8. Hyperbolic diffeomorphisms

Let $\mathcal{M}$ be a Riemannian manifold, and let $\mathcal{M}_0 \subset \mathcal{M}$ be an open subset.

DEFINITION 8.1. A compact set $\Lambda \subset \mathcal{M}$ such that $f(\Lambda) = \Lambda$ is said to be *hyperbolic* (we also say that f is *hyperbolic on* Λ) if the restriction of the tangent bundle $T\mathcal{M}$ to Λ can be decomposed into a direct sum $E^s \oplus E^u$ of sub-bundles so that there exist $C > 0$ and $\lambda \in (0, 1)$ such that for every every $n \geq 0$ we have

- $\|Df^n \vec{v}\| \leq C\lambda^n \|\vec{v}\|$ for all $\vec{v} \in E^s$;
- $\|Df^{-n} \vec{v}\| \leq C\lambda^n \|\vec{v}\|$ for all $\vec{v} \in E^u$.

8.1. Smale horseshoe. Let $R = (0, 1) \times (0, 1) \subset \mathbb{R}^2$, and consider a diffeomorphism $f : R \longrightarrow \mathbb{R}^2$ mapping R to a "horseshoe" shown on Figure... We can choose it in such a way that restriction of f to the sub-rectangle $(0, 1) \times (1/5, 2/5)$ is given by $f(x, y) = (x/5 + 1/5, 5y - 1)$ and restriction to $(0, 1) \times (3/5, 4/5)$ is given by $f(x, y) = (-x/5 + 4/5, -5y + 4)$.

Then $f^{-1}(R)$ is equal to $R_1 = (0, 1) \times (1/5, 2/5) \cup (0, 1) \times (3/5, 4/5)$, its image is $R_{-1} = (1/5, 2/5) \times (0, 1) \cup (3/5, 4/5) \times (0, 1)$. The map f contracts the tangent vectors to points of R_{-1} parallel to the x-axis by 5 and expands the tangent vectors parallel to the y-axis. It follows that f restricted to the maximal f-invariant subset of R is hyperbolic.

Let us try to understand what is the maximal f-invariant subset of R. Denote, for $n \in \mathbb{Z}$, by R_n the set of points $(x, y) \in R$ such that $f^n(x, y)$ is defined. We have seen that $R_1 = (0, 1) \times (1/5, 2/5) \cup (0, 1) \times (3/5, 4/5)$. It follows from the definition of f that R_n, for $n \geq 1$ is a union of 2^n rectangles of the form $(0, 1) \times (k/5^n, (k+1)/5^n)$, where $k \in 0, 1, \ldots, 5^n - 1$ is a number that has only digits 1 and 3 in its base 5 expansion. Similarly, R_{-n}, for $n \geq 1$, is the union of the rectangles of the form $(k/5^n, (k+1)/5^n) \times (0, 1)$ with the same condition on k. Consequently, the maximal f-invariant set is the set of points $(x, y) \in (0, 1) \times (0, 1)$ such that in the base 5 expansions $.a_1a_2\ldots$ of the coordinates x and y only the digits 1 and 3 appear.

PROBLEM 8.1. Show that the action of f on its maximal invariant set is topologically conjugate to the full shift $\{0, 1\}^{\mathbb{Z}}$.

The above construction is very flexible, and similar dynamical sets appear in many situations.

8.2. Solenoid. The solenoid, described in 3.6 is an example of a hyperbolic set. Let us give an explicit formula for a diffeomorphism hyperbolic on a solenoid. Consider the torus obtained by rotating around the z-axis the circle C in the xz-plane of radius $1/2$ with center in $(1, 0, 0)$. Let D be the interior part of the torus. If a point with coordinates $(u + 1, 0, v)$ of the circle C is rotated around the z-axis by θ, then we get a point with coordinates

$$\begin{aligned} x &= (u+1)\cos\theta, \\ y &= (u+1)\sin\theta, \\ z &= v. \end{aligned}$$

Let us use the coordinates (u, v, θ). The above formulas show us the relation between these coordinates and (x, y, z). The set D is the set of points satisfying

$u^2 + v^2 < 1/4$. Consider the map $f : D \longrightarrow D$ given by

$$f(u, v, \theta) = \left(\frac{u + \cos\theta}{4}, \frac{v + \sin\theta}{4}, 2\theta\right),$$

or, in the usual coordinates:

$$f(x, y, z) = \left(\frac{A(x^2 - y^2)}{4}, \frac{Axy}{2}, \frac{z + \frac{y}{\sqrt{x^2+y^2}}}{4}\right),$$

where $A = \frac{x^2+y^2+3\sqrt{x^2+y^2}+x}{(x^2+y^2)^{3/2}}$ It will wind the torus D twice around itself, see Figure... It is not hard to prove that the action of f on $S = \bigcap_{n\geq 0} f^n(D)$ is topologically conjugate to the solenoid dynamical system described in 3.6, and that f is hyperbolic on S. The set S is an example of an *attractor*: for every neighborhood U of S we have $\bigcap_{n\geq 1} f^n(U) = S$.

8.3. Hénon maps. A *Hénon map* is given by

$$f_{a,b}(x, y) = (1 - ax^2 + y, bx),$$

where (a, b) are some fixed parameters. Note that it is a homeomorphism, if $b \neq 0$, since the inverse map is $f_{a,b}^{-1}(x, y) = \left(\frac{y}{b}, -1 + \frac{a}{b^2}y^2 + x\right)$.

We can consider it either as a map from $\mathbb{R}^2$ to itself, or as a map of $\mathbb{C}^2$ to itself. In the latter case, a conjugate version is usually used. Namely, let $\phi(x, y) = (-ax, -\frac{a}{b}y)$. Then $\phi \circ f_{a,b} \circ \phi^{-1}(x, y) = \phi(1 - x^2/a - \frac{b}{a}y, -\frac{b}{a}x) = (-a + x^2 + by, x)$. Replacing $-a$ by c we get a map

$$H_{b,c}(x, y) = (x^2 + c + by, x).$$

It was proved by ... that if $z^2 + c$ is a hyperbolic quadratic polynomial, and b is sufficiently small non-zero number, then a restriction of $H_{b,c}$ to a hyperbolic subset of $\mathbb{C}^2$ is topologically conjugate to the natural extenstion of the action of $z^2 + c$ on its Julia set.

9. Smale spaces

9.1. Spaces with local product structure.

DEFINITION 9.1. A *rectangle* is a topological space R with a continuous binary operation $[\cdot, \cdot] : R \times R \longrightarrow R$ satisfying

(1) $[x, x] = x$;
(2) $[x, [y, z]] = [x, z]$;
(3) $[[x, y], z]] = [x, z]$.

If a space R is decomposend into a direct product $R = A \times B$ of spaces, then we have a natural structure of a rectangle on it given by $[(a_1, b_1), (a_2, b_2)] = (a_1, b_2)$. Let us show that all rectangles are like this.

Write $x \sim_1 y$ if $[x, y] = x$. Note that $[x, y] = x$ implies that $[y, x] = [y, [x, y]] = [y, y] = y$, i.e, $y \sim_1 x$. If $x \sim_1 y$ and $z \sim_1 y$, then $[x, z] = [x, [z, y]] = [x, y] = x$, i.e., $x \sim_1 z$. We have shown that $\sim_1$ is an equivalence relation.

Write $x \sim_2 y$ if $[x, y] = y$. In the same way as above, we see that $\sim_2$ is an equivalence relation. Denote by $L_i(x)$ the $\sim_i$ equivalence class of x.

Define $P_1 : R \longrightarrow L_1(x)$ by $P_1(y) = [y, x]$. Then $[x, P_1(y)] = [x, [y, x]] = x$, hence $P_1(y) \in L_1(x)$. Similarly, the map $P_2(y) = [x, y]$ is a continuous map from R to $L_2(x)$.

Define a map $F : L_1(x) \times L_2(x) \longrightarrow R$ by $F(a, b) = [a, b]$. We have $F(P_1(a), P_2(a)) = [[a, x], [x, a]] = [a, [x, a]] = a$. Consequently, $(P_1, P_2) : R \longrightarrow L_1(x) \times L_2(x)$ is inverse to F, which implies that F is a homeomorphism. We have $[F(a_1, b_1), F(a_2, b_2)] = [[a_1, b_1], [a_2, b_2]] = [a_1, b_2] = F(a_1, b_2)$. Consequently, if we identify R with $L_1(x) \times L_2(x)$, then $[\cdot, \cdot]$ is given by the usual formula $[(a_1, b_1), (a_2, b_2)] = (a_1, b_2)$.

Definition 9.2. A *atlas of a local product structure* on a topological space $\mathcal{X}$ is an open cover of $\mathcal{X}$ by rectangles $(R_i, [\cdot, \cdot]_i)$ such that for every point $x \in \mathcal{X}$ and for every pair of rectangles R_i, R_j there exists a neighborhood U of x such that $[a, b]_1 = [a, b]_2$ for all $a, b \in U$ such that the corresponding expressions are defined.

Two atlases of a local product structure on $\mathcal{X}$ define the same *local product structure* on $\mathcal{X}$ if their union is also an atlas of a local product structure.

Note that the condition of the definition is vacuous if x does not belong to the intersection of the closures of R_i and R_j.

If $\mathcal{X}$ is a space with a local product structure, then a *rectangle* on $\mathcal{X}$ is an open subset $R \subset \mathcal{X}$ and a structure of rectangle $[\cdot, \cdot]$ on R such that if we add $(R, [\cdot, \cdot])$ to an atlas defining the local product structure, then we get an atlas of the local product structure.

9.2. Smale spaces.

Definition 9.3. A *Smale space* is a compact metrizable space $\mathcal{X}$ and a homeomorphism $f : \mathcal{X} \longrightarrow \mathcal{X}$ such that there exists an atlas $\mathcal{U} = \{(R_i, [\cdot, \cdot]_i)\}$ of a local product structure on $\mathcal{X}$, a metric d on $\mathcal{X}$, and a number $\lambda \in (0, 1)$, such that f preserves the local product structure, and for every rectangle $(R_i, [\cdot, \cdot]_i)$ we have

$$d(f(x), f(y)) \leq \lambda d(x, y), \quad \text{if } y \in L_1(x)$$

and

$$d(f^{-1}(x), f^{-1}(y) \leq \lambda d(x, y), \quad \text{if } y \in L_2(x).$$

Example 9.1. Every Anosov diffeomorphism is a Smale space...

Example 9.2. Natural extension of an expanding self-covering is a Smale space. We can take the direct products $U \times \partial T_U$, where U is an open set of diameter less than some sufficiently small δ, as rectangles...

9.3. Algebraic examples. Theorem of M. Shub and M. Gromov, generalized...

10. Holonomy groupoid of a Smale space

Definition of the holonomy pseudogroup, Arnold cat map example, equivalence to circle rotation...

Fibonacci and Penrose tiling...

Ruelle groupoids, their full group, finitely presented extensions of iterated monodromy group...