6. Self-similar groups

6.1. Definition.

DEFINITION 6.1. Let G be a group acting faithfully on the tree X^* (the right Cayley graph of the free monoid). We say that it is *self-similar* if for every $x \in X$ and $g \in G$ there are $y \in X$ and $h \in G$ such that

$$g(xw) = yh(w)$$

for all $w \in X^*$.

We have seen (in Proposition 5.17) that $\mathrm{IMG}(f, \iota)$ can be always realized as a self-similar group.

DEFINITION 6.2. The self-similar actions of $\mathrm{IMG}(f, \iota)$ defined in Proposition 5.17 are called *standard*.

Let us try to understand better self-similar groups as algebraic object.

It follows from the definition that for every $v \in X^*$ and $g \in G$ there exists an element $h \in G$ such that

$$g(vw) = g(v)h(w)$$

for all $w \in X^*$. We will denote h by $g|_v$, and call it *section* of g in v. The section is uniquely defined, since we assume that the action of G on X^* is faithful.

We have the following properties of the section, which follow directly from the definition:

$$g|_{v_1 v_2} = g|_{v_1}|_{v_2} \tag{5}$$

for all $g \in G$ and $v_1, v_2 \in X^*$, and

$$(g_1 g_2)|_v = g_1|_{g_2(v)} g_2|_v \tag{6}$$

for all $g_1, g_2 \in G$ and $v \in X^*$.

Every element $g \in G$ acts as a permutation on the first level $X \subset X^*$ of the tree. Denote by $\sigma_g \in \mathrm{Symm}(X)$ the corresponding permutation. Then $y = \sigma_g(x)$ in the conditions of Definition 6.1.

We have also the function $X \longrightarrow G : x \mapsto g|_x$, i.e., an element of the direct product G^X. We get hence a map from G to the set $\mathrm{Symm}(X) \times G^X$ mapping g to the pair (σ_g, f), where $f(x) = g|_x$. Consider the set $\mathrm{Symm}(X) \times G^X$ with the structure of the *semidirect product* $\mathrm{Symm}(X) \ltimes G^X$ of the group $\mathrm{Symm}(X)$ and the direct power G^X. The group structure is given by the multiplication rule

$$\sigma_1 f_1 \cdot \sigma_2 f_2 = \sigma_1 \sigma_2 f_1' f_2,$$

where $f_1' \in G^X$ is given by $f_1'(x) = f_1(\sigma_2(x))$.

Let us identify X with $\{1, 2, \ldots, d\}$ for $d = |X|$. Then elements of $\mathrm{Symm}(X) \ltimes G^X$ are written as sequences $\sigma(g_1, g_2, \ldots, g_d)$ for $\sigma \in \mathrm{Symm}(d)$ and $g_i \in G$. The multiplication rule is

$$\sigma_1(g_1, g_2, \ldots, g_d)\sigma_2(h_1, h_2, \ldots, h_d) = \sigma_1\sigma_2(g_{\sigma_2(1)}h_1, g_{\sigma_2(2)}h_2, \ldots, g_{\sigma_2(d)}h_d). \tag{7}$$

LEMMA 6.1. *Let G be a self-similar group. The map*

$$G \longrightarrow \mathrm{Symm}(X) \ltimes G^X : g \mapsto \sigma_g \cdot f,$$

where $f(x) = g|_x$, is a homomorphism of groups.

PROOF. It is easy to see that (7) agrees with (6). $\square$

The group $\mathrm{Symm}(X) \ltimes G^X$ is the *wreath product* $G \wr \mathrm{Symm}(X)$ of $\mathrm{Symm}(X)$ and G, according to the standard definition of (permutational) wreath products.

DEFINITION 6.3. We call the homomorphism from Lemma 6.1 the *wreath recursion* associated with the self-similar group.

Note that the wreath recursion is an injective homomorphism. The wreath recursion $G \longrightarrow \mathrm{Symm}(X) \ltimes G^X$ uniquely determines the action of G on X^*. The wreath recursion is a compact way of writing the recurrent definitions of action of elements of self-similar group on words.

EXAMPLE 6.1. We have seen that the generator a of the iterated monodromy group of the angle doubling map acts by the rule

$$a(0w) = 1w, \qquad a(1w) = 0a(w).$$

We write this definition in terms of the wreath recursion as

$$\psi(a) = \sigma(1, a),$$

where $\sigma = (01)$ is the transposition, and 1 in $(1, a)$ denotes the identity element of the group.

EXAMPLE 6.2. The group $\mathrm{IMG}\left(z^2 - 1\right)$, as computed in 5.8.2 is generated by

$$a = \sigma(b, 1), \qquad b = (a, 1).$$

Note that we usually omit the identity element of $\mathrm{Symm}(d)$ when writing elements of the wreath product.

The wreath recursion can be used to find relations between elements of $\mathrm{IMG}\left(z^2 - 1\right)$. For example, we have the following equalities:

$$\psi(a^{-1}ba) = (b^{-1}, 1)\sigma(a, 1)\sigma(b, 1) = (b^{-1}, 1)(1, a)(b, 1) = (1, a),$$

which implies

$$\psi([a^{-1}ba, b]) = ([1, a], [a, 1]) = (1, 1),$$

hence $[a^{-1}ba, b] = 1$, as the homomorphism ψ is injective.

PROPOSITION 6.2. *Let $\psi_1, \psi_2 : G \longrightarrow \mathrm{Symm}(d) \ltimes G^d$ be the wreath recursions on $G = \mathrm{IMG}(f, \iota)$ associated with two standard actions. Then there exists an element $h \in \mathrm{Symm}(d) \ltimes G^d$ such that $\psi_1(g) = h^{-1}\psi_2(g)h$ for all $g \in G$.*

PROOF. Denote $X = \{1, 2, \ldots, d\}$. Let $\Lambda : X \longrightarrow f^{-1}(t)$ and ℓ_x, and $\tilde{\Lambda} : X \longrightarrow f^{-1}(t)$ and $\tilde{\ell}_x$ be two bijections and connecting paths. Let ψ and $\tilde{\psi}$ be the corresponding wreath recursions.

Denote, for $z \in f^{-1}(t)$, by γ_z the lift of $\gamma \in \pi_1(\mathcal{M}_0, t)$ by f starting in z. By Proposition 5.17, the wreath recursions are given by

$$\psi(\gamma) = \sigma_\gamma(g_1, g_2, \ldots, g_d), \qquad \tilde{\psi}(\gamma) = \tilde{\sigma}_\gamma(\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_d),$$

where $\sigma_\gamma(x)$ is the end of $\gamma_{\Lambda(x)}$, $\tilde{\sigma}_\gamma(x)$ is the end of $\gamma_{\tilde{\Lambda}(x)}$, and

$$g_x = \ell_{\sigma_\gamma(x)}^{-1}\gamma_{\Lambda(x)}\ell_x, \qquad \tilde{g}_x = \tilde{\ell}_{\tilde{\sigma}_\gamma(x)}^{-1}\gamma_{\tilde{\Lambda}(x)}\tilde{\ell}_x.$$

Let $\rho : X \longrightarrow X$ be the permutation $\rho = \tilde{\Lambda}^{-1} \circ \Lambda$. Then $\tilde{\Lambda}(\rho(x)) = \Lambda(x)$ for all $x \in X$.

Choose $x \in X$, and denote $z = \Lambda(x) = \tilde{\Lambda}(\rho(x))$, $\tilde{x} = \rho(x)$. Then $\Lambda(\sigma_\gamma(x)) = \tilde{\Lambda}(\tilde{\sigma}_\gamma(\rho(x)))$ is the end of γ_z. It follows that $\rho\sigma_\gamma = \tilde{\sigma}_\gamma\rho$, i.e., that $\sigma_\gamma = \rho^{-1}\tilde{\sigma}_\gamma\rho$. We also have

$$g_x = \ell_{\sigma_\gamma(x)}^{-1}\gamma_z\ell_x = \ell_{\sigma_\gamma(x)}^{-1}\tilde{\ell}_{\tilde{\sigma}_\gamma(\tilde{x})} \cdot \tilde{\ell}_{\tilde{\sigma}_\gamma(\tilde{x})}^{-1}\gamma_z\tilde{\ell}_{\tilde{x}} \cdot \tilde{\ell}_{\tilde{x}}^{-1}\ell_x = \ell_{\sigma_\gamma(x)}^{-1}\tilde{\ell}_{\tilde{\sigma}_\gamma(\tilde{x})} \cdot \tilde{g}_{\tilde{x}} \cdot \tilde{\ell}_{\tilde{x}}^{-1}\ell_x. \tag{8}$$

Note that since $\Lambda(x) = \tilde{\Lambda}(\tilde{x}) = z$ and $\Lambda(\sigma_\gamma(x)) = \tilde{\Lambda}(\tilde{\sigma}_\gamma(\tilde{x}))$ is the end of γ_z, the paths $\tilde{\ell}_{\tilde{x}}^{-1}\ell_x$ and $\ell_{\sigma_\gamma(x)}^{-1}\tilde{\ell}_{\tilde{\sigma}_\gamma(\tilde{x})}$ are elements of $\pi_1(\mathcal{M}_0, t)$. Denote for all $x \in X$:

$$h_x = \tilde{\ell}_{\rho(x)}^{-1}\ell_x.$$

Then $\ell_{\sigma_\gamma(x)}^{-1}\tilde{\ell}_{\tilde{\sigma}_\gamma(\tilde{x})} = \left(\tilde{\ell}_{\tilde{\sigma}_\gamma(\tilde{x})}^{-1}\ell_{\sigma_\gamma(x)}\right)^{-1} = \left(\tilde{\ell}_{\rho(\sigma_\gamma(x))}^{-1}\ell_{\sigma_\gamma(x)}\right)^{-1} = h_{\sigma_\gamma(x)}^{-1}$, and (8) becomes

$$g_x = h_{\sigma_\gamma(x)}^{-1}\tilde{g}_{\rho(x)}h_x,$$

which implies (using $\sigma_\gamma = \rho^{-1}\tilde{\sigma}_\gamma\rho$)

$$\sigma_\gamma(g_1, g_2, \dots, g_d) = (\rho(h_1, h_2, \dots, h_d))^{-1}\tilde{\sigma}_\gamma(\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_d)\rho(h_1, h_2, \dots, h_d).$$

□

PROPOSITION 6.3. *Let* $\psi : G \longrightarrow \mathrm{Symm}(d) \ltimes G^d$ *be a wreath recursion. Let* $h \in \mathrm{Symm}(d) \ltimes G^d$*, and consider the wreath recursion* $\tilde{\psi}$ *equal to composition of* ψ *with the inner automorphism defined by* h*:*

$$\tilde{\psi}(g) = h^{-1}\psi(g)h.$$

Then the actions of G *on* X^* *defined by* ψ *and* $\tilde{\psi}$ *are conjugate.*

PROOF. Let $h = \pi(h_1, h_2, \dots, h_d)$. Define an automorphism α of the tree X^* by the recursive formula

$$\alpha(xw) = \pi(x)h_x \circ \alpha(w)$$

for all $x \in X$ and $w \in X^*$.

Then

$$\alpha^{-1}(yv) = \pi^{-1}(y)\alpha^{-1}h_{\pi^{-1}(y)}^{-1}(v)$$

for all $y \in X$ and $v \in X^*$. (Just apply the definition of α for $y = \pi(x)$ and $v = h_x\alpha(w)$.) $u = h_x(v)$

Suppose that $\psi(g) = \sigma(g_1, g_2, \dots, g_d)$. Then

$$g\alpha(xw) = g(\pi(x)h_x\alpha(w)) = \sigma\pi(x)g|_{\pi(x)}h_x\alpha(w),$$

hence

$$\alpha^{-1}g\alpha(xw) = \pi^{-1}\sigma\pi(x)\alpha^{-1}h_{\pi^{-1}\sigma\pi(x)}^{-1}g_{\pi(x)}h_x\alpha(w).$$

It follows that the elements of $\alpha^{-1}G\alpha$ satisfy the wreath recursion

$$\begin{aligned}\alpha^{-1}g\alpha \mapsto \\ &\pi^{-1}\sigma\pi(\alpha^{-1}h_{\pi^{-1}\sigma\pi(1)}g_{\pi(1)}h_1\alpha, \dots, \alpha^{-1}h_{\pi^{-1}\sigma\pi(d)}g_{\pi(d)}h_d\alpha) = \\ &(\alpha, \dots, \alpha)^{-1}\left(\pi(h_1, \dots, h_d))^{-1}\sigma(g_1, g_2, \dots, g_d)\pi(h_1, \dots, h_d)\right)(\alpha, \dots, \alpha),\end{aligned}$$

hence the action of G defined by the wreath recursion $g \mapsto h^{-1}\psi(g)h$ coincides with the action of $\alpha^{-1}G\alpha$. □

DEFINITION 6.4. We say that two self-similar groups G_1, G_2 acting on X_1^* and X_2^* are *equivalent* if there exists an isomorphism $\phi : G_1 \longrightarrow G_2$ and a bijection $F : X_1 \longrightarrow X_2$ such that if $\psi_i : G_i \longrightarrow \mathrm{Symm}\,(X_i) \ltimes G_i^{X_i}$ are the wreath recursions associated with the self-similar groups, then there exists an element $h \in \mathrm{Symm}\,(X_2) \ltimes G_2^{X_2}$ such that $h^{-1} \cdot \psi_2(\phi(g)) \cdot h = \tilde{\phi}(\psi_1(g))$ for all $g \in G_1$, where $\tilde{\phi} : \mathrm{Symm}\,(X_1) \ltimes G_1^{X_1} \longrightarrow \mathrm{Symm}\,(X_2) \ltimes G_2^{X_2}$ is the natural isomorphism induced by the bijection $F : X_1 \longrightarrow X_2$ and the isomorphism $\phi : G_1 \longrightarrow G_2$.

For example, standard actions of the same iterated monodromy group are pairwise equivalent.

EXAMPLE 6.3. Self-similar actions of $\mathbb{Z}$ equivalent to the action generated by the binary adding machine $\psi(a) = \sigma(1, a)$ come from binary numeration actions with non-standard sets of digits. For example, let us conjugate $\psi(a) = \sigma(1, a)$ by $(1, a)$:

$$\psi'(a) = (1, a^{-1})\sigma(1, a)(1, a) = \sigma(a^{-1}, a^2).$$

It describes adding 1 to binary integers

$$n = a_0 + 2a_1 + 2^2 a_2 + 2^3 a_3 + \cdots$$

where $a_i \in \{0, 3\}$. Namely, if n is even, then $n = 0 + 2a_1 + 2^2a_2 + 2^3a_3 + \cdots$, and

$$n + 1 = 3 + 2(-1 + a_1 + 2a_2 + 2^2 a_3 + \cdots),$$

hence we carry -1 when we add 1. If n is odd, then $n = 3 + 2a_1 + 2^2a_2 + 2^3a_3 + \cdots$, so that

$$n + 1 = 0 + 2(2 + a_1 + 2a_2 + 2^2 a_3 + \cdots),$$

hence we carry 2 when we add 1. This agrees with the recursion $a = \sigma(a^{-1}, a^2)$.

6.2. Homotopy invariance of iterated monodromy groups.

THEOREM 6.4. *Let $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ and $f', \iota' : \mathcal{M}_1' \longrightarrow \mathcal{M}_0'$ be topological correspondences such that $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_0'$, and $\mathcal{M}_1'$ are locally path connected and path connected. Let $\phi_1 : \mathcal{M}_1' \longrightarrow \mathcal{M}_1$ and $\phi_0 : \mathcal{M}_0' \longrightarrow \mathcal{M}_0$ be continuous maps such that the diagrams*

$$\begin{array}{ccc} \mathcal{M}_1' & \xrightarrow{\phi_1} & \mathcal{M}_1 \\ \downarrow f' & & \downarrow f \\ \mathcal{M}_0' & \xrightarrow{\phi_0} & \mathcal{M}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{M}_1' & \xrightarrow{\phi_1} & \mathcal{M}_1 \\ \downarrow \iota' & & \downarrow \iota \\ \mathcal{M}_0' & \xrightarrow{\phi_0} & \mathcal{M}_0 \end{array}$$

are commutative up to homotopy (i.e., the corresponding compositions are homotopic rather than equal). Suppose that $\phi_{0,*} : \pi_1(\mathcal{M}_0') \longrightarrow \pi_1(\mathcal{M}_0)$ is surjective and $\deg f = \deg f'$. *Then the homomorphism $\phi_{0,*}$ induces an isomorphism of the groups* $\mathrm{IMG}\,(f', \iota')$ *and* $\mathrm{IMG}\,(f, \iota)$ *implementing an equivalence of self-similar groups.*

In the general case (when we do not assume that $\deg f = \deg f'$ and that $\phi_{0,*}$ is onto) the maps ϕ_1 and $\phi_{0,*}$ will induce a map between the alphabets and a homomorphism of groups that agree with the corresponding wreath recursions, i.e., they induce a *morphism of self-similar groups*. But in general this morphism will be neither injective nor surjective neither on the alphabets nor on the groups.

PROOF. Let us show at first that we can always assume that the first diagram is commutative (not just up to homotopy).

LEMMA 6.5. *There exists a continuous map $\tilde{\phi}_1 : \mathcal{M}'_1 \longrightarrow \mathcal{M}_1$ homotopic to ϕ_1 such that the diagram*

$$\begin{array}{ccc} \mathcal{M}'_1 & \xrightarrow{\tilde{\phi}_1} & \mathcal{M}_1 \\ \Big\downarrow f' & & \Big\downarrow f \\ \mathcal{M}'_0 & \xrightarrow{\phi_0} & \mathcal{M}_0 \end{array}$$

is commutative.

PROOF. There exists a homotopy $H : \mathcal{M}'_1 \times [0,1] \longrightarrow \mathcal{M}_0$ such that $H(x,0) = f \circ \phi_1(x)$ and $H(x,1) = \phi_0 \circ f'(x)$ for all $x \in \mathcal{M}'_1$. Then by the Homotopy Lifting Theorem, we can lift the homotopy H by the covering $f : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ to get a homotopy $\tilde{H} : \mathcal{M}'_1 \times [0,1] \longrightarrow \mathcal{M}_1$ satisfying

$$\tilde{H}(x,0) = \phi_1(x), \qquad f(\tilde{H}(x,1)) = \phi_0 \circ f'(x).$$

Then $\tilde{\phi}_1(x) = \tilde{H}(x,1)$ satisfies the conditions of the lemma. □

We will assume therefore, that the first diagram in Theorem 6.4 is commutative.

Let X be such that $|X| = \deg f = \deg f'$. Choose $t' \in \mathcal{M}'_0$, and a bijection $\Lambda' : X \longrightarrow (f')^{-1}(t')$. Choose a collection of paths ℓ'_x from t to $\iota'(\Lambda'(x))$. Consider the corresponding standard action of IMG (f', ι') on X^*.

Let $t = \iota_0(t')$. Since the first diagram is commutative, we have $\phi_1((f')^{-1}(t')) \subset f^{-1}(t)$. Since we assume that the spaces $\mathcal{M}'_1$ and $\mathcal{M}_1$ are path-connected, the natural actions of $\pi_1(\mathcal{M}'_0, t')$ and $\pi_1(\mathcal{M}_0, t)$ on $(f')^{-1}(t')$ and $f^{-1}(t)$ are transitive. Suppose that $\phi_1 : (f')^{-1}(t') \longrightarrow f^{-1}(t)$ is not onto. Then there exists an element $\gamma \in \pi_1(\mathcal{M}_0, t)$ and a lift δ of γ by f such that δ starts in $\phi_1((f')^{-1}(t'))$ and ends outside of it. But then γ can not belong to $\phi_{0,*}(\pi_1(\mathcal{M}'_0, t'))$, which is a contradiction.

It follows that $\phi_1 : (f')^{-1}(t') \longrightarrow f^{-1}(t)$ is a bijection, since we assume that $\deg f = \deg f'$. Let $\Lambda = \phi_1 \circ \Lambda'$.

Let $H : \mathcal{M}'_1 \times [0,1] \longrightarrow \mathcal{M}_0$ be the homotopy such that $H(x,0) = \phi_0 \circ \iota'(x)$ and $H(x,1) = \iota \circ \phi_1(x)$ for all $x \in \mathcal{M}'_1$. For $x \in X$, consider the points $\iota'(\Lambda'(x))$ and $\iota(\Lambda(x))$. We have $\iota \circ \Lambda(x) = \iota \circ \phi_1 \circ \Lambda'(x) = H(\Lambda'(x), 1)$. Hence, $\delta_x(t) = H(\Lambda'(x), t) : [0,1] \longrightarrow \mathcal{M}_0$ is a path from $\phi_0(\iota'(\Lambda(x)))$ to $\iota(\Lambda(x))$. Set $\ell_x = \delta_x \phi_0(\ell'_x)$, i.e., continue the path $\phi_0(\ell'_x)$ by δ. Consider the standard action of IMG (f, ι) constructed using the chosen Λ, ℓ_x, t.

Let us show that the defined standard actions of the iterated monodromy groups coincide. Let $\gamma' \in \pi_1(\mathcal{M}'_0, t')$, and let $\gamma = \phi_0(\gamma')$ be the corresponding element of $\pi_1(\mathcal{M}_0, t)$. Let $x \in X$, and let γ'_x be the lift of γ by f' starting at $\Lambda'(x)$. Let $y \in X$ be such that $\Lambda'(y)$ is the end of γ'_x. By commutativity of the diagram with f and f', we get that the path $\phi_1(\gamma'_x)$ starts at $\Lambda(x)$, ends in $\Lambda(y)$, and is a lift of γ by f. Let us denote $\gamma_x = \phi_1(\gamma'_x)$. Then $H(\cdot, t) \circ \gamma$ is a homotopy from $\phi_0(\iota'(\gamma'_x))$ to $\iota \circ \phi_1(\gamma'_x) = \iota(\gamma_x)$. It follows that the loops $\phi_0((\ell'_y)^{-1}\iota'(\gamma'_x)\ell'_x)$ and $\ell_y^{-1}\iota(\gamma_x)\ell_x$ are homotopic.

Consequently, the action of the elements γ' on X^* coincides with the action of the elements $\phi_0(\gamma')$. Since $\phi_{0,*}$ is surjective, this implies that the set of automorphisms of the tree X^* defined by the elements of $\pi_1(\mathcal{M}'_0)$ coincides with the set of automorphisms defined by the elements of $\pi_1(\mathcal{M}_0)$. □

As an example of application of Theorem 6.4, consider the following interpretation of standard actions of IMG (f, ι). Let $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ be a topological correspondence such that $\mathcal{M}_0, \mathcal{M}_1$ are path connected and locally path connected. Let S be a generating set of $\pi_1(\mathcal{M}_0, t)$. Consider a bouquet of circles Γ_0, where the set of circles is in a bijection with the elements of S. Let $\phi_0 : \Gamma_0 \longrightarrow \mathcal{M}_0$ be the natural map such that the image of a circle of Γ_0 is the corresponding element of the generating set S, where the common point of the circles is mapped to t. Lift ϕ_0 by the covering $f : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$, i.e., close the pull-back diagram

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\phi_1} & \mathcal{M}_1 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \Gamma_0 & \xrightarrow{\phi_0} & \mathcal{M}_0 \end{array}$$

The graph Γ_1 is the lift of the graph $\phi_0(\Gamma_0)$ by f. In particular, its vertices are in a bijection (by ϕ_1) with the points of $f^{-1}(t)$, and its edges correspond to lifts of the generators $\gamma \in S$ by f.

See, for example, Figure... where a graph Γ_0 for $f(z) = z^2 - 1$ is shown...

Consider now the map $\iota : \phi_1(\Gamma_1) \longrightarrow \mathcal{M}_0$. Since $\pi_1(\mathcal{M}_0, t)$ is generated by S, the map $\iota : \phi_1(\Gamma_1) \longrightarrow \mathcal{M}_0$ is homotopic to a map defined on graphs $\Gamma_1 \longrightarrow \Gamma_0$, i.e., there exists a cellular map $\iota' : \Gamma_1 \longrightarrow \Gamma_0$ such that the diagram

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\phi_1} & \mathcal{M}_1 \\ \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota} \\ \Gamma_0 & \xrightarrow{\phi_0} & \Gamma_1. \end{array}$$

Namely, choose connecting paths ℓ_z from t to $\iota(z)$ for each $z \in f^{-1}(t)$. For every edge e of Γ_1 connect the images of the endpoints of e under $\iota \circ \phi_1$ with t by the paths e_z. The obtained loop is an element of $\pi_1(\mathcal{M}_0, t)$, hence is homotopic to the image by ϕ_0 of a path in Γ_0. Choose such a path, and map e to it by ι'. It is easy to see that defined $\iota' : \Gamma_1 \longrightarrow \Gamma_0$ will satisfy our conditions.

For example ...

Problem 6.1. Let $\mathcal{M}$ be a compact metric space. Let $\mathcal{U}$ be a cover of $\mathcal{M}$ by open sets, and let Γ be the (geometric realization of the) nerve of the covering. Show that there exists a continuous map $\phi : \mathcal{M} \longrightarrow \Gamma$ such that if A is the set of all elements of $\mathcal{U}$ containing $x \in \mathcal{M}$, then $\phi(x)$ is contained in the simplex A. Hint: use a partition of unity subordinate to $\mathcal{U}$.

Problem 6.2. Let $f : \mathcal{M} \longrightarrow \mathcal{M}$ be an expanding covering map. Let $\mathcal{U}$ be a semi-markovian cover of $\mathcal{M}$ by small open sets. Let $\mathcal{U}_1$ be the set of components of $f^{-1}(U)$ for $U \in \mathcal{U}$, and let Γ_0, Γ_1 be the nerves of $\mathcal{U}$ and $\mathcal{U}_1$, respectively. Let $f_0, \iota_0 : \Gamma_1 \longrightarrow \Gamma_0$ be the topological correspondence as constructed in 5.3. Show that there exist maps $\phi_0 : \mathcal{M} \longrightarrow \Gamma_0$ and $\phi_1 : \mathcal{M} \longrightarrow \Gamma_1$ such that the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi_1} & \Gamma_1 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} \\ \mathcal{M} & \xrightarrow{\phi_0} & \Gamma_0 \end{array}$$

and $\iota_0 \circ \phi_1 : \mathcal{M} \longrightarrow \Gamma_0$ is homotopic to $\phi_0 : \mathcal{M} \longrightarrow \Gamma_0$.

6.3. Contracting self-similar groups.

Definition 6.5. Let G be a self-similar group acting on X^*. It is said to be *contracting* if there exists a finite subset $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all $v \in X^*$ such that $|v| \geq n$.

Note that if the action is contracting, then for every $g \in G$ the set $\{g|_v \ : \ v \in X^*\}$ of all sections of g is finite.

Proposition 6.6. *If the group G is contracting, then the set*

$$\mathcal{N} = \cup_{g \in G} \cap_{n \geq 1} \{g|_v \ : \ v \in X^*, |v| \geq n\}$$

is the smallest set satisfying the conditions of Definition 6.5.

We call the smallest set $\mathcal{N}$ satisfying the conditions of Definition 6.5 the *nucleus* of the group.

If G is a contracting finitely generated group, then it has a finite generating set S such that $S = S^{-1}$, and for every $g \in G$ and $v \in X^*$ we have $g|_v \in S$ (just add all sections $g|_v$ of all elements of any symmetric finite generating set of G). Note that then S contains the nucleus.

Proposition 6.7. *Let G be a self-similar group acting on X^*. Suppose that G is finitely generated, let $l(g)$ denote the length of $g \in G$ with respect to a fixed generating set, and let*

$$\rho = \limsup_{n \to \infty} \sqrt[n]{\limsup_{l(g) \to \infty} \max_{v \in X^n} \frac{l(g|_v)}{l(g)}}.$$

Then ρ does not depend on the choice of the generating set, and the action is contracting if and only if $\rho < 1$.

We call the number ρ the *contraction coefficient* of G.

Proof. Let $S = S^{-1}$ be a finite generating set such that $g|_v \in S$ for every $g \in S$ and $v \in X^*$. There exists n_1 such that $(g_1 g_2)|_v \in \mathcal{N} \subset S$ for all $g_1, g_2 \in S$ and $v \in X^*$, $|v| \geq n_1$. Then for every product $g_1 g_2 \ldots g_{2n}$ of length $2n$ of elements of S, and for every $v \in X^*$, $|v| \geq n_1$, the section

$$(g_1 g_2 \ldots g_{2n})|_v = (g_1 g_2)|_{g_3 g_4 \ldots g_{2n}(v)} (g_3 g_4)|_{g_5 g_6 \ldots g_{2n}(v)} \cdots (g_1 g_2)|_v$$

is a product of at most n elements of S. Similarly, for every product $g_1 g_2 \ldots g_{2n+1}$ the section $(g_1 g_2 \ldots g_{2n+1})|_v$ is a product of length at most $n+1$ for every word $v \in X^*$ of length at least n_1. It follows that

$$l(g|_v) \leq \frac{l(g) + 1}{2}$$

for all $g \in G$ and $v \in X^*$ of length at least n_1. This implies

$$\rho \leq \sqrt[n_1]{1/2}.$$

Conversely, suppose that $\rho < 1$. Let ρ_1 be such that $\rho < \rho_1 < 1$. Then there exists n_1 such that $\sqrt[n]{\limsup_{l(g) \to \infty} \max_{v \in X^n} \frac{l(g|_v)}{l(g)}} < \rho_1$ for all $n \geq n_1$. Hence, there exists m_1 such that $\frac{l(g|_v)}{l(g)} < \rho_1^{n_1}$ for all $v \in X^{n_1}$ and all $g \in G$ such that $l(g) \geq m_1$. It follows that $l(g|_v) < \rho_1^{n_1} l(g) + m_1$ for all $g \in G$ and all $v \in X^{n_1}$.

Let $g \in G$ and $v \in X^*$ be arbitrary. Write $v = v_1 v_2 \dots v_k u$, where $|u| < n_1$ and $|v_i| = n_1$ for all i. Then

$$l(g|_{v_1 v_2 \dots v_k}) < \rho_1^{n_1} l(g|_{v_1 v_2 \dots v_{k-1}}) + m_1 < \rho_1^{2n_1} l(g|_{v_1 v_2 \dots v_{k-2}}) + \rho_1^{n_1} m_1 + m_1 < \dots < \rho_1^{kn_1} l(g) + \rho_1^{(k-1)n_1} m_1 + \dots + \rho_1^{n-1} m_1 + m_1.$$

If k is big enough, then $\rho_1^{kn_1} l(g) < 1$, and $l(g|_{v_1 v_2 \dots v_k})$ is less than $1 + \frac{m_1}{1-\rho_1^{n_1}}$. It follows that the set of sections in words of length $< n_1$ of elements of G of length at most $1 + \frac{m_1}{1-\rho_1^{n_1}}$ satisfies the conditions of Definition 6.5. □

From now on we will write recursive definitions of elements of self-similar groups just $g = \sigma(g_1, g_2, \dots, g_d)$, instead of $\psi(g) = \sigma(g_1, g_2, \dots, g_d)$.

EXAMPLE 6.4. For the adding machine $a = \sigma(1, a)$ we have $a^2 = (a, a)$, hence $a^{2n} = (a^n, a^n)$, and $a^{2n+1} = \sigma(a^n, a^{n+1})$. It follows that $\rho = 1/2$.

EXAMPLE 6.5. Consider the group generated by $a = \sigma(1, a)$ and $b = (a, b)$. Then a and b have infinite order (a is the adding machine, hence has infinite order; and $b^n = (a^n, b^n) \neq (1, 1)$ for all n, hence b also has infinite order). It follows that the group $G = \langle a, b\rangle$ is not contracting, since $b^n|_{\underbrace{11\dots1}_{k \text{ times}}} = b^n$ for all $n, k \in \mathbb{N}$.

PROBLEM 6.3. Let $G = \langle a, b\rangle$ be as in Example 6.5. Describe the components of the Schreier graph of the action of G on $X^{\mathbb{N}}$.

PROBLEM 6.4. Let G be as in Example 6.5. Show that for every $g \in G$ there exists n such that $g|_v \in \mathcal{N}$ for every $v \in X^*$ of length $|v| \geq n$, where

$$\mathcal{N} = \{a^n, b^n, ab^n, b^n a^{-1}, ab^n a^{-1} \ : \ n \in \mathbb{Z}\}.$$

PROPOSITION 6.8. *Suppose that the action of G on X^* is contracting with contraction coefficient ρ. Then the growth $\gamma(r) = B_w(r)$ of the Schreier graphs of the action of G on the boundary $X^{\mathbb{N}}$ of X^* satisfies*

$$\limsup_{r\to\infty} \frac{\log \gamma(r)}{\log r} \leq \frac{\log |X|}{-\log \rho}.$$

PROOF. Let $S = S^{-1}$ be a generating set such that $g|_v \in S$ for all $g \in S$ and $v \in X^*$. As in the proof of Proposition 6.7, let ρ_1 be such that $1 > \rho_1 > \rho$. Let n_1 and m_1 be such that $l(g|_v) < \rho_1^{|v|} l(g)$ for all $v \in X^{n_1}$ and $g \in G$ such that $l(g) \geq m_1$.

Let $w = x_1 x_2 \dots \in X^{\mathbb{N}}$, and consider the ball $B_w(r)$ in the Schreier graph of the action of G on $X^{\mathbb{N}}$. It consists of all elements of the form $g(w)$ for $g \in G$ such that $l(g) \leq r$.

Consider the map

$$y_1 y_2 \dots \mapsto (y_1 y_2 \dots y_{n_1}, y_{n_1+1} y_{n_1+2} \dots)$$

from $X^{\mathbb{N}}$ to $X^{n_1} \times X^{\mathbb{N}}$. It is obviously a bijection. It maps every ball $B_{x_1 x_2 \dots}(r)$ injectively into the direct product of X^{n_1} with the ball $B_{x_{n_1+1} x_{n_1+2} \dots}(\rho_1^{n_1} r + m_1)$. It follows that the growth function $\gamma_w(r)$ satisfies

$$\gamma_w(r) \leq |X|^{n_1} \gamma_{w'}(\rho_1^{n_1} r + m_1),$$

where w' is the shift of w by n_1 positions.

Let $k = \left\lceil \frac{\log r}{-n_1 \log \rho_1} \right\rceil$. Then $\rho_1^{kn_1} r \le 1$, and we get (for $d = |X|$)

$$\gamma_w(r) \le d^{kn_1} \gamma_{w'} (1+\rho_1^{(k-1)n_1} m_1 + \rho_1^{(k-2)n_1} m_1 + \cdots \rho_1^{n_1} m_1 + m_1) \le d^{kn_1} \gamma_{w'} \left(1 + m_1 \frac{1}{1-\rho_1^{n_1}}\right),$$

where w' is some shift of w. There is a uniform constant C_1 (not depending on w') bounding from above $\gamma_{w'} \left(1 + m_1 \frac{1}{1-\rho_1^{n_1}}\right)$. We get

$$\gamma_w(r) \le C_1 d^{(\frac{\log r}{-n_1 \log \rho_1}+1)n_1} = (C_1 d^{n_1}) r^{\frac{\log d}{-\log \rho_1}}.$$

It follows that for every $\epsilon > 0$ there exists $C > 0$ such that

$$\gamma_w(r) \le C r^{\frac{\log d}{-\log \rho} + \epsilon}$$

for all $w \in X^{\mathbb{N}}$ and $r > 0$. □

COROLLARY 6.9. *The contraction coefficient of an infinite contracting group satisfies*

$$\rho \ge 1/|X|.$$

PROOF. Let G be a contracting group acting on X^*. One can show that either the size of orbits of the action of G on X^* are bounded and then the group G is finite, or there exists an infinite G-orbit on the boundary $X^{\mathbb{N}}$ of the tree X^*.

But if orbit of $w \in X^{\mathbb{N}}$ is infinite, then $|B_w(r)| \ge r$, hence $1 \le \limsup_{r\to\infty} \frac{\log |B_w(r)|}{\log r} \le \frac{\log |X|}{-\log \rho} \le \frac{\log |X|}{-\log \rho}$, which implies that $\rho \ge 1/|X|$. □

Not much is known about contracting self-similar groups. Let us list some of the properties without proofs, and formulate some open problems.

THEOREM 6.10. *Let G be a contracting finitely generated self-similar group acting on X^*. If ρ is its contraction coefficient, then for every $\epsilon > 0$ there exists an algorithm solving the word problem in G of polynomial complexity of degree $\le \frac{\log |X|}{-\log \rho} + \epsilon$.*

Contracting groups are usually infinitely presented, except for some virtually nilpotent examples. It is an open question if a finitely presented contracting group is virtually nilpotent.

THEOREM 6.11. *A contracting group has no non-abelian free subgroups.*

It is not known if all contracting groups are amenable.

6.4. Contracting topological correspondences.

DEFINITION 6.6. Let $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ be a topological correspondence, where $\mathcal{M}_1, \mathcal{M}_0$ are compact metrizable spaces. We say that it is *contracting* if there exist metrics d_1 and d_0 on $\mathcal{M}_1$, $\mathcal{M}_0$, respectively, and $\epsilon > 0$, $\rho < 1$, such that

$$d_0(f(x), f(y)) = d_1(x, y), \qquad d_0(\iota(x), \iota(y)) \le \rho d_1(x, y),$$

for all $x, y \in \mathcal{M}_1$ such that $d_1(x, y) < \epsilon$.

EXAMPLE 6.6. Let f be a hyperbolic complex rational function. Then the closure P of the post-critical set of f is disjoint from the Julia set, and the identical embedding $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$, for $\mathcal{M}_0 = \widehat{\mathbb{C}} \setminus P$ and $\mathcal{M}_1 = f^{-1}(\mathcal{M}_0)$, is expanding with respect to the Poincaré metrics on the corresponding domains. The function f is a local isometry. Expansion is non-uniform, but restricting the Poincaré metric to compact neighborhoods of the Julia set, we will get a contracting topological correspondence.

PROPOSITION 6.12. *Let $f : \mathcal{M} \longrightarrow \mathcal{M}$ be an expanding covering map, and let $\iota : \mathcal{M} \longrightarrow \mathcal{M}$ be the identity map. Then the correspondence $f, \iota : \mathcal{M} \longrightarrow \mathcal{M}$ is contracting.*

PROOF. Let d be a metric on $\mathcal{M}$, and $\epsilon > 0$, $L > 1$, be such that $d(f(x), f(y)) \geq Ld(x, y)$ for all $x, y \in \mathcal{M}$ such that $d(x, y) < \epsilon$. Suppose that $\delta > 0$ satisfies the conditions of Lemma 5.1. Let $\mathcal{U}$ be an open cover of $\mathcal{M}$ by sets of diameter less than δ, and let $\mathcal{U}_1$ be the set of components of $f^{-1}(U)$ for $U \in \mathcal{U}$.

Let $d_0 = d$, and let $d_1(x, y)$ be equal to the infimum of values of $d(f(x_1), f(x_2))+ d(f(x_3), f(x_4)) + \cdots + d(f(x_{n-1}), f(x_n))$ over all sequences $x_1, x_2, \ldots, x_n$, where $x_1 = x$, $x_n = y$, and $d(x_i, x_{i+1}) < \epsilon$ for every i. The function d_1 is obviously symmetric and it satisfies the triangle inequality.

We have

$$d(f(x_1), f(x_2)) + d(f(x_3), f(x_4)) + \cdots + d(f(x_{n-1}), f(x_n)) \geq \\ L(d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)) \geq Ld(x_1, x_n),$$

which implies

$$d_1(x, y) \geq Ld(x, y)$$

for all $x, y \in \mathcal{M}$, which implies that d_1 is a metric, and the identity map $\iota : (\mathcal{M}, d_1) \longrightarrow (\mathcal{M}, d)$ is expanding.

We have $d_1(x, y) \geq d(f(x), f(y))$ for all $x, y \in \mathcal{X}$. If $d(x, y) < \epsilon$, then $x_1 = x$ and $x_2 = y$ is a sequence satisfying the conditions of the definition of d_1, hence we have $d_1(x, y) \leq d(f(x), f(y))$, hence $d_1(x, y) = d(f(x), f(y))$ for all $x, y \in \mathcal{M}$ such that $d(x, y) < \epsilon$. It implies that f is a local isometry, and that d_1 and d define the same topology on $\mathcal{M}$. □

Let $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ be a contracting topological correspondence. Let d_0, d_1, ϵ, and ρ be as in the definition of a contracting correspondence. Let δ be a number in the interval $(0, \epsilon)$ such that for every subset $U \subset \mathcal{M}_0$ of diameter less than δ there is a decomposition $f^{-1}(U) = U_1 \cup U_2 \cup \cdots \cup U_m$ such that $f : U_i \longrightarrow U$ are homeomorphisms and any two points belonging to different components U_i, U_j are on distance more than δ. Note that $f : U_i \longrightarrow U$ are isometries, and that $\iota : U_i \longrightarrow \iota(U_i)$ are uniform contractions. As in the case of expanding maps, we call U_i the *components* of $f^{-1}(U)$.

Let $A \subset \mathcal{M}_0$ be a subset of diameter less than δ. Similarly to the case of expanding coverings, we can construct the tree of preimages T_A. Namely, the root of T_A corresponds to A, the first level $\mathcal{L}_1$ of T_A consists of the components of $f^{-1}(A)$, and if $\mathcal{L}_{n-1}$ is the $(n-1)$-st level, then the nth level consists of components of $f^{-1}(\iota(B))$ for $B \in \mathcal{L}_{n-1}$. We connect each of the components of $f^{-1}(\iota(B))$ to B by an edge.

For every vertex $B \in \mathcal{L}_n$ we have a unique sequence $B_1, B_2, \ldots, B_n = B$ such that B_1 is a component of $f^{-1}(A)$, and B_{i+1} is a component of $f^{-1}(\iota(B_i))$. It follows that we can identify B_n with the set $\tilde{B}_n = \{(x_n, x_{n-1}, \ldots, x_1) \ : \ x_i \in B_i, f(x_i) = \iota(x_{i-1})\} \subset \mathcal{M}_n$ (in the sense that $(x_n, x_{n-1}, \ldots, x_1) \mapsto x_n$ is a homeomorphism form $\tilde{B}_n$ to B_n). Hence, the tree T_A is the tree of preimages of A under the covering maps

$$\mathcal{M}_0 \xleftarrow{f} \mathcal{M}_1 \xleftarrow{f_1} \mathcal{M}_2 \xleftarrow{f_2} \cdots .$$

If A, B are two subsets of $\mathcal{M}_0$ of diameter less than δ such that $A \cap B \neq \emptyset$, then there exists a unique isomorphism of rooted trees $S_{A,B} : T_A \longrightarrow T_B$, defined in the same way as for expanding coverings.

Consequently, we can define iterated monodromy groups of contracting topological correspondences using covers by small sets in the same way as we defined iterated monodromy groups of expanding covering maps. This definition will agree with the definition using paths.

The standard actions of the iterated monodromy groups of expanding topological correspondences also can be defined using covers by small sets, which is essentially the usual definition using paths, if we pass to the nerve of the cover.

Theorem 6.13. *Let $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ be a contracting topological correspondence. Suppose that $\mathcal{M}_0$ is connected. Then* $\mathrm{IMG}\,(f, \iota)$ *is a contracting group (with respect to any standard self-similar action).*

Proof. Let us prove the theorem for the path connected and locally path connected space. The general case, using covers by small sets, is very similar, but notationally a bit more cumbersome (one has replace paths by chains of small subsets).

Let $\mathcal{U}$ be a finite cover of $\mathcal{M}_0$ by open sets of diameters less than δ. Let δ_0 be the Lebesgue's number of $\mathcal{U}$, i.e., such that for every subset A of diameter less than δ_0 there exists $U \in \mathcal{U}$ such that $A \subset U$.

We say that a path γ in $\mathcal{M}_0$ is K-small, where K is a positive real number, if γ can be split into product of paths $\gamma = \gamma_1\gamma_2 \ldots \gamma_m$ such that sum of diameters of the images of γ_i is not more than K and diameter of each γ_i is less than ϵ_0.

Lemma 6.14. *For every $K > 0$ the set of elements S_γ of* $\mathrm{IMG}\,(f, \iota)$ *defined by K-small loops is finite.*

Proof. If γ is K-small, then we can split in into a product $\gamma_1\gamma_2 \ldots \gamma_n$ such that length of each γ_i is less than ϵ_0, the sum of diameters is not more than K, and no two neighboring paths γ_i, γ_{i+1} both have diameter less than $\epsilon_0/2$. Then $n \leq 4K/\epsilon_0$.

Find for every γ_i an element $A_i \in \mathcal{U}$ such that the image of γ_i is contained in A_i. Then $S_\gamma = S_{A_1,A_2} S_{A_2,A_3} \cdots S_{A_{n-1},A_n}$. We see that S_γ is a composition of not more then $4K/\epsilon_0$ elements of the form $S_{A,B}$ for $A, B \in \mathcal{U}$. It follows that the number of possible values of S_γ is finite. □

Note that for every path $\gamma : [0, 1] \longrightarrow \mathcal{M}_0$ there is K such that γ is K-small. If γ_1 is K_1-small, and γ_2 is K_2-small, then $\gamma_1\gamma_2$ is $K_1 + K_2$-small. If γ is K-small, and γ' is a lift of γ by f, then $\iota(\gamma)$ is ρK-small.

Let C be such that every connecting path ℓ_x is C-small. Suppose that $\gamma \in \pi_1(\mathcal{M}, t)$ is K-small. Then it follows from Proposition 5.17 that the section $\gamma|_x$ of the corresponding element of $\mathrm{IMG}\,(f, \iota)$, for every $x \in X$, is defined by a loop which is $(2C + \rho K)$-small. Consequently, any section in a word of length n is $(2C + 2C\rho + 2C\rho^2 + \cdots + 2C\rho^{n-1} + \rho^n K)$-small. For all n sufficiently big we have

$\rho^n K < 1$. Hence, for all n sufficiently big the sections of γ in words of length more than n are all defined by loops that are $\left(1+\frac{2C}{1-\rho}\right)$-small. This implies, by Lemma 6.14, that $\mathrm{IMG}\,(f,\iota)$ is contracting. □

EXAMPLE 6.7. Consider the polynomial $f(z) = -z^3/2 + 3z/2$, discussed in ... Let Γ_0 be the graph consisting of two circles in $\mathbb{C}$ going around 1 and -1 as it is shown on Figure... Let Γ_1 be the preimage of Γ_0 by f, shown on Figure... We have a covering map $f : \Gamma_1 \longrightarrow \Gamma_0$. For map $\iota : \Gamma_1 \longrightarrow \Gamma_0$ homotopic in $\mathbb{C} \setminus \{1, -1\}$ to the identical embedding $\Gamma_1 \longrightarrow \mathbb{C}$, the iterated monodromy group $\mathrm{IMG}\,(f,\iota)$ is equivalent as a self-similar group to $\mathrm{IMG}\left(-z^3/2 + 3z/2\right)$. Let us endow Γ_0 with a length metric such that the circles have length 1. Lift this metric to Γ_1 by f. Then the semicircles of Γ_1 have length 1, and the loops ... have length 1 each. Let us map the cycles of Γ_1 going around 1 and -1 to the corresponding circles of Γ_0 by a map dividing all distances by 2, and contract the extra circles to points. Then ι is homotopic to the identical embedding and is contracting with $\rho = 1/2$. It follows that $\mathrm{IMG}\left(-z^3/2 + 3z/2\right)$ is contracting.

One can show that contraction coefficient ρ in the definition of a contracting topological correspondence is an upper bound on the contraction coefficient of its iterated monodromy group. It follows that in the last example the contraction coefficient of $\mathrm{IMG}\,(f)$ is not more than $1/2$. On the other hand, it is easy to see from the recursive definition of the generators a and b of $\mathrm{IMG}\,(f)$ that $a^{2n} = (a^n, a^n, 1)$, which implies that the contraction coefficient is not less than $1/2$, i.e., it is equal to $1/2$.

6.5. Absence of free subgroups.

LEMMA 6.15. *Let F be a free non-abelian group and let*

$$\phi : F \longrightarrow G_1 \times G_2 \times \cdots \times G_n$$

be a homomorphism to a direct product of groups. If for every i the composition of ϕ with the projection $G_1 \times G_2 \times \cdots \times G_n \longrightarrow G_i$ has non-trivial kernel, then ϕ has non-trivial kernel.

PROOF. It is enough to prove the lemma for $n = 2$. Let $r_1, r_2 \in F$ be non-trivial elements such that $\phi(r_1) = (g_1, 1)$ and $\phi(r_2) = (1, g_2)$ for some $g_i \in G_i$. Then $\phi([r_1, r_2]) = (1, 1)$, hence $[r_1, r_2]$ is an element of the kernel of ϕ. If $[r_1, r_2] \neq 1$, then we are done. Suppose that $[r_1, r_2] = 1$. Then r_1 and r_2 belong to one cyclic subgroup of F, i.e., there exist $r \in F$ and $n_1, n_2 \in \mathbb{Z}$ such that $r_i = r^{n_i}$. But then $r_1^{n_2} = r_2^{n_1} = r^{n_1 n_2} = (g_1^{n_2}, 1) = (1, g_2^{n_1})$ is a non-trivial element of the kernel of ϕ. □

LEMMA 6.16. *Let F be a free non-abelian subgroup, and let $H < F$ be a cyclic subgroup. Then there exists a free non-abelian subgroup $\tilde{F} \leq F$ such that $\tilde{F} \cap H$ is trivial.*

PROOF. One can find a subgroup of F freely generated by the generator h of H and two other elements $g_1, g_2 \in F$. Then we can take $\tilde{F} = \langle g_1, g_2 \rangle$. □

DEFINITION 6.7. Let G be a group acting by homeomorphisms on a space $\mathcal{X}$. A *germ* (or a *G-germ*) is equivalence class of a pair $(g, x) \in G \times \mathcal{X}$, where two pairs (g_1, x_1) and (g_2, x_2) are equivalent if $x_1 = x_2$ and there exists a neighborhood U of x_1 such that $g_1|_U = g_2|_U$.

Note that if (g_1, x_1) and (g_2, x_2) are germs such that $g_2(x_2) = x_1$, then the germ (g_1g_2, x_2) depends only on (g_1, x_1) and (g_2, x_2). We write $(g_1g_2, x_2) = (g_1, x_1)(g_2, x_2)$, and thus get a partially defined multiplication on the set of all G-germs. This multiplication behaves almost as multiplication in a group, except that it is not everywhere defined (and there is a unit $(1, x)$ for every $x \in \mathcal{X}$). But the set of all germs (g, x) such that $g(x) = x$ is a group, which we will denote $G_{(x)}$.

The group $G_{(x)}$ is the quotient of the stabilizer G_x of $x \in \mathcal{X}$ in G by the subgroup of elements $h \in G$ such that there exists a neighborhood U_h of x such that h acts trivially on U_h.

Theorem 6.17. *Let G be a group acting faithfully on a locally finite rooted tree T. Then one (or more) of the statements following holds.*

(1) *G has no free non-abelian subgroups.*
(2) *There exists a free non-abelian subgroup $F \le G$ and a point $w \in \partial T$ such that the stabilizer F_w is trivial.*
(3) *There exists a point $w \in \partial T$ and a free non-abelian subgroup of $G_{(w)}$.*

Proof. Suppose that theorem is not true. Then there exists a group G with a free subgroup F, for every $w \in \partial T$ the group $G_{(w)}$ has no free subgroup, and for every free subgroup $\tilde{F} \le G$ and every $w \in \partial T$ the stabilizer $\tilde{F}_w$ is non-trivial.

For every $w \in \partial T$ the stabilizer F_w is non-cyclic, since otherwise, using Lemma 6.16, we can find a free subgroup $\tilde{F}$ of F such that $\tilde{F}_w = \tilde{F} \cap F_w$ is trivial, which is impossible by our assumption. The group $G_{(w)}$ has no free subgroups, hence the natural homomorphism $F_w \longrightarrow G_{(w)}$ has non-trivial kernel. Hence there exists a vertex v_w on the path w, and an element $g_w \in F$ acting trivially on the subtree T_{v_w}.

We get a cover of ∂T by open subsets ∂T_{v_w} such that there exist non-trivial elements g_w acting trivially on ∂T_{v_w}. Let us find a finite sub-cover $\{\partial T_v : v \in V\}$, where V is a finite set of vertices of T. Note that we can assume that this cover is by disjoint sets. For every $v \in V$ there exists non-trivial elements of F acting trivially on T_v.

Let F_V be the intersection of the stabilizers of elements of V in F. It is a subgroup of finite index in F. We know that for every $v \in V$ there exists a non-trivial, hence infinite, subgroup $H_v \le F$ acting trivially on T_v. Its intersection with F_V will have finite index in H_v, hence will be also infinite.

We have an injective homomorphism $F_V \longrightarrow \prod_{v \in V} \mathrm{Aut}(T_v)$ coming from restricting the action of F_V onto each tree T_v. But its projection onto any factor $\mathrm{Aut}(T_v)$ has non-trivial kernel, which is a contradiction, by Lemma 6.15. □

Definition 6.8. Let G be a group acting by homeomorphisms on a topological space $\mathcal{X}$, and let $S = S^{-1}$ be a generating set of G. For $x \in \mathcal{X}$, the *graph of germs* Γ_x is the graph with the set of vertices equal to the set of germs (g, x) for $g \in G$, in which two vertices (g_1, x) and (g_2, x) are connected by an edge if and only if there exists $s \in S$ such that $(sg_1, x) = (g_2, x)$.

Problem 6.5. Show that $(g, x) \mapsto g(x)$ is a covering map from the graph of germs to the Schreier graph of the action of G on the orbit of x.

Theorem 6.18. *Contracting groups have no free non-abelian subgroups.*

Proof. One can prove, exactly in the same way as Proposition 6.8, that graphs of germs of the action of a contracting group on the boundary $X^{\mathbb{N}}$ of the tree X^*

have polynomial growth. This rules out cases (2) and (3) of Theorem 6.17, since the graph of the action on the orbit is covered by the graph of germs, while the Cayley graph of $G_{(w)}$ is contained in the graph of germs. In fact, it is easy to see that $G_{(w)}$ is finite. $\square$