})) = \phi_n(v_2, v_3, \ldots, v_{n+1}))$, as we defined $\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$ as a component of $f^{-1}(\phi_n(v_2, v_3, \ldots, v_{n+1}))$.

The vertex $\iota_n(\phi_{n+1}(v_1, v_2, \ldots, v_{n+1}))$ is, by definition, the component of $f^{-1}(\iota_{n-1} \circ f \circ \phi_{n+1}(v_1, v_2, \ldots, v_{n+1}))$ containing $\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$. We have $\iota_{n-1} \circ f \circ \phi_{n+1}(v_1, v_2, \ldots, v_{n+1}) = \iota_{n-1}(\phi_n(v_2, v_3, \ldots, v_{n+1})) = \phi_{n-1}(v_2, v_3, \ldots, v_n)$. Consequently, $\iota_n(\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$ is the component of $f^{-1}(\phi_{n-1}(v_2, v_3, \ldots, v_n))$ containing $\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$. The set $\phi_n(v_1, v_2, \ldots, v_n)$ satisfies these conditions, since $f(\phi_n(v_1, v_2, \ldots, v_n)) = \phi_{n-1}(v_2, \ldots, v_n)$, by the inductive assumption, and $\phi_n(v_1, v_2, \ldots, v_n) \supset \phi_{n+1}(v_1, v_2, \ldots, v_{n+1})$, by the definition of ϕ_{n+1}. It follows that $\iota_n(\phi_{n+1}(v_1, v_2, \ldots, v_{n+1})) = \phi_n(v_1, v_2, \ldots, v_n)$. The case $n = 1$ is similar.

Let us show (also by induction) that ϕ_{n+1} is simplicial. Suppose that

$$\Delta = \{(v_{1,i}, v_{2,i}, \ldots, v_{n+1,i}) \ : \ i = 1, \ldots, k\}$$

is a simplex of $\widetilde{\Gamma}_{n+1}$. Then $\{\phi_n(v_{2,i}, v_{3,i}, \ldots, v_{n+1,i})\}$ and $\{\phi_n(v_{1,i}, v_{2,i}, \ldots, v_{n,i})\}$ are simplices of $\widetilde{\Gamma}_n$, since ϕ_n is simplicial. It means that $\bigcap_{i=1,\ldots,k} \phi_n(v_{2,i}, v_{3,i}, \ldots, v_{n+1,i})$ and $\bigcap_{i=1,\ldots,k} \phi_n(v_{1,i}, v_{2,i}, \ldots, v_{n,i})$ are non-empty. Then it follows from the definition (4) of ϕ_{n+1} and Lemma 5.2 that $\{\phi_{n+1}(v_{1,i}, v_{2,i}, \ldots, v_{n+1,i})\}_{i=1,\ldots,k}$ is a simplex of Γ_{n+1}. The case $n = 1$ is similar.

It remains to show that ϕ_{n+1} has an inverse simplicial map. If $n = 1$, then it is checked directly that the inverse map is $\phi_2^{-1}(v) = (\iota(v), f(v))$.

For every $v \in \Gamma_{n+1}$ we have $f_{n-1}(\iota_n(v)) = \iota_{n-1}(f_n(v))$, hence $\phi_n^{-1}(\iota_n(v)) = (v_1, v_2, \ldots, v_n)$ and $\phi_n^{-1}(f_n(v)) = (v_2, v_3, \ldots, v_{n+1})$ for some $v_i \in \Gamma_1$. Define $\phi'_{n+1} = (v_1, v_2, \ldots, v_{n+1})$. It is checked then directly that ϕ'_{n+1} is the inverse of ϕ_{n+1}. It is obvious that ϕ'_{n+1} □

Consider the sequence of complexes and morphisms:

$$\Gamma_0 \xleftarrow{\iota} \Gamma_1 \xleftarrow{\iota_1} \Gamma_2 \xleftarrow{\iota_2} \cdots,$$

and let $\lim_\iota \Gamma_n$ be the inverse limit. It can be considered as a simplicial complex: its set of vertices is the inverse limit of the sets of vertices of Γ_n; and its set of simplices is the inverse limit of the sets of simplices of Γ_n. Note that both sets are compact topologica spaces (homeomorphic to the Cantor sets, if the set of edges is non-empty). As an abstract complex (without topology), the complex $\lim_\iota \Gamma_n$ has uncountably many connected components.

A vertex of $\lim_\iota \Gamma_n$ is a sequence $(V_0, V_1, V_2, \ldots)$ of vertices $V_n \in \mathcal{U}_n$ of Γ_n such that $\iota_n(V_{n+1}) = V_n$ for all n. Then $V_{n+1} \subset V_n$. Diamenter of V_n is less than $L^{-n}\delta$. It follows that every sequence of points $x_n \in V_n$ is converging and the limit does not depend on the choice of x_n. Let us denote it by $\Phi(V_0, V_1, \ldots)$.

Lemma 5.10. *If vertices u, v of $\lim_\iota \Gamma_n$ are adjacent, then $\Phi(u) = \Phi(v)$.*

Proof. Let $u = (A_0, A_1, \ldots)$ and $v = (B_0, B_1, \ldots)$. If u and v are adjacent, then $A_n \cap B_n \neq \emptyset$, and we can choose $x_n \in A_n \cap B_n$. Then $\Phi(u) = \Phi(v) = \lim_{n\to\infty} x_n$. □

Lemma 5.11. *The map Φ is onto.*

Proof. Let $x \in \mathcal{X}$ be an arbitrary point. For every n there exists $A_n \in \mathcal{U}_n$ such that $x \in A_n$. Then x belongs to every element of the sequence $\iota_0 \circ \iota_1 \circ \cdots \circ \iota_{n-1}(A_n), \iota_1 \circ \iota_2 \circ \cdots \circ \iota_{n-1}(A_n), \ldots, \iota_{n-1}(A_n), A_n$. Consider the sequence of such sequences as $n \to \infty$. Since every complex Γ_n is finite, we can find a convergent sub-sequence, and its limit will be a vertex $(A_0, A_1, \ldots)$ of $\lim_\iota \Gamma_n$ such that $x \in A_n$ for all n. Then $\Phi(A_0, A_1, \ldots) = x$. □

Proposition 5.12. *If elements of $\mathcal{U}$ are closed and u, v are vertices of $\lim_\iota \Gamma_n$ such that $\Phi(u) = \Phi(v)$, then u and v are adjacent.*

If elements of $\mathcal{U}$ are open and $\Phi(u) = \Phi(v)$, then there exists combinatorial distance from u to v in the graph $\lim_\iota \Gamma_n$ is not more than 2.

Proof. If elements of $\mathcal{U}$ are closed (resp., open), then all elements of $\mathcal{U}_n$ are closed (resp., open).

Let $u = (A_0, A_1, \ldots)$ and $v = (B_0, B_1, \ldots)$. Suppose that $x = \Phi(u) = \Phi(v)$. We have $A_0 \supset A_1 \supset A_2 \supset \ldots$, $B_0 \supset B_1 \supset B_2 \supset \ldots$, and x is an accumulation point on both sequences. It follos that x is an accumulation point of each set A_n and B_n for all n. If all A_n, B_n are closed, then this implies that u and v are adjacent.

Suppose that the covers $\mathcal{U}_n$ are open. Then, by the proof of Lemma 5.11, there exists a vertex $(C_0, C_1, \ldots)$ such that $x \in C_n$ for all n. Since x belongs to the closure of each set A_n and B_n, we have $C_n \cap A_n \neq \emptyset$ and $C_n \cap B_n \neq \emptyset$. It follows that $(C_0, C_1, \ldots)$ is adjacent both to $(A_0, A_1, \ldots)$ and to $(B_0, B_1, \ldots)$. □

Lemma 5.13. *The map $\Phi : \lim_\iota \Gamma_n \longrightarrow \mathcal{X}$ is continuous on the space of vertices of* $\lim_\iota$.

Proof. Define a metric d on the set of vertices of $\lim_\iota \Gamma_n$ by the condition that $d((A_0, A_1, \ldots), (B_0, B_1, \ldots)) = \frac{1}{m+1}$, where m is the minimal index such that $A_m \neq B_m$.

Suppose that $v = (A_0, A_1, \ldots)$ and $u = (B_0, B_1, \ldots)$, and $d(v, u) = \frac{1}{m+1}$. Then $A_m = B_m$, and $\Phi(A_0, A_1, \ldots)$ and $\Phi(B_0, B_1, \ldots)$ both belong to the closure of A_m. The closure of A_m has diameter less than $L^{-m}\delta$, hence

$$d(\Phi(v), \Phi(u)) \leq L^{-m}\delta = L^{1-1/d(v,u)}\delta,$$

which implies that Φ is continuous. □

Note that it follows from commutativity of the diagram (3) that if $(A_0, A_1, \ldots)$ is a vertex of $\lim_\iota \Gamma_n$, then $(f(A_1), f(A_2), \ldots)$ is also a vertex of $\lim_\iota \Gamma_n$. Let us denote $f_\infty(A_0, A_1, \ldots) = (f(A_1), f(A_2), \ldots)$. It is easy to see that f_∞ is a continuous simplicial map.

Theorem 5.14. *Suppose that the elements of the cover $\mathcal{U}$ are either all closed or all open. Consider the space of connected components of the graph $\lim_\iota \Gamma_n$ with the topology of the quotient of the space of vertices. Then there exists a homeomorphism of the quotient space with $\mathcal{X}$ that conjugates f with the map induced by f_∞.*

In other words, the topological dynamical system $(\mathcal{X}, f)$ is uniquely (up to topological conjugacy) determined by the pair of maps $f, \iota : \Gamma_1 \longrightarrow \Gamma_0$. Note also that we used only the 1-skeleta of Γ_1 and Γ_0.

Proof. The map Φ induces a continuous bijection between the space of connected components and $\mathcal{X}$. The equivalence relation of belonging to one component is, by Proposition 5.12, equal to the relation of adjacency (if the elements of the cover are closed) or to the relation of being on distance less or equal to 2 (if the elements of the cover are open). In both cases the equivalence relation is a closed subset of the direct square of the space of vertices. It follows that the space of connected components is compact Hausdorff. But any continuous bijection between compact Hausdorff spaces is a homeomorphism (since image of a closed, hence compact, set is compact, hence closed). □

As an example, consider the angle doubling map $f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$, $f(x) = 2x$. Let $\mathcal{U}$ be the cover of the circle $\mathbb{R}/\mathbb{Z}$ by the arcs $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$, $[3/4, 1]$. Then $\mathcal{U}_n$ consists of arcs of the form $\left[\frac{k}{2^{n+2}}, \frac{k+1}{2^{n+2}}\right]$ for $k = 0, 1, \ldots, 2^{n+2} - 1$. It follows that the graphs Γ_n are cycles of length 2^{n+2}. There is only one choice for the map $\iota_n : \Gamma_{n+1} \longrightarrow \Gamma_n$, since an arc $\left[\frac{k}{2^{n+2}}, \frac{k+1}{2^{n+2}}\right]$ is contained in exactly one arc of the form $\left[\frac{l}{2^{n+1}}, \frac{l+1}{2^{n+1}}\right]$. Namely, $l = k/2$ if k is even and $(k-1)/2$ if k is odd.

Figure... illustrates the graphs Γ_n and the maps ι_n. One can show that the set of vertices of $\lim_{\iota_n} \Gamma_n$ can be realized as a subset of the circle homeomorphic to the Cantor set, so that edges of $\lim_{\iota_n} \Gamma_n$ connect the endpoints of the components of the complement of the Cantor set (i.e., "filling the gaps" in the Cantor set). It follows that the space of connected components is homeomorphic to the circle.

5.4. Iterated monodromy group. Let $f_0, \iota_0 : \Gamma_1 \longrightarrow \Gamma_0$ be a pair of simplicial maps between simplicial complexes such that f is a covering map. As above, this defines a sequence of complexes Γ_n and maps $f_n, \iota_n : \Gamma_{n+1} \longrightarrow \Gamma_n$.

Consider the sequence

$$\Gamma_0 \xleftarrow{f_0} \Gamma_1 \xleftarrow{f_1} \Gamma_2 \xleftarrow{f_2} \cdots .$$

For simplex (in particular a vertex) v we get the associated rooted tree T_v given by the sequence

$$v \xleftarrow{f_0} f_0^{-1}(v) \xleftarrow{f_1} (f_0 \circ f_1)^{-1} \xleftarrow{f_2} (f_0 \circ f_1 \circ f_2)^{-1}(v) \xleftarrow{f_2} \cdots ,$$

(the levels are the sets of the sequence, and two vertices are adjacent if one is the image of the other and the corresponding map of the sequence).

Every oriented edge $e = (v_1, v_2)$ defines an isomorphism $S_e : T_{v_1} \longrightarrow T_{v_2}$ uniquely defined by the condition that for every vertex u of T_{v_1}, $\{u, S_e(u)\}$ is an edge of T_e.

For every path $\gamma = (e_1, e_2, \ldots, e_n)$ in Γ_0, i.e., for a sequence of edges of the form $e_i = (v_{i-1}, v_i)$ for some sequence $(v_0, v_1, \ldots, v_n)$ of vertices, the product

$$S_\gamma = S_{e_n} S_{e_{n-1}} \cdots S_{e_1}$$

is an isomorphism from T_{v_0} to T_{v_n}.

If $\{v_1, v_2, v_3\}$ is a simplex, then $S_{(v_2,v_3)} \circ S_{(v_1,v_2)} = S_{(v_1,v_3)}$. Therefore, for any two homotopic (rel. to their endpoints) paths γ_1, γ_2, we have $S_{\gamma_1} = S_{\gamma_2}$.

Definition 5.2. Let $f_0, \iota_0 : \Gamma_1 \longrightarrow \Gamma_0$ be as above. Choose a vertex v of Γ_0. The *iterated monodromy group* $\mathrm{IMG}(f_0, \iota_0, v)$ is the group of automorphisms of T_v of the form S_γ, where γ runs through $\pi_1(\Gamma_0, v)$.

The following is straightforward.

Proposition 5.15. *Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be an expanding covering map. Let $\mathcal{U}$ be a semi-Markovian cover by sets of small diameter. Let Γ_1 and Γ_0 be the nerves of $\mathcal{U}_1$ and $\mathcal{U}$, let $f : \Gamma_1 \longrightarrow \Gamma_0$ be the map induces by f, and let $\iota : \Gamma_1 \longrightarrow \Gamma_0$ be such that $\iota(A) \supset A$. Then* $\mathrm{IMG}(f, \iota, V)$ *is isomorphic (as a group acting on a rooted tree) with* $\mathrm{IMG}(f, \mathcal{U}, V)$.

5.5. General definition of a topological correspondence.

Definition 5.3. A *topological correspondence* (or *topological automaton*) is a pair $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$, where $\mathcal{M}_1, \mathcal{M}_0$ are topological spaces, $f : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ is a finite degree covering map, and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ is continuous.

Example 5.1. Let f be a post-critically finite rational function. Define $\mathcal{M}_0$ to be the Riemann sphere minus the post-critical set, and let $\mathcal{M}_1 = f^{-1}(\mathcal{M}_0)$. Then $\mathcal{M}_1 \subset \mathcal{M}_0$, and we can take $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ to be the identical embedding.

Let $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ be a topological correspondence. Define $\mathcal{M}_n$ as the subspace of $\mathcal{M}_1^n$ consisting of all sequences $(x_1, x_2, \ldots, x_n)$ such that

$$f(x_i) = \iota(x_{i+1})$$

for all $i = 1, 2, \ldots, n-1$. Define

$$\begin{aligned} f_n(x_1, x_2, \ldots, x_{n+1}) &= (x_2, x_3, \ldots, x_{n+1}) \\ \iota_n(x_1, x_2, \ldots, x_{n+1}) &= (x_1, x_2, \ldots, x_n) \end{aligned}$$

and $f_0 = f$, $\iota_0 = \iota$. It is easy to check that $f_{n-1} \circ \iota_n = \iota_{n-1} \circ f_n$ for all $n \geq 1$.

If $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ is the identical embedding, then points $(x_1, x_2, \ldots, x_n) \in \mathcal{M}_n$ satisfies $f(x_i) = x_{i+1}$, hence they are just orbits of length n of the partial map $f : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ and are uniquely determined by x_1. It follows that $\mathcal{M}_n$ is naturally homeomorphic to the domain of the nth iteration of the map f.

LEMMA 5.16. *The maps f_n are coverings.*

PROOF. Let $(x_1, x_2, \ldots, x_n) \in \mathcal{M}_n$. Let U be an evenly covered by f neighborhood of $\iota(x_1) \in \mathcal{M}_0$. Let $U_1, U_2, \ldots, U_d$ be the decomposition of $f^{-1}(U)$ into disjoint sets such that $f : U_i \longrightarrow U$ is a homeomorphism.

Let W be the set of points $(y_1, y_2, \ldots, y_n) \in \mathcal{M}_n$ such that $\iota(y_1) \in U$. It is open in $\mathcal{M}_n$. Let W_i, for $i = 1, 2, \ldots, d$, be the set of points $(y_0, y_1, \ldots, y_n) \in \mathcal{M}_{n+1}$ such that $y_0 \in U_i$. Then the sets W_i are disjoint and open, and their union is the set of points $(y_0, y_1, \ldots, y_n)$ such that $y_0 \in f^{-1}(U)$, i.e., the set of points such that $f(y_0) \in U$, or equivalently, the set of points such that $\iota(y_1) \in U$, since $f(y_0) = \iota(y_1)$ for all points of $\mathcal{M}_{n+1}$. It follows that $\bigcup W_i$ is equal to $f_n^{-1}(W)$. The map $f_n : W_i \longrightarrow W$ is continuous, and has continuous inverse given by

$$(y_1, y_2, \ldots, y_n) \mapsto (y_0, y_1, y_2, \ldots, y_n),$$

where y_0 is defined by the conditions $y_0 \in U_i$ and $f(y_0) = \iota(y_1)$. □

Suppose that $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ is a topological correspondence, and suppose that $\mathcal{M}_0$ is path connected and locally path connected. For $t \in \mathcal{M}_0$ denote by T_t the rooted tree given by the sequence

$$\{t\} \xleftarrow{f_0} f_0^{-1}(t) \xleftarrow{f_1} (f_0 \circ f_1)^{-1}(t) \xleftarrow{f_2} (f_0 \circ f_1 \circ f_2)^{-1}(t) \xleftarrow{f_3} \cdots$$

If γ is a path in $\mathcal{M}_0$ from t_1 to t_2, then for every vertex $v \in (f_0 \circ f_1 \circ \cdots \circ f_n)^{-1}(t_1)$ of T_{t_1} there is a unique lift of γ by the covering map $f_0 \circ f_1 \circ \cdots \circ f_n$ starting at v. Its end is a vertex $S_\gamma(v) \in (f_0 \circ f_1 \circ \cdots \circ f_n)^{-1}(t_2)$ of T_{t_2}. The map $S_\gamma : T_{t_1} \longrightarrow T_{t_2}$ is an isomorphism of rooted graphs.

DEFINITION 5.4. Let $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ be a topological correspondence such that $\mathcal{M}_0$ is locally path connected and path connected. Its *iterated monodromy group* $\mathrm{IMG}(f, \iota)$ is the group of all automorphisms S_γ of T_v, where γ runs through $\pi_1(\mathcal{M}_0, v)$.

If $f : \mathcal{X} \longrightarrow \mathcal{X}$ is a self-covering map, and $\iota : \mathcal{X} \longrightarrow \mathcal{X}$ is the identity map, then the definition of $\mathrm{IMG}(f, \iota)$ coincides with the definition of $\mathrm{IMG}(f)$ given in 5.2.

5.6. Trees of words. Let X be a finite set (alphabet). Denote by $X^* = \bigcup_{n\geq 0} X^n$ the set of finite words over X, including the empty word $\varnothing$. We denote $X^0 = \{\varnothing\}$. In other terms, X^* is the *free monoid* generated by X.

The *right Cayley graph* of X^* is the graph with the set of vertices X^* in which two vertices are connected by an edge if and only if they are of the form v and vx for $v \in X^*$ and $x \in X$. The right Cayley graph is a rooted trees with the root $\varnothing$ and levels X^n. From now on we will consider X^* as a rooted tree.

For every $x \in X$ the map $v \mapsto xv$ is an isomorphism of X^* with the sub-tree of words starting with letter x.

Let now T be an abstract rooted tree, i.e., a tree with a marked vertex called *root*. Then the nth level of the tree T is the set of vertices on distance n from the root. For a vertex v of T we denote by T_v the sub-graph spanned by all vertices u such that the path from the root to u passes through v. It is a rooted tree with the root v. If v belongs to the kth level of T, then the nth level of the tree T_v is a subset of the $(n+k)$th level of the tree T.

Suppose that for every vertex x of the first level of the tree T we have found an isomorphism $S_x : T \longrightarrow T_x$. Then the monoid H generated by the transformations S_x is free. If v is the root of T, then the map $\lambda : g \mapsto g(v)$ is an isomorphism from the right Cayley graph of H to T. Taking X equal to the set of maps S_x for x in the first level of the tree T, we get an isomorphism $\lambda : X^* \longrightarrow T$.

5.7. Computation of the iterated monodromy groups. Let $f_0, \iota_0 : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ be a topological correspondence. Assume that $\mathcal{M}_0$ is path connected and locally path connected. Choose $t \in \mathcal{M}_0$ and consider the iterated monodromy group $\mathrm{IMG}(f, \iota)$ acting on T_t.

The set $f_0^{-1}(t)$ is the first level of the tree T_t. Let X be a finite set of cardinality $\deg f = |f_0^{-1}(t)|$. For every $x \in X$ choose a path ℓ_x in $\mathcal{M}_0$ starting in t and ending in $\iota(\Lambda(x))$. It will define an isomorphism $S_{\ell_x} : T_t \longrightarrow T_{\iota(\Lambda(x))}$.

The union of the maps $\iota_n : (f_1 \circ \cdots \circ f_n)^{-1}(z) \longrightarrow (f_0 \circ f_1 \circ \cdots \circ f_n)^{-1}(t)$ is an isomorphism $\iota_* : T_{\Lambda(x)} \longrightarrow T_{\iota(\Lambda(x))}$, where $T_{\Lambda(x)}$ is the subtree

$$T_{\Lambda(x)} = \{\Lambda(x)\} \cup f_1^{-1}(z) \cup (f_1 \circ f_2)^{-1}(\Lambda(x)) \cup \cdots$$

of T_t.

We get isomorphisms $\iota_*^{-1} \circ S_{\ell_x} : T_t \longrightarrow T_{\Lambda(x)} \subset T_t$. Then, as in the previous subsection, the isomorphisms $\iota_*^{-1} \circ S_{\ell_x} : T_t \longrightarrow T_{\Lambda(x)}$ define an isomorphism $\Lambda : X^* \longrightarrow T_t$ of rooted trees. It is defined inductively by the rule

$$\Lambda(xv) = \iota_*^{-1} \circ S_{\ell_x}(\Lambda(v))$$

for $x \in X$ and $v \in X^*$. Equivalently,

$$\Lambda(x_1 x_2 \ldots x_n) = \iota_*^{-1} \circ S_{\ell_{x_1}} \circ \iota_*^{-1} \circ S_{\ell_{x_2}} \circ \cdots \circ \iota_*^{-1} \circ S_{\ell_{x_1}}(t).$$

Let us conjugate the action of the iterated monodromy group on T_t to an action on the tree X^* by the isomorphism Λ. Namely, we set, for every $\gamma \in \pi_1(\mathcal{M}_0, t)$ and $v \in X^*$:

$$\gamma(v) = \Lambda^{-1} \gamma \Lambda(v).$$

PROPOSITION 5.17. *Let $x \in X$, $v \in X^*$, and $\gamma \in \pi_1(\mathcal{M}_0, t)$. Let γ_x be the lift of γ by f starting in $\Lambda(x)$. Let $\Lambda(y)$ be the end of γ_x. Then we have*

$$\gamma(xv) = y(\ell_y^{-1} \iota(\gamma_x) \ell_x)(v).$$

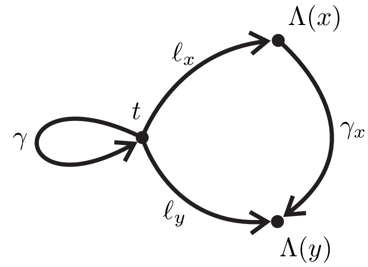

FIGURE 18. Computation of the iterated monodromy group

See Figure 18. Here and in the sequel we multiply paths as maps: in a product $\gamma_1\gamma_2$ the path γ_2 is passed before γ_1.

PROOF. Consider the composition $(\iota_*^{-1}S_{\ell_y})^{-1}S_\gamma(\iota_*^{-1}S_{\ell_x}) : T_t \longrightarrow T_t$. It is equal to the composition of $\iota_*^{-1}S_{\ell_x} : T_t \longrightarrow T_{\Lambda(x)}$ with the restriction of S_γ to an isomorphism $S'_\gamma : T_{\Lambda(x)} \longrightarrow T_{\Lambda(y)}$ and with the isomorphism $(\iota_*^{-1}S_{\ell_y})^{-1} : T_{\Lambda(y)} \longrightarrow T_t$. The restriction S'_γ is defined by taking lifts of the path $\gamma_x \subset \mathcal{M}_1$ by the coverings $f_1, f_2, \ldots$. Applying ι_*, we see that $\iota_* S'_\gamma \iota_*^{-1} : T_{\iota(\Lambda(x))} \longrightarrow T_{\iota(\Lambda(y))}$ is equal to $S_{\iota(\gamma_x)}$. Therefore,

$$(\iota_*^{-1}S_{\ell_y})^{-1}S_\gamma(\iota_*^{-1}S_{\ell_x}) = S_{\ell_y}^{-1}\iota_* S'_\gamma \iota_*^{-1}S_{\ell_x} = S_{\ell_y^{-1}}S_{\iota(\gamma_x)}S_{\ell_x} = S_{\ell_y^{-1}\iota(\gamma_x)\ell_x}.$$

Moving everything to X^* using Λ, we get that $S_y^{-1}\gamma S_x$ (where $S_x(v) = xv$) is equal to $(\ell_y^{-1}\iota(\gamma_x)\ell_x)$. In other words, appending x to the beginning of a word, acting by γ, and then erazing y is equivalent to acting by $(\ell_y^{-1}\iota(\gamma_x)\ell_x)$. □

5.8. Examples.

5.8.1. *Angle doubling.* Consider the angle doubling map $f : x \mapsto 2x \pmod 1$ on $\mathbb{R}/\mathbb{Z}$. Take $t = 0$ as the basepoint, $X = \{0, 1\}$. Then $f^{-1}(0) = \{0, 1/2\}$. Set $\Lambda(0) = 0$, $\Lambda(1) = 1/2$. Choose ℓ_0 to be the trivial (constant) path at 0, and ℓ_1 to be the image of the segment $[0, 1/2]$.

The group $\pi_1(\mathbb{R}/\mathbb{Z}, 0)$ is generated by the loop a equal to the image of the positively oriented loop $[0, 1] \subset \mathbb{R}/\mathbb{Z}$. Then lifts of a are the paths $\gamma_0 = [0, 1/2]$ and $\gamma_1 = [1/2, 1]$. We get

$$a(0w) = 1(\ell_1^{-1}\gamma_0\ell_0)(w) = 1w,$$

$$a(1w) = 0(\ell_0^{-1}\gamma_1\ell_1)(w) = 0a(w),$$

since $\ell_1^{-1}\gamma_0\ell_0 = [0, 1/2]^{-1}[0, 1/2]$ is trivial, and $\ell_0^{-1}\gamma_1\ell_1 = [1/2, 1][0, 1/2] = [0, 1] = a$. See Figure 19.

5.8.2. $z^2 - 1$. Consider the polynomial $z^2 - 1$. It has two critical points: ∞ and 0. Infinity is a fixed point, and the orbit of 0 is

$$0 \mapsto -1 \mapsto 0.$$

Consequently, $z^2 - 1$ is post-critically finite, and we can take $\mathcal{M}_0 = \mathbb{C} \setminus \{0, -1\}$, $\mathcal{M}_1 = f^{-1}(\mathcal{M}_0) = \mathbb{C} \setminus \{0, 1, -1\}$, and ι the identical inclusion.

Let us take t to be the fixed point $t = \frac{1-\sqrt{5}}{2}$. Then $f^{-1}(t) = \{t, -t\}$. The fundamental group $\pi_1(\mathcal{M}_0)$ is freely generated by loops around the punctures 0 and -1. Let us take the generators a and b shown on the top part of Figure 20.

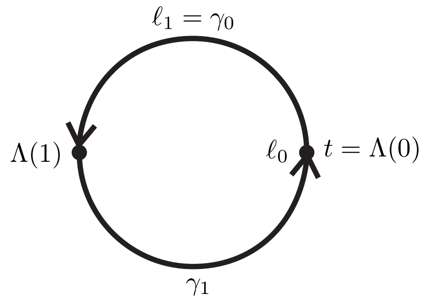

FIGURE 19. IMG $\left(z^2\right)$

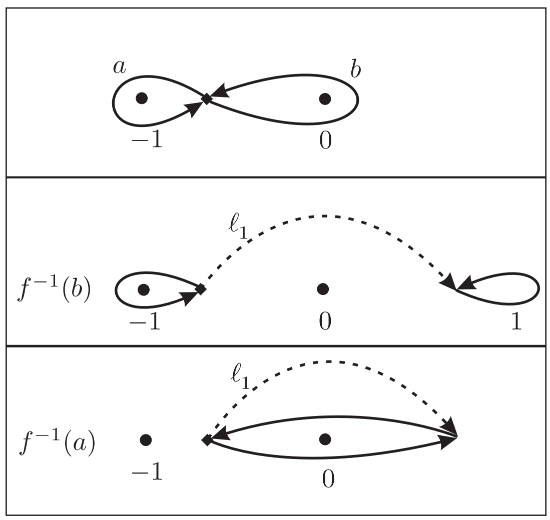

FIGURE 20. Computation of IMG $\left(z^2 - 1\right)$

Take $X = \{0, 1\}$, $\Lambda(0) = t$, $\Lambda(1) = -t$. Let ℓ_0 be the trivial path at t, and let ℓ_1 be the path from t to $-t$ shown on the two lower parts of Figure 20.

Lower parts of Figure 20 show the lifts of a and b by f. We get

$$a(0w) = 1b(w), \quad a(1w) = 0w$$

and

$$b(0w) = 0a(w), \quad b(1w) = 1w.$$

5.8.3. $-z^3/2+3z/2$. Let $f(z) = -\frac{z^3}{2} + \frac{3z}{2}$. Its critical points are (together with ∞) the solutions of the equation $-3z^2/2+3/2 = 0$, i.e., $z = \pm 1$. We have $f(1) = 1$ and $f(-1) = -1$. Hence all critical points are fixed, and f is post-critically finite. Its Julia set is shown on Figure 21.

Choose $t = 0$. Then $f^{-1}(0) = \{0, \sqrt{3}, -\sqrt{3}\}$. Take $X = \{0, 1, 2\}$, $\Lambda(0) = 0, \Lambda(1) = \sqrt{3}, \Lambda(2) = -\sqrt{3}$. Let ℓ_0 be the trivial path at 0, and let ℓ_1 and ℓ_2 connect 0 to $\sqrt{3}$ and $-\sqrt{3}$ as it is shown on the bottom part of Figure 22.

The fundamental group $\pi_1(\mathcal{M}_0, 0)$ for $\mathcal{M}_0 = \mathbb{C} \setminus \{1, -1\}$, is generated by loops a and b going around 1 and -1 in positive direction, as it is shown of the bottom

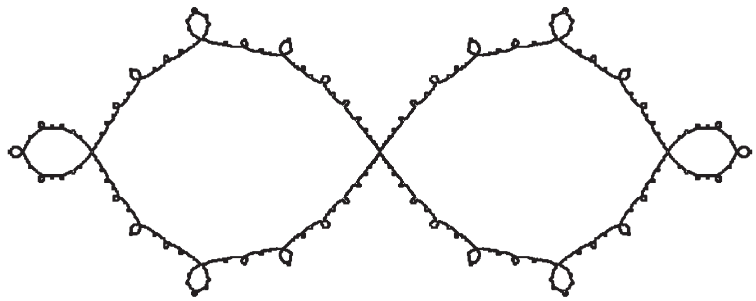

FIGURE 21. Julia set of $-z^3/2 + 3z/2$

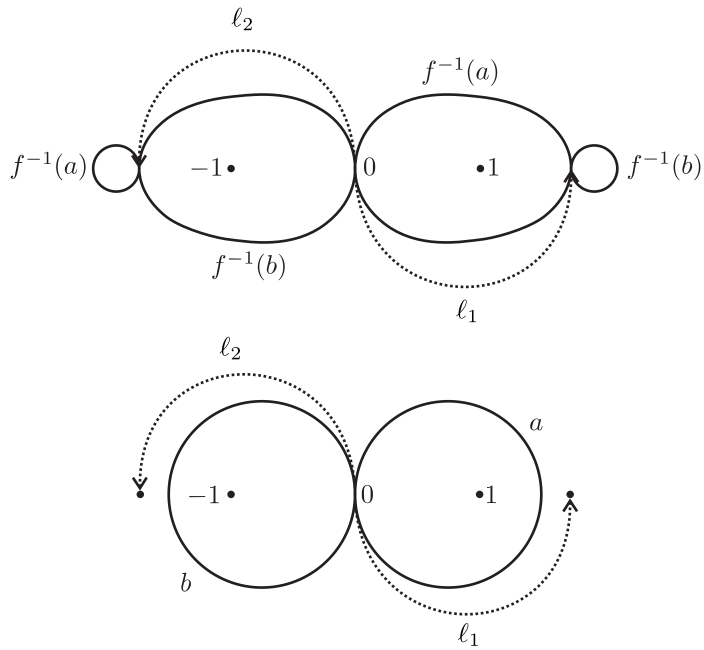

FIGURE 22. Computation of IMG $\left(-z^3/2 + 3z/2\right)$

part of Figure 22. The top part of the figure shows the lifts of a and b by f. We conclude that

$$a(0w) = 1w, \quad a(1w) = 0a(w), \quad a(2w) = 2w,$$

and

$$b(0w) = 2w, \quad b(1w) = 1w, \quad b(2w) = 0b(w).$$

PROBLEM 5.1. Compute iterated monodromy groups of the following post-critically finite complex polynomials and rational functions:

(1) $z^2 - 2$;
(2) $z^2 + i$;
(3) $z^2 + c$ for every c such that 0 belongs to a cycle of length 3: $0 \mapsto c \mapsto c^2 + c \mapsto 0$.
(4) $z^2 - \frac{16}{27z}$;

(5) $\left(\frac{2-z}{z}\right)^2$.

Problem 5.2. Describe the iterated monodromy groups of the Tchebyshev polynomials.