7. Limit dynamical systems of contracting groups

7.1. Limit dynamical system. Let G be a contracting self-similar group acting on X^*. Denote by $X^{-\omega}$ the space of left-infinite sequences $\ldots x_2x_1$ of elements of X with the direct product topology.

DEFINITION 7.1. We say that $\ldots x_2x_1, \ldots y_2y_1 \in X^{-\omega}$ are *asymptotically equivalent* if there exists a finite set $N \subset G$ and a sequence $g_k \in N$ such that

$$g_k(x_kx_{k-1}\ldots x_1) = y_ky_{k-1}\ldots y_1$$

for all $k \geq 1$.

In other words, the sequences are asymptotically equivalent if and only if their endings $x_k \ldots x_1$ and $y_k \ldots y_1$ are on a uniformly bounded distance from each other in the Schreier graphs of the action of G on X^*.

It is obvious that asymptotic equivalence is an equivalence relation. It is also easy to see that if $\ldots x_2x_1 \sim \ldots y_2y_1$, then $\ldots x_3x_2 \sim \ldots y_3y_2$, i.e., that this equivalence relation is shift-invariant.

PROPOSITION 7.1. *Let $\mathcal{N}$ be the nucleus of G. If $\ldots x_2x_1 \sim \ldots y_2y_1$, then there exists a sequence $g_k \in \mathcal{N}$ such that $g_k(x_k) = y_k$ and $g_k|_{x_k} = g_{k-1}$.*

Draw a graph with the set of vertices $\mathcal{N}$, where for every $g \in \mathcal{N}$ and $x \in X$ there is an arrow from g to $g|_x$ labeled by $(x, g(x))$. Then two sequences $\ldots x_2x_1$ and $\ldots y_2y_1$ are asymptotically equivalent if and only if there exists a sequence of edges $e_1, e_2, \ldots$, such that beginning of e_n is the end of e_{n+1}, and e_n is labeled by (x_n, y_n).

In particular, it follows that the asymptotic equivalence relation is a closed subset of $X^{-\omega} \times X^{-\omega}$. This in turn implies that the quotient space is Hausdorff (in fact, metrizable).

The quotient of $X^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of G, and is denoted $\mathcal{J}_G$. Since the equivalence relation is shift-invariant, the shift $\ldots x_2x_1 \mapsto \ldots x_3x_2$ induces a continuous self-map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$, which we call the *limit dynamical system.*

7.2. Limit dynamical system of an iterated monodromy group.

THEOREM 7.2. *Let $f, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ be a contracting topological correspondence, where $\mathcal{M}_0$ is connected. Let $\mathcal{M}_\infty$ be the inverse limit of the spaces $\mathcal{M}_n$ with respect to the maps $\iota_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$, and let $f_\infty : \mathcal{M}_\infty \longrightarrow \mathcal{M}_\infty$ be the map on the inverse limit induced by $f_n : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$. Then the dynamical system $f_\infty : \mathcal{M}_\infty \longrightarrow \mathcal{M}_\infty$ is topologically conjugate to the limit dynamical system of the iterated monodromy group* $\mathrm{IMG}\,(f, \iota)$.

COROLLARY 7.3. *Let $f : \mathcal{M} \longrightarrow \mathcal{M}$ be an expanding covering map, where $\mathcal{M}$ is connected. Then it is topologically conjugate to the limit dynamical system of its iterated monodromy group.*

Let us prove the corollary. The proof of the theorem is very similar. We will consider only the case when $\mathcal{M}$ is path connected and locally path connected.

PROOF. Let $t \in \mathcal{M}$ be a basepoint, and let $\Lambda : X \longrightarrow f^{-1}(t)$, and ℓ_x be a bijection and a set of connecting paths, defining a standard action of $\mathrm{IMG}\,(f)$.

There exist constants $C > 0$ and $\rho \in (0,1)$ such that for every $x \in X$ every lift of ℓ_x by f^n is of diameter less than $C\rho^n$, see the proof of Proposition 6.12.

Let $\ldots x_2x_1x_0 \in \mathcal{X}^{-\omega}$. Let $\ell^{(0)} = \ell_{x_0}$, and define $\ell^{(n)}$ as the lift of ℓ_{x_n} by f^{-n} starting in the end of $\ell^{(n-1)}$. Then $\ell^{(n)}\ell^{(n-1)}\ldots\ell^{(0)}$ is a path from t to $\Lambda(x_nx_{n-1}\ldots x_0)$, see the definition of Λ in 5.7.

Since diameter of $\ell^{(n)}$ is less than $C\rho^n$, the sequence $\Lambda(x_nx_{n-1}\ldots x_0)$ is Cauchy. Denote its limit by $\Lambda(\ldots x_2x_1x_0)$, and denote by $\ell_{\ldots x_2x_1x_0}$ the path $\ldots\ell^{(2)}\ell^{(1)}\ell^{(0)}$.

Suppose that $\ldots x_2x_1x_0$ and $\ldots y_2y_1y_0$ are asymptotically equivalent. Let $g_n \in \mathcal{N}$ be a sequence such that $g_n(x_n\ldots x_0) = y_n\ldots y_0$ for all n. There exists $K > 0$ such that diameter of any lift of any $g \in \mathcal{N}$ (as a loop based at t) by f^n is less than $K\rho^n$. Then $g_n(x_n\ldots x_0) = y_n\ldots y_0$ means that there exists a lift of g_n by f^{n+1} starting in $\Lambda(x_n\ldots x_0)$ and ending in $\Lambda(y_n\ldots y_0)$. Consequently, the distance from $\Lambda(x_n\ldots x_0)$ to $\Lambda(y_n\ldots y_0)$ is less than $K\rho^n$. Consequently, the limits $\Lambda(\ldots x_1x_0)$ and $\Lambda(\ldots y_1y_0)$ coincide.

On the other hand, if $\Lambda(\ldots x_1x_0) = \Lambda(\ldots y_1y_0)$, then the loops $g_n = \ell^{-1}_{\ldots y_{n+2}y_{n+1}}\ell_{\ldots x_{n+2}x_{n+1}}$ satisfy $g_n(x_n) = y_n$ and $g_n|_{x_n} = g_{n-1}$. Using Lemma 6.14 we can show that the set $\{g_n\} \subset \mathrm{IMG}(f)$ is finite. Consequently, $\ldots x_1x_0$ and $\ldots y_1y_0$ are asymptotically equivalent.

We have shown that the map Λ induces a well defined injective map $\Lambda : \mathcal{J}_G \longrightarrow \mathcal{M}$.

Its continuity easily follows from the exponentially decreasing estimates on the diameters of the paths $\ell^{(n)}$.

There exists K such that for every point $x \in \mathcal{M}$ there exists a K-small path γ starting in t and ending in x. Lifting a path connecting $f^n(x)$ to t, we conclude that for every n and $x \in \mathcal{M}$ there exists $\Lambda(v) \in f^{-n}(t)$, $v \in X^n$, on distance less than $K\rho^n$ from x. It follows (using compactness of $X^{-\omega}$ that Λ is onto.

The map $\Lambda : \mathcal{J}_G \longrightarrow \mathcal{M}$ is a continuous bijection between compact Hausdorff spaces, hence it is a homeomorphism.

It follows directly from the definition of Λ that it satisfies $\Lambda \circ \mathsf{s} = f \circ \Lambda(w)$, i.e., that it is a conjugacy of the dynamical systems. ☐

In the general proof of Theorem 7.2 we construct in a similar way continuous maps $\Lambda_n : \mathcal{J}_G \longrightarrow \mathcal{M}_n$ agreeing with the maps ι_n, and show that these maps realize a homeomorphism between the inverse limit and $\mathcal{J}_G$. The proof is notationally a bit more technical, but the ideas are exactly the same.

7.3. Tiles and their adjacency graphs. The Schreier graphs of the action of a contracting group G on the levels X^n of the tree X^* are good visualizations of the limit spaces. More precisely, suppose that a generating set S is *state-closed*, i.e., satisfies $g|_x \in S$ for all $g \in S$ and $x \in X$. Let Γ_n be the graph of the action of G on X^n. Recall that is set of vertices is X^n, and its set of edges is $S \times X^n$, where an edge (s, v) starts in v and ends in $s(v)$.

Then the map $\iota_n : \Gamma_{n+1} \longrightarrow \Gamma_n$ given by $\iota_n(s, xv) = (s|_x, v)$ for $s \in S$, $x \in X$, and $v \in X^n$ is a morphism of graphs. In fact, it is easy to see that the maps ι_n and the coverings $f_n : \Gamma_{n+1} \longrightarrow \Gamma_n : vx \mapsto v$ are obtained in the standard way from the topological correspondence $f_0, \iota_0 : \Gamma_1 \longrightarrow \Gamma_0$.

It follows directly from the definition of the asymptotic equivalence relation that $\mathcal{J}_G$ is naturally homeomorphic to the space of connected components of the graph $\lim_{\substack{\longleftarrow\\ \iota_n}} \Gamma_n$. The map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ is induced by the maps f_n.

In the case $S = \mathcal{N}$, the graphs Γ_n can be interpreted as the (one-skeleta of the) nerves of natural closed covers of $\mathcal{J}_G$.

Definition 7.2. Let G be a contracting group, and let $X^{-\omega} \longrightarrow \mathcal{J}_G$ be the natural quotient map. For $v \in X^*$, denote by $\mathcal{T}_v \subset \mathcal{J}_G$ the image of the set $X^{-\omega}v$ of sequences ending with v under the quotient map.

We obviously have $\mathcal{T}_\emptyset = \mathcal{J}_G$, $\mathcal{T}_v = \bigcup_{x\in X} \mathcal{T}_{xv}$, and $\mathsf{s}(\mathcal{T}_{vx}) = \mathcal{T}_v$.

Proposition 7.4. *Let $v_1, v_2 \in X^n$. The sets $\mathcal{T}_{v_1}$ and $\mathcal{T}_{v_2}$ have non-empty intersection if and only if there exists $g \in \mathcal{N}$ such that $g(v_1) = v_2$.*

Proof. If $\mathcal{T}_{v_1}$ and $\mathcal{T}_{v_2}$ have a non-empty intersection, then there exist sequences $\ldots x_2x_1v_1, \ldots y_2y_1v_2 \in X^{-\omega}$ that are asymptotically equivalent. Then, by Proposition 7.1, there exists $g \in \mathcal{N}$ such that $g(v_1) = v_2$.

Let us prove the other direction.

Lemma 7.5. *For every $g \in \mathcal{N}$ there exists $x \in X$ and $h \in \mathcal{N}$ such that $h|_x = g$.*

Proof. Suppose that it is not true. Then g does not belong to the set $\mathcal{N}|_X = \{h|_x : h \in \mathcal{N}, x \in X\}$. But $\mathcal{N}|_X$ satisfies the conditions of Definition 6.5. This contradicts to the condition that $\mathcal{N}$ is the smalles set satisfying the conditions of Definition 6.5. $\square$

Let $v_1, v_2 \in X^n$ and $g \in \mathcal{N}$ be such that $g(v_1) = v_2$. By the lemma above, there exists $g_1 \in \mathcal{N}$ and $x_1 \in X$ such that $g_1|_{x_1} = g$. By induction, there exist $g_n \in \mathcal{N}$ and $x_n \in X$ such that $g_n|_{x_n} = g_{n-1}$. Then $\ldots x_2x_1v_1$ and $\ldots y_2y_1v_2$ are asymptotically equivalent, where $y_n = g_n(x_n)$. $\square$

Example 7.1. Let G be the group generated by $a = \sigma(1,1)$ and $b = (a, b)$. It is easy to see that $a^2 = 1$ and hence $b^2 = 1$. It follows that G is infinite dihedral. (Infinite, since $ab = \sigma(a, b)$ has infinite order, as $(ab)^2 = (ba, ab)$.)

It is also easy to see that the graph Γ_n of the action of G on X^n is a chain of length 2^n with endpoints $1^n = 1\ldots111$ and $1^{n-1}0 = 1\ldots110$ (the only points fixed by b). This implies that the limit space $\mathcal{J}_G$ is homeomorphic to a closed interval. The map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ folds the interval in two.

7.4. Orbispace structure on $\mathcal{J}_G$. The limit dynamical system $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ is not a covering map in general, see Example 7.1. Other examples include the iterated monodromy groups of *sub-hyperbolic* rational functions, such as $z^2 + i$.

Definition 7.3. A self-similar group G acting on X^* is said to be *regular* if for every $w \in X^{\mathbb{N}}$ the group of germs $G_{(w)}$ is trivial. Equivalently, if for every $g \in G$ and $w \in X^{\mathbb{N}}$ such that $g(w) = w$ there exists a beginning v of w such that $g|_v = 1$.

Note that it is enough to check the last condition of the definition for the elements g of the nucleus only.

Example 7.2. In Example 7.1 the only fixed point of b on $X^{\mathbb{N}}$ is the sequence $111\ldots$. It is isolated, hence the group $G = \langle a, b\rangle$ is not regular.

Example 7.3. The iterated monodromy group of $z^2 - 1$ is generated by $a = \sigma(b, 1)$ and $b = (a, 1)$. The nucleus is $\{1, a, a^{-1}, b, b^{-1}, ab^{-1}, ba^{-1}\}$. It is easy to see that $a, a^{-1}, ab^{-1}, ba^{-1}$ have no fixed points on $X^{\mathbb{N}}$, while the set of fixed points of b and b^{-1} is $1X^{\mathbb{N}}$, which is open. Consequently, the iterated monodromy group of $z^2 - 1$ is regular.

PROPOSITION 7.6. *The iterated monodromy group of an expanding covering is regular.*

PROOF. Let $f : \mathcal{M} \longrightarrow \mathcal{M}$ be an expanding covering map. There exists $\delta > 0$ such that every subset $A \subset \mathcal{M}$ of diameter less than δ is evenly covered and components of $f^{-1}(A)$ all have diameters less than δ. Then all lifts of a loop of diameter less than δ by any iteration f^n are loops. This easily implies (using the formula for a standard action of the iterated monodromy group) that $\mathrm{IMG}(f)$ is regular. □

We have the following correspondence between contracting groups and expanding maps.

We say that a group acting on a rooted tree T is *level-transitive* if it acts transitively on each level of the tree.

THEOREM 7.7. *If $f : \mathcal{M} \longrightarrow \mathcal{M}$ is an expanding covering and $\mathcal{M}$ is connected, then the iterated monodromy group of f is a contracting regular level-transitive group.*

If G is a contracting regular level-transitive group, then its limit dynamical system $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ is an expanding covering map, $\mathcal{J}_G$ is connected, and G is the iterated monodromy group of s.

In order to generalize the last theorem to non-regular groups, we need to introduce a structure of an *orbispace* on $\mathcal{J}_G$.

DEFINITION 7.4. Let G be a contracting group acting on X^*. Consider the direct product $X^{-\omega} \times G$, where X and G are discrete. We say that $\ldots x_2x_1 \cdot g, \ldots y_2y_1 \cdot h \in X^{-\omega} \times G$ are *asymptotically equivalent* if there exists a finite set $N \subset G$ and a sequence $g_n \in N$, $n \geq 1$, such that

$$g_n(x_n \cdots x_2x_1) = y_n \cdots y_2y_1, \qquad g_n|_{x_n \cdots x_2x_1} g = h$$

for all n. The quotient of $X^{-\omega} \times G$ is called the *limit G-space*, and is denoted $\mathcal{X}_G$.

PROPOSITION 7.8. *Two sequences $\ldots x_2x_1 \cdot g, \ldots y_2y_1 \cdot h$ are asymptotically equivalent if and only if there exists a sequence g_n, $n \geq 0$, of elements of the nucleus such that $g_n|_{x_n} = g_{n-1}$, $g(x_n) = y_n$, for all $n \geq 1$, and $g_0g = h$.*

The group G naturally acts on $X^{-\omega} \times G$ by right multiplication. It is easy to see that the asymptotic equivalence relation is invariant under this action. Consequently, we get an action of G on $\mathcal{X}_G$.

PROPOSITION 7.9. *The action of G on $\mathcal{X}_G$ is proper, i.e., for every compact set $K \subset \mathcal{X}_G$ the set $\{g \in G \ : \ Kg \cap K \neq \emptyset\}$ is finite.*

The map $\ldots x_2x_1 \cdot g \mapsto \ldots x_2x_1$ induces a homeomorphism of the space of orbits $\mathcal{X}_G/G$ with $\mathcal{J}_G$.

Spaces of orbits of proper actions are called sometimes *orbispaces*. Usually, orbispaces are given by representing neighborhoods of points as quotients of finite groups acting on topological spaces. These representations have to agree on their overlaps, so that we get an *atlas* of the orbispace.

EXAMPLE 7.4. The limit G-space of the infinite dihedral group $a = \sigma(1, 1), b = (a, b)$ is homeomorphic to the real line $\mathbb{R}$, where the generators a and b act on $\mathbb{R}$ by reflections (e.g., $x \mapsto -x$ and $x \mapsto 1 - x$). It follows that the orbispace

$\mathcal{J}_G$ is homeomorphic to the interval, in which neighborhoods of the endpoints are represented as quotients of a segment by the action of a group of order 2.

Arbitrary contracting groups can be represented as iterated monodromy groups of a correspondence $\mathsf{s}, \iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$, where $\mathcal{M}_1$ and $\mathcal{M}_0$ are orbispaces with identical underlying topological spaces; $\mathsf{s} : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ is the limit dynamical system on the underlying spaces, and a covering of the orbispaces; and $\iota : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$ is the identity map on the underlying spaces and a morphism of orbispaces.

7.5. Functoriality. The most natural algebraic interpretation of self-similar groups is given by the notion of a *G-biset*.

Let G be a self-similar group acting on X^*. Consider the set $\mathfrak{M}$ of maps from X^* to itself of the form

$$w \mapsto xg(w),$$

where $x \in X$ and $g \in G$. We will denote this map by xg.

The set $\mathfrak{M}$ is invariant under the actions of G on it given by pre- and post-compositions. Namely, for any $xg \in \mathfrak{M}$, and $h \in G$, the transformations

$$xg \cdot h : w \mapsto xgh(w)$$

and

$$h \cdot xg : w \mapsto h(xg(w)) = h(x)(h|_x g)(w)$$

belong to $\mathfrak{M}$. We get hence left and right actions of G on $\mathfrak{M}$. These actions obviously commute.

Definition 7.5. A *G-biset* is a set $\mathfrak{M}$ together with left and right commuting actions of G on it. We say that it is a *covering biset* if the right action is free (i.e., if $x \cdot g = x$ implies $g = 1$ for $x \in \mathfrak{M}, g \in G$), and has a finite number of orbits.

Let $\mathfrak{M}$ be a covering G-biset. Let $X \subset \mathfrak{M}$ be a set such that every right G-orbit contains precisely one element of X. Then for all $x \in X$ and $g \in G$ there exist unique $y \in X$ and $h \in G$ such that

$$g \cdot x = y \cdot h.$$

We can use these formulas do define an action of G on X^* by the rule

$$g(xw) = yh(w)$$

for all $w \in X^*$ whenever $g \cdot x = y \cdot h$ in $\mathfrak{M}$. We call this action *standard action associated with the biset (and the transversal X)*. It is easy to see that if $\mathfrak{M} = X \cdot G$ is the biset associated with a self-similar group, as above, then the standard action associated with $\mathfrak{M}$ and $X = X \cdot 1 \subset \mathfrak{M}$ is the original action.

It is also not hard to show that two self-similar actions of G are equivalent if and only if the corresponding bisets $\mathfrak{M}_1$, $\mathfrak{M}_2$ are isomorphic, i.e., there exists a bijection $F : \mathfrak{M}_1 \longrightarrow \mathfrak{M}_2$ such that $F(g_1 \cdot x \cdot g_2) = g_1 \cdot F(x) \cdot g_2$.

Example 7.5. Let $f : \mathcal{M} \longrightarrow \mathcal{M}$ be a covering map such that $\mathcal{M}$ is path connected and locally path connected. Let $t \in \mathcal{M}$, and consider the set $\mathfrak{M}_f$ of homotopy classes of paths from t to a point of $f^{-1}(t)$. The set $\mathfrak{M}_f$ is a $\pi_1(\mathcal{M}, t)$-biset with respect to the actions

$$[\ell] \cdot [\gamma] = [\ell\gamma]$$

and

$$[\gamma] \cdot [\ell] = [\gamma' \ell]$$

where γ' is the lift of γ by f starting at the end of ℓ. (Recall that in a product $\alpha\beta$ the path β is passed before the path α.) Here $[\cdot]$ denots the homotopy class of a path. It follows directly from the definition of a standard action that standard actions associated with $\mathfrak{M}_f$ coincide with the standard actions of the iterated monodromy group $\mathrm{IMG}(f)$.

DEFINITION 7.6. Let $\mathfrak{M}_1, \mathfrak{M}_2$ be G-bisets. Their *tensor product* $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ is the quotient of the set $\mathfrak{M}_1 \times \mathfrak{M}_2$ by the identifications

$$x_1 \cdot g \otimes x_2 = x_1 \otimes g \cdot x_2.$$

It is a G-biset with respect to the actions

$$g_1 \cdot (x_1 \otimes x_2) \cdot g_2 = (g_1 \cdot x_1) \otimes (x_2 \cdot g_2).$$

Operation of tensor product is associative, i.e., the biset $(\mathfrak{M}_1 \otimes \mathfrak{M}_2) \otimes \mathfrak{M}_3$ is isomorphic to $\mathfrak{M}_1 \otimes (\mathfrak{M}_2 \otimes \mathfrak{M}_3)$ in the natural way.

If $\mathfrak{M}$ is the biset associated with a self-similar action, then $\mathfrak{M}^{\otimes n}$ is the biset of transformations of the form

$$w \mapsto vg(w)$$

for $g \in G$ and $v \in X^n$.

The limit spaces $\mathcal{X}_G$ and $\mathcal{J}_G$ can be defined using in terms of bisets in the following way.

DEFINITION 7.7. Let $\mathfrak{M}$ be a G-biset associated with a contracting group. For a finite set $A \subset \mathfrak{M}$ consider the direct power $A^{-\omega}$, and let Ω be the inductive limit of the topological spaces $A^{-\omega}$ over all finite sets $A \subset \mathfrak{M}$ (with respecto to the inclusions $A_1^{-\omega} \hookrightarrow A_2^{-\omega}$ induced by inclusions $A_1 \hookrightarrow A_2$. We say that two sequences $\ldots x_2x_1, \ldots y_2y_1 \in \Omega$ are equivalent if there exists a finite set $N \subset G$ and a sequence $g_n \in N$, such that $g_n \cdot x_n \otimes x_{n-1} \otimes \cdots \otimes x_1 = y_n \otimes y_{n-1} \otimes \cdots \otimes y_1$ in $\mathfrak{M}^{\otimes n}$ for all $n \geq 1$.

One can show that the quotient of Ω by the defined equivalence relation is naturally homeomorphic to $\mathcal{X}_G$. It is easy to see that the right action of G on $\mathfrak{M}$ induces a right action on Ω, and that the equivalence relation is invariant under this action. The quotient of $\mathcal{X}_G$ by the action of G is $\mathcal{J}_G$.

DEFINITION 7.8. Let $\mathfrak{M}_i$ be G_i-bisects, for $i = 1, 2$. A *morphism* $(G_1, \mathfrak{M}_1) \longrightarrow (G_2, \mathfrak{M}_2)$ of bisets is given by a group homomorphism $\phi : G_1 \longrightarrow G_2$ and a map $F : \mathfrak{M}_1 \longrightarrow \mathfrak{M}_2$ such that $F(g \cdot x \cdot h) = \phi(g) \cdot F(x) \cdot \phi(h)$ for all $g, h \in G_1$ and $x \in \mathfrak{M}_1$.

PROPOSITION 7.10. *Every morphism $(G_1, \mathfrak{M}_1) \longrightarrow (G_2, \mathfrak{M}_2)$ of bisets associated with contracting groups induces a continuous map $f : \mathcal{J}_{G_1} \longrightarrow \mathcal{J}_{G_2}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{J}_{G_1} & \xrightarrow{f} & \mathcal{J}_{G_2} \\ \downarrow \mathsf{s} & & \downarrow \mathsf{s} \\ \mathcal{J}_{G_1} & \xrightarrow{f} & \mathcal{J}_{G_2} \end{array}$$

is commutative.

The converse is also true, any continuous map f making the above diagram commutative induces a morphism of bisets.

Example 7.6. Consider the iterated monodromy group of z^2-1. It is generated by

$$a = \sigma(b, 1), \qquad b = (a, 1).$$

We have $ab = \sigma(ba, 1)$. Conjugate the right-hand side by $\sigma(b, b)$. We get an equivalent self-similar group

$$a = \sigma(1, b), \qquad b = (1, b^{-1}ab),$$

and then $ab = \sigma(1, ab)$. It follows that $\langle ab \rangle \hookrightarrow \langle a, b \rangle$ together with the identity map on X is a morphism of bisets. The group $\langle ab \rangle$ is the iterated monodromy group of the double self-covering of the circle. The morphism of bisets induces a semi-conjugacy of the circle double covering with the action of z^2-1 on its Julia set. It coincides with the Caratheodori loop. It is the extension to the boundary of the biholomorphic isomorphism of the complement of the unit circle with the complement of the basin of attraction to infinity of z^2-1. We pick the extension that is tangent to the identity at infinity. See Figure 23.

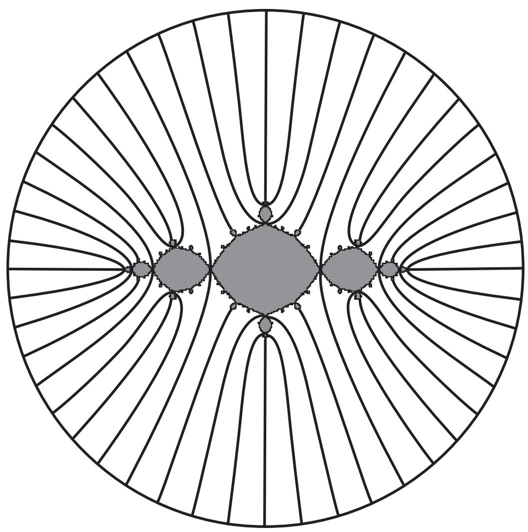


Figure 23. Caratheodori loop

Example 7.7. Let $f(z) = z^2 + c$, where $c = c_{1/4} \approx -0.2282 + 1.1151i$ be the quadratic polynomial such that $f^2(c)$ is a fixed point of f. The parameter c is a root of the polynomial $x^3 + 2x^2 + 2x + 2$.

One of the standard actions of $\mathrm{IMG}(f)$ is

$$\begin{aligned} \alpha &= \sigma(\beta^{-1}\alpha^{-1}, \alpha\beta), \\ \beta &= (\alpha, 1), \\ \gamma &= (\gamma, \beta). \end{aligned}$$

It corresponds to choosing the basepoint $t = +\infty$, and connecting paths ℓ_0 and ℓ_1 trivial, and the upper semicircle at infinity.

The Caratheodori loop is represented by the element

$$\gamma\alpha\beta = \sigma(1, \gamma\alpha\beta).$$

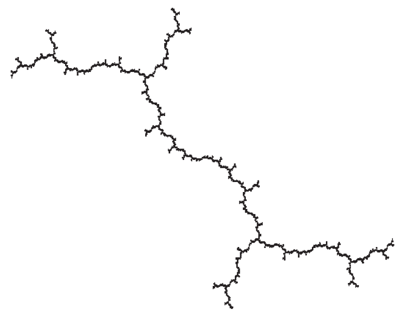


FIGURE 24. Julia set of $z^2 - 0.2282\ldots + 1.1151\ldots i$

Let us take two copies of the Julia set of f and glue one to the other along the Caratheodori loop, flipping one of them using the complex conjugation.

Alternatively, take two copies of the plane with the action of f on both of them and glue them together along the circle at infinity identifying the points of the circles which are symmetric with respect to the real axis.

In terms of the category of self-similar groups (bisets) this operation can be interpreted as the amalgam of the embeddings $a \mapsto \gamma\alpha\beta$ and $a \mapsto (\gamma\alpha\beta)^{-1}$ of the adding machine $a = \sigma(1, a)$ into $\mathrm{IMG}(f)$, i.e., the universal object H making the diagram

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathrm{IMG}(f) \\ \downarrow & & \downarrow \\ \mathrm{IMG}(f) & \longrightarrow & H \end{array}$$

commutative.

Let us compute the iterated monodromy group of the result. Choose in one plane the generators α, β, γ and the right orbit transversal ℓ_0, ℓ_1, as before. Let α', β', γ' and ℓ_0', ℓ_1' be the paths defined by the same way in the second plane. Note that $\gamma\alpha\beta = (\gamma'\alpha'\beta')^{-1} = \beta'\alpha'\gamma'$. Then we have $\ell_0 = \ell_0'$ and $\ell_1 = \ell_1' \cdot \beta'\alpha'\gamma'$.

The action of α', β', γ' defined with respect to the basis ℓ_0', ℓ_1' is the same as for α, β, γ with respect to ℓ_0, ℓ_1 (with primes added everywhere). Hence, the action of α', β', γ' associated with the transversal ℓ_0, ℓ_1 is obtained by pos-conjugating the recursion by $(1, \beta'\alpha'\gamma')$. Hence, we get

$$\begin{aligned} \alpha' &= \sigma(\gamma', \gamma'), \\ \beta' &= (\alpha', 1), \\ \gamma' &= (\gamma', \gamma'\alpha'\beta'\alpha'\gamma'). \end{aligned}$$

Let us change the transversal to $\ell_0, \ell_1 \cdot \beta\alpha$, i.e., conjugate the recursion by $(1, \beta\alpha)$:

$$\begin{aligned} &\alpha = \sigma, \\ &\beta = (\alpha, 1), \qquad \beta' = (\gamma'\beta\alpha\gamma\beta', 1), \\ &\gamma = (\gamma, \alpha\beta\alpha), \quad \gamma' = (\gamma', \gamma\beta'\gamma), \end{aligned}$$

denote $a = \alpha, b = \alpha\beta\alpha, c = \gamma, b' = \gamma\beta'\gamma, c' = \gamma'$:

$$\begin{aligned} a &= \sigma, \\ b &= (1, a), \quad b' = (cc'abb', 1), \\ c &= (c, b), \quad c' = (c', b'), \end{aligned}$$

and consider the subgroup generated by $a, B = bb', C = cc'$ which are given then by

$$\begin{aligned} a &= \sigma, \\ B &= (CaB, a), \\ C &= (C, B). \end{aligned}$$

PROPOSITION 7.11. *The simplicial Schreier graphs of* $\mathrm{IMG}\,(F) = \langle a, b, c, b', c' \rangle$ *and* $G = \langle a, B, C \rangle$ *coincide. Namely, if* $b(v) \neq v$, $b'(v) \neq v$, $c(v) \neq v$, *or* $c'(v) \neq v$, *then* $b(v) = B(v)$, $b'(v) = B(v)$, $c(v) = C(v)$ *and* $c'(v) = C(v)$, *respectively.*

COROLLARY 7.12. *The inclusion* $G < \mathrm{IMG}\,(F)$ *induces a homeomorphism of the limit spaces. The inclusion* $\mathrm{IMG}\,(f) = \langle a, b, c \rangle < \mathrm{IMG}\,(F)$ *induces a surjective continuous map.*

One can show that the group generated by a, B and C is isomorphic to the group of affine transformations of $\mathbb{C}$ of the form $z \mapsto \pm z + (m + ni)$, where $m + ni \in \mathbb{Z}[i]$. The generators act on the complex plane by the rules $z \cdot a = -z - i$, $z \cdot B = -z - i + 1$ and $z \cdot C = -z$ and the associated virtual endomorphism is induced by the linear map $z \mapsto z/(1 - i)$ on $\mathbb{C}$.

Hence the limit space $\mathcal{J}_G = \mathcal{J}_{\mathrm{IMG}(F)}$ is homeomorphic to the sphere and the limit G-space $\mathcal{X}_G$ is the complex plane $\mathbb{C}$ with the described affine action of G (the limit space $\mathcal{X}_{\mathrm{IMG}(F)}$ is more complicated).

See a part of the Cayley graph Γ of G on Figure 25 (the highlighted edges will be explained later). The Schreier graph of the action of G on X^n is the quotient of the graph Γ by the action of the group $(1 - i)^n G$.

The Schreier graphs of $\mathrm{IMG}\,(f)$ are trees and sub-graphs of the corresponding Schreier graphs of G.

Figure 25 shows these edges for $n = 8$ (one has however, to wrap it around the sphere, taking quotient of the picture by the action of the group $(1 - i)^8 G$).

The Schreier graphs of $\mathrm{IMG}\,(f)$ approximate the limit space $\mathcal{J}_{\mathrm{IMG}(f)}$, i.e., the Julia set of f. The embedding of the Schreier graph into the Schreier graph of G approximates the continuous map from the dendrite $\mathcal{J}_{\mathrm{IMG}(f)}$ onto the sphere $\mathcal{J}_G = \mathcal{J}_{\mathrm{IMG}(F)}$ induced by the embedding $\mathrm{IMG}\,(f) \hookrightarrow \mathrm{IMG}\,(F)$.

An interesting observation is the fact that the Schreier graphs of $\mathrm{IMG}\,(f)$ can be constructed using the classical paper folding procedure. Take a strip of paper and fold it n times, each time in the same way (say, put it horizontally and rotate the right half around the middle point by π in the positive direction). Then unfold it so that every bend is a right angle. See the top of Figure 26 where the resulting broken lines for $n = 2, \ldots, 7$ are shown. Properly rescaled lines will converge to the Heighway dragon curve.

Take now two copies of such a broken line (for the same n) and put them together in such a way that they have common endpoints and one is rotation by π of the other. We get a closed broken line of 2^{n+1} segments. The internal part of this line will consist of 2^{n-1} squares which are connected by corridors in the

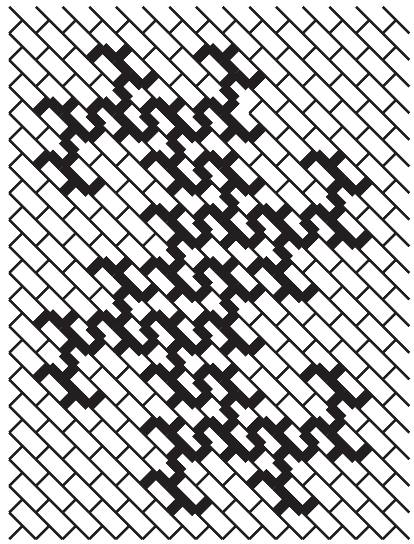
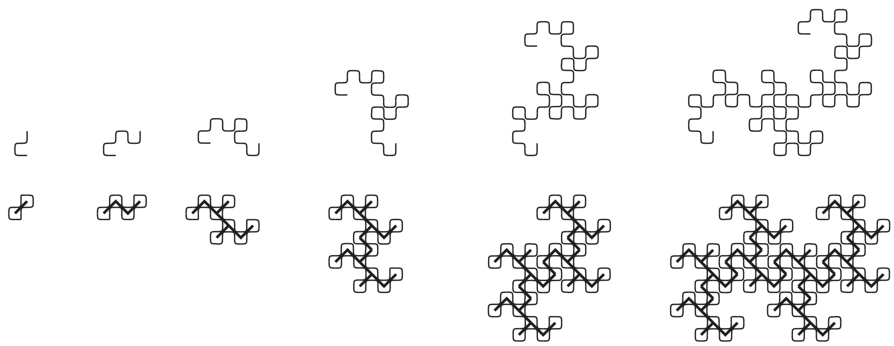


FIGURE 25. Schreier graph of IMG $\left(z^2 - 0.2282\ldots + 1.1151\ldots i\right)$



[h]

FIGURE 26. Paper folding and IMG (f)

same way as the vertices of the Schreier graph of the action of IMG (f) on X^{n-1} are connected by edges. The lower part of Figure 26 shows the corridors and the corresponding Schreier graphs.