Groups and topological dynamics

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ABSTRACT. ...

Contents

Chapter 1

Dynamical systems

A *topological dynamical system* $H \curvearrowright \mathcal{X}$ is an action of a semigroup H on a topolgical space $\mathcal{X}$ by continuous transformations.

Classically, $\mathcal{X}$ is the *phase space*, the semigroup H represents time, and the action describes time evolution of the system. Accordingly, the acting semigroup is typically a subsemigroup of the additive group of real numbers (e.g., the semigroup of non-negative reals, the group of integers, or the semigroup of natural numbers).

The subsequent chapters of the book will mostely deal with more “exotic” groups. But even in such cases, the groups often will be associated in a natural way to classical dynamical systems. The first section of this chapter is a short overview of well known examples of dynamical systems. It introduces concepts that will be developed and generalized in the later parts of the book. The subsequent sections deal with more specialized topics in dynamical systems: subshifts, minimal homeomorphisms of Cantor sets, basic notions of hyperbolic dynamics, symbolic encoding of dynamical systems, and basic facts of holomorphic dynamics.

1.1. Introduction by examples

1.1.1. Irrational rotation. Consider the circle $\mathbb{R}/\mathbb{Z}$ of real numbers modulo 1. It can be naturally identified with the complex unit circle $T \subset \mathbb{C}$ by the map $x \mapsto e^{2\pi i x}$.

The circle $\mathbb{R}/\mathbb{Z}$ is a group with respect to the addition. The above identification of $\mathbb{R}/\mathbb{Z}$ with the unit circle is an isomorphism of the additive group $\mathbb{R}/\mathbb{Z}$ with the multiplicative group T.

Suppose that $\theta \in \mathbb{R}/\mathbb{Z}$ is irrational. Consider the corresponding rotation

$$R_\theta : x \mapsto x + \theta.$$

It is a homeomorphism of the circle, hence it generates an action of the infinite cyclic group $\mathbb{Z}$ by homeomorphisms. It is given by $(n, x) \mapsto n\theta + x$ for $n \in \mathbb{Z}$ and $x \in \mathbb{R}/\mathbb{Z}$.

The central topic of topological dynamics is the study of topological properties of the *orbits* of a dynamical system $H \curvearrowright \mathcal{X}$, i.e., the sets of the form $Hx = \{hx \ : \ h \in H\}$ for $x \in \mathcal{X}$. For example, we may be interested in the cases when Hx is finite (or compact for a topological semigroup H), or in properties of the closure of Hx, etc.. If G is a group acting on a space $\mathcal{X}$ by homeomorphisms, then it naturally defines the *orbit equivalence* relation on $\mathcal{X}$. It is given by

$$x \sim y \Longleftrightarrow \exists g \in G \ : \ g(x) = y.$$

The orbits of the action are the equivalence classes of this relation.

It is natural to consider then the set $G\backslash\mathcal{X}$ (or $\mathcal{X}/G$ for right actions) of orbits of the action. We introduce on it the smallest (coarsest) topology (i.e., topology with smallest set of open sets) for which the natural map $\mathcal{X} \longrightarrow G\backslash\mathcal{X}$ is continuous. In other words, a subset $A \subset G\backslash\mathcal{X}$ is open if and only if its full preimage in $\mathcal{X}$ is open.

The space $G\backslash\mathcal{X}$ is frequently non-Hausdorff. For example, the following classical theorem of Kronecker [**Kro84**] implies that in the case of the irrational rotation R_θ the space of orbits of the action of $\mathbb{Z}$ on the circle has trivial topology (i.e., the only open sets are the empty set and the whole space).

Theorem 1.1.1. *Every orbit* $\{x + n\theta \ : \ n \in \mathbb{Z}\}$ *is dense in* $\mathbb{R}/\mathbb{Z}$.

Proof. Denote, for a real number x, by $\mathrm{frac}(x)$ the fractional part of x, i.e., the unique number from the interval $[0, 1)$ such that $a - \mathrm{frac}(a) \in \mathbb{Z}$.

It is enough to prove that the orbit of 0 is dense in $\mathbb{R}/\mathbb{Z}$, since the orbit of an arbitrary point $\alpha \in \mathbb{R}/\mathbb{Z}$ is obtained from the orbit of 0 by the rotation R_α.

Consider an arbitrary positive integer N, and the arcs $[0, 1/N)$, $[1/N, 2/N)$, $\ldots [(N-1)/N, 1)$ of the circle $\mathbb{R}/\mathbb{Z}$. Since the orbit $\{\mathrm{frac}(n\theta) \ : \ n \in \mathbb{Z}\}$ is infinite, there exist integers $n_1 < n_2$ such that $\mathrm{frac}(n_1\theta)$ and $\mathrm{frac}(n_2\theta)$ belong to the same arc. Then $\mathrm{frac}(|(n_1 - n_2)\theta|) < 1/N$. This proves that for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{Z}$ such that $\mathrm{frac}(n_\epsilon\theta) < \epsilon$. Then the difference between consecutive entries of the sequence $0, \mathrm{frac}(n_\epsilon\theta), \mathrm{frac}(2n_\epsilon\theta), \ldots$ is less than ϵ, hence for every $\alpha \in \mathbb{R}/\mathbb{Z}$ there exists k such that $\mathrm{frac}(|\alpha - kn_\epsilon\theta|) < \epsilon$. Consequently, the orbit of 0 is dense in $\mathbb{R}/\mathbb{Z}$. $\square$

Definition 1.1.2. An action $H \curvearrowright \mathcal{X}$ is *minimal* if every H-orbit Hx is dense in $\mathcal{X}$.

It is easy to see that a group action $G \curvearrowright \mathcal{X}$ is minimal if and only if the space $G\backslash\mathcal{X}$ is antidiscrete.

Minimality is one of possible notions of irreducibility of topological dynamical systems, as the following lemma shows. (It also explains the origin of the term "minimal", which referred to minimal invariant closed subsets.)

Lemma 1.1.3. *A system $H \curvearrowright \mathcal{X}$ is minimal if and the only closed subsets $\mathcal{Y} \subset \mathcal{X}$ such that $h(\mathcal{Y}) \subset \mathcal{Y}$ for every $h \in H$ are $\mathcal{X}$ and the empty set.*

Proof. If $H \curvearrowright \mathcal{X}$ is not minimal, then there exists $x \in \mathcal{X}$ such that the orbit Hx is not dense. Then its closure $\mathcal{Y} = \overline{Hx}$ satisfies $h(\mathcal{Y}) \subset \mathcal{Y}$ for all $h \in H$, and is not equal neither to $\mathcal{X}$ nor to $\emptyset$.

In the other direction, if $\mathcal{Y}$ is H-invariant and closed, then for every $x \in \mathcal{Y}$ the closure of the orbit Hx is contained in $\mathcal{Y}$. So, if $\mathcal{Y}$ is non-emtpy and different from $\mathcal{X}$, then no H orbit of a point of $\mathcal{Y}$ is dense in $\mathcal{X}$. $\square$

The circle is the quotient of $\mathbb{R}$ by the natural action of $\mathbb{Z}$, and the rotation R_θ is lifted to the action on $\mathbb{R}$ of the transformation $x \mapsto x + \theta$. The orbits of the irrational rotation are therefore equal to the images under the natural quotient map $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ of the orbits of the action $\mathbb{Z}^2 \curvearrowright \mathbb{R}$ generated by the transformations $x \mapsto x + 1$ and $x \mapsto x + \theta$.

Consider the natural action $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ generated by the transformations

$$a : (x, y) \mapsto (x + 1, y), \qquad b : (x, y) \mapsto (x, y + 1),$$

and consider the projection $P : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $P(x, y) = x + \theta y$. We have $P(a(\vec{v})) = P(\vec{v}) + 1$ and $P(b(\vec{v})) = P(\vec{v}) + \theta$. In other words, the map P projects the natural action $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ to the action $\mathbb{Z}^2 \curvearrowright \mathbb{R}$ generated by $x \mapsto x + 1$ and $x \mapsto x + \theta$.

Denote by Q the composition of P with the natural quotient $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$. The orbits of the rotation $R_\theta \circlearrowleft \mathbb{R}/\mathbb{Z}$ are the Q-images of the sets of the form $\mathbb{Z}^2 + v \subset \mathbb{R}^2$. The segment $[0, 1) \subset \mathbb{R}$ is a natural *fundamental domain* of the quotient map $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$, and we can consider the part of $\mathbb{Z}^2 + v$ projected by P to $[0, 1)$. Every point of the R_θ-orbit of $\alpha \in \mathbb{R}/\mathbb{Z}$ is represented then by exactly one point of the set $P^{-1}([0, 1)) \cap (\mathbb{Z}^2 + v)$ for $v \in Q^{-1}(\alpha)$.

See Figure 1.1, where an orbit of a point under a rotation is represented in this way. The grid is the set $\mathbb{Z}^2 + v$ for some $v \in \mathbb{R}^2$. The transformation P is the projection onto the horizontal coordinate axis along lines parallel to the slanted lines shown on the picture. The strip between the slanted lines is the set $P^{-1}([0, 1))$. The points of the grid inside the strip represent the points of the orbit of the rotation. The segments connecting neighboring

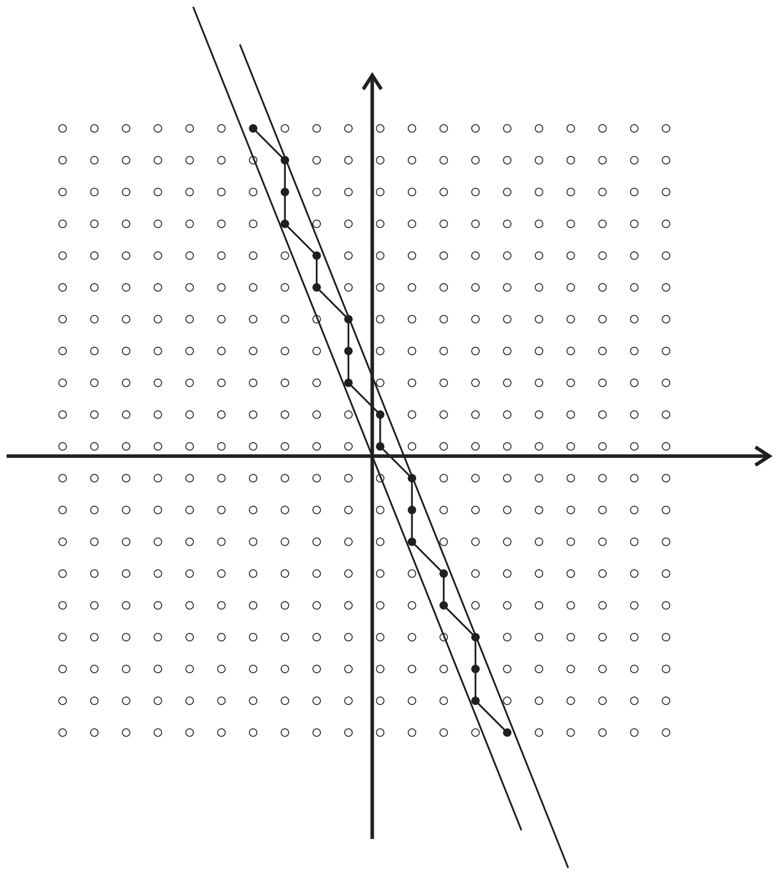

Figure 1.1. Irrational rotation

points of the orbit represent the action of R_θ. Let $\theta \in [0,1)$ (which we always can assume). Then for $x \in [0,1)$ the point $R_\theta(x)$ is represented either by $x+\theta$ or by $x+\theta-1$ in $[0,1)$. Therefore, on Figure 1.1, we move from the point representing x to the point representing $R_\theta(x)$ either by adding $(0,1)$ or by adding $(-1,1)$. One can see these two cases as two types of edges of the broken line inside the strip on Figure 1.1: the vertical edges (corresponding to adding $(0,1)$) and the diagonal edges (corresponding to adding $(-1,1)$).

We can record the shape of the broken line by writing a two-sided infinite sequence

$$\dots dvvdvdvvdvdvvdvdvvd\dots,$$

where d stands for "diagonal" and v for "vertical". More formally, let x_n be the point of $[0,1)$ representing $R_\theta^n(x)$ (i.e., $x_n = \operatorname{frac}(x+n\theta)$). Define the

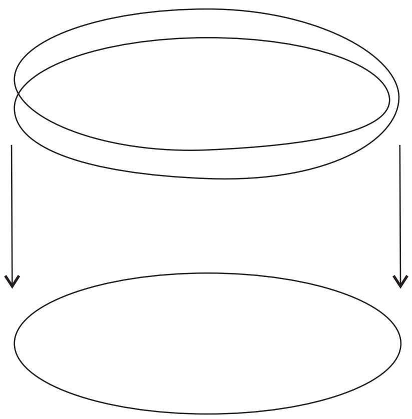

Figure 1.2. Angle doubling

sequence $(a_n)_{n\in\mathbb{Z}}$ by the rule

$$a_n = \begin{cases} v & \text{if } x_{n+1} = x_n + \theta, \\ d & \text{if } x_{n+1} = x_n + \theta - 1. \end{cases}$$

We will study such sequences and their generalizations in subsequent sections of the book (see 1.2.5 and 3.5.2, for example). A multi-dimensional generalization of these sequences are quasi-crystals (see...) and, more generally, Cayley graphs of groupoids (see...).

1.1.2. Angle doubling map and one-sided shift. Consider the map $f : x \mapsto 2x \pmod 1$ on the circle $\mathbb{R}/\mathbb{Z}$. If we identify the circle $\mathbb{R}/\mathbb{Z}$ with the complex unit circle by the map $x \mapsto e^{2\pi i x}$, then f becomes $z \mapsto z^2$.

The map $f \circlearrowleft \mathbb{R}/\mathbb{Z}$ is a degree two covering. Every point $x \in \mathbb{R}/\mathbb{Z}$ has two preimages: $x/2$ and $(x+1)/2$. Accordingly, as a dynamical system, we consider it to be an action of the semigroup of non-negative integers. In particular, the orbit of x is, by definition, the set $\{f^n(x)\}_{n=0,1,2,\ldots}$. Here f^0 is the identity map.

If x is a rational number, then f does not increase the denominator of x (it either does not change it or divides it by 2). Since we always can represent points of $\mathbb{R}/\mathbb{Z}$ by points of $[0,1)$, it follows that the f-orbits of rational points of $\mathbb{R}/\mathbb{Z}$ are finite. Conversely, if $x \in \mathbb{R}/\mathbb{Z}$ has finite orbit, then there exist positive integers $m < n$ such that $2^m x = 2^n x \pmod 1$, which implies that x is rational.

If x has a finite orbit under the action of a map f, then either x belongs to a *cycle* (is *periodic*) and f acts on the orbit of x as a cyclic permutation, or it is *pre-periodic* and the orbit contains two points of the form $y/2$ and

$(y+1)/2$ mapped by f to the same point. In the first case there exists a non-negative integer n such that $f^n(x) = x$. The smallest such n is called the *period* or the *length* of the cycle. We have then that $2^n x = x \pmod 1$, hence $x = \frac{m}{2^n-1}$, so that x is a fraction with an odd denominator. Conversely, if $x = \frac{p}{q}$ for $p, q \in \mathbb{N}$, where q is odd, then the orbit of x is finite and can not be pre-periodic, since for every rational y one of the fractions $y/2$, $(y+1)/2$ has an even denominator. It follows that x belongs to an f-cycle if and only if x is rational and has an odd denominator.

In the pre-perodic case there exist smallest $n > 0$ such that $f^n(x)$ belongs to a cycle (i.e., has an odd denominator). We call this n the *pre-period* of x. Then x is of the form $\frac{p}{2^n q}$, where p and q are odd.

Let us represent the points of $\mathbb{R}/\mathbb{Z}$ by their binary expansions. Namely, a point $x \in \mathbb{R}/\mathbb{Z}$ is represented by a sequence $.a_1a_2a_3\ldots$ of zeros and ones so that

$$x = \sum_{k=1}^{\infty} \frac{a_k}{2^k} \pmod 1.$$

Denote by $\{0,1\}^\omega$ the set of all infinite sequences of zeros and ones, and denote by $\Phi : \{0,1\}^\omega \longrightarrow \mathbb{R}/\mathbb{Z}$ the natural map given by the binary numeration system:

$$\Phi(a_1a_2\ldots) = \sum_{k=1}^{\infty} \frac{a_k}{2^k} \pmod 1.$$

It is well known that the binary representation of real numbers is almost one-to-one, i.e., $\Phi : \{0,1\}^\omega \longrightarrow \mathbb{R}/\mathbb{Z}$ is almost a bijection. The only ambiguity is

$$.a_1a_2\ldots a_n10000\ldots = .a_1a_2\ldots a_n01111\ldots. \tag{1.1}$$

The space $\{0,1\}^\omega$ comes with a natural *direct product topology*. It is defined by the basis of open sets consisting of all sets of the form

$$C_{a_1a_2\ldots a_n} = \{x_1x_2\ldots \in \{0,1\}^\omega \ : \ x_1x_2\ldots x_n = a_1a_2\ldots a_n\},$$

called sometimes *cylindrical sets*. Note that the sets $\Phi(C_{a_1a_2\ldots a_n})$ are closed intervals of the form $\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right]$ for $k \geqslant 0$ and $0 \leqslant m \leqslant 2^k - 1$.

A natural metric on $\{0,1\}^\omega$ is given by $d(w_1, w_2) = 2^{-n}$, where n is the maximal length of a common beginning of w_1 and w_2. It is easy to see that we have $|\Phi(w_1) - \Phi(w_2)| \leqslant d(w_1, w_2)$, hence the map Φ is continuous. As we have seen above, $\Phi^{-1}(x)$ consists of one or two sequences, and all instances when $\Phi^{-1}(x)$ has two elements are described by (1.1). Namely, $|\Phi^{-1}(x)| = 2$ if and only if x is of the form $\frac{m}{2^n}$ for some natural numbers m and n.

If $\Phi(a_1a_2a_3\ldots) = x$, then $f(x) = \Phi(a_2a_3a_4\ldots)$. We get what is called a *semiconjugacy* implemented by Φ between two dynamical systems: $f \circlearrowleft \mathbb{R}/\mathbb{Z}$ and the *shift* map $\mathsf{s} : a_1a_2a_3\ldots \mapsto a_2a_3a_4\ldots$ on $\{0,1\}^\omega$.

Definition 1.1.4. Let $H \curvearrowright \mathcal{X}$ and $H \curvearrowright \mathcal{Y}$ be two actions of the same semigroup on topological spaces. A *semiconjugacy* from the first topological dynamical system to the second one is a continuous map $\Phi : \mathcal{X} \longrightarrow \mathcal{Y}$ such that $\Phi(hx) = h\Phi(x)$ for all $h \in H$ and $x \in \mathcal{X}$. If, additionally, Φ is a homeomorphism, then we say that it is a (*topological*) *conjugacy*.

In our case, the statement that Φ is a semiconjugacy is equivalent to the statement that the diagram

$$\begin{array}{ccc} \{0,1\}^\omega & \xrightarrow{\mathsf{s}} & \{0,1\}^\omega \\ \big\downarrow{\scriptstyle\Phi} & & \big\downarrow{\scriptstyle\Phi} \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{f} & \mathbb{R}/\mathbb{Z} \end{array}$$

commutes. The dynamical system $\mathsf{s} \circlearrowleft \{0,1\}^\omega$ is called the *one-sided shift*.

This semiconjugacy can be used to prove many facts about the angle doubling map. For example, consider the following description of typical orbits of $f \circlearrowleft \mathbb{R}/\mathbb{Z}$.

Proposition 1.1.5. *The set of points $x \in \mathbb{R}/\mathbb{Z}$ such that the orbit of x under f is dense is of full measure and co-meager in $\mathbb{R}/\mathbb{Z}$.*

Recall that a set is called *co-meager* (or *residual*) if it is equal to intersection of a countable collection of open dense sets. By Bair Category Theorem ... such sets are non-empty for locally compact Hausdorff spaces and for complete metric spaces. In these cases the notion of a co-meager set is a topological version of the notion of a set of full measure.

Proof. Consider the uniform Bernoulli measure μ on $\{0,1\}^\omega$. It is uniquely determined by the condition $\mu(C_{a_1a_2\ldots a_n}) = \frac{1}{2^n}$. Note that the length of the subinterval $\Phi(C_{a_1a_2\ldots a_n}) \subset [0,1]$ is also equal to $\frac{1}{2^n}$. It follows that $\Phi : \{0,1\}^\omega \longrightarrow \mathbb{R}/\mathbb{Z}$ maps the Bernoulli measure on $\{0,1\}^\omega$ to the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$. In fact, Φ is an isomorphism of the corresponding measure spaces, since it is a bijection modulo a set of measure zero.

Let $w \in \{0,1\}^\omega$. If the orbit of w with respect to the shift action is dense in $\{0,1\}^\omega$, then its Φ-image is dense in $\mathbb{R}/\mathbb{Z}$. The orbit of w is dense if and only if every finite word $v \in \{0,1\}^*$ appears as a subword in w. Equivalently, if the orbit is not dense, then there exists a finite word $v \in \{0,1\}^*$ that does not appear in w. Let $P_v \subset \{0,1\}^\omega$ be the set of sequences $w \in \{0,1\}^\omega$ not containing v as a subword. The set P_v is obviously closed.

If a finite word u contains v, then $C_u \cap P_v = \emptyset$. For every n the number of words $u \in \{0,1\}^{n|v|}$ not containing v as a subword is not more than $(2^{|v|}-1)^n$. It follows that the measure of P_v is not more than $\frac{(2^{|v|}-1)^n}{2^{|v|n}} = \left(\frac{2^{|v|}-1}{2^{|v|}}\right)^n \to 0$ as $n \to \infty$. Consequently, P_v has measure zero. It follows that the set of points w with non-dense orbits has measure zero, as a countable union $\bigcup_{v\in\{0,1\}^*} P_v$ of sets of measure zero. Consequently, the set of points $w \in \{0,1\}^*$ with dense orbit has measure one, which implies the same statement about $\mathbb{R}/\mathbb{Z}$.

Note also that P_v has empty interior, since it has measure zero. Consequently, P_v is a closed nowhere dense set, and hence $\bigcup_{v\in\{0,1\}^*} P_v$ is meager. The set $\Phi(P_v)$ is also closed as a continuous image of a compact set. Suppose that $\Phi(P_v)$ has non-empty interior. Then $\Phi^{-1}(\Phi(P_v))$ has non-empty interior. But for any subset $A \subset \{0,1\}^\omega$, the set $\Phi^{-1}(\Phi(A))$ is contained in the union of A with the countable set of sequences eventually equal to $000\ldots$ or to $111\ldots$ (since this is the set of points where Φ is not one-to-one). It follows that $\Phi^{-1}(\Phi(P_v))$ is contained in a union of a countable set with a nowhere dense closed set, hence it can not contain an open subset. Consequently, $\Phi(P_v)$ is closed and nowhere dense, and hence is meager, which implies that the set $\bigcup_{v\in\{0,1\}^*} \Phi(P_v)$ is meager. □

A particular corollary of Proposition 1.1.5 is that there exists a point $x \in \mathbb{R}/\mathbb{Z}$ with a dense f-orbit. Existence of a dense orbit is another irreducibility notion for topological dynamical systems, weaker than minimality. We call it (in the case of group actions) *topological transitivity*. We will see later (Proposition 2.1.17) that a group action $G \curvearrowright \mathcal{X}$ on a second-countable completely metrizable space is topologically transitive if and only if every two non-empty open subsets of the space of orbits $G\backslash\mathcal{X}$ have non-empty intersection.

The angle doubling map $f \circlearrowleft \mathbb{R}/\mathbb{Z}$ is very far from being minimal. Since the set of rational numbers with odd denominator is dense in $\mathbb{R}$, we have the following property.

Proposition 1.1.6. *The union of all finite cycles under the angle doubling map is dense in* $\mathbb{R}/\mathbb{Z}$.

Density of the union of finite cycles is usually one of the ingredients of different definitions of "chaos", see [**LY75, Dev89, AH03**].

1.1.3. Horseshoe and two-sided shift. Let S be a stadium-shaped region of $\mathbb{R}^2$ formed by a square and two half-discs shown on the left hand side part of Figure 1.3. Let Q be the square, and denote by D_1, D_2 the left and the right half-discs, as it is shown on the figure.

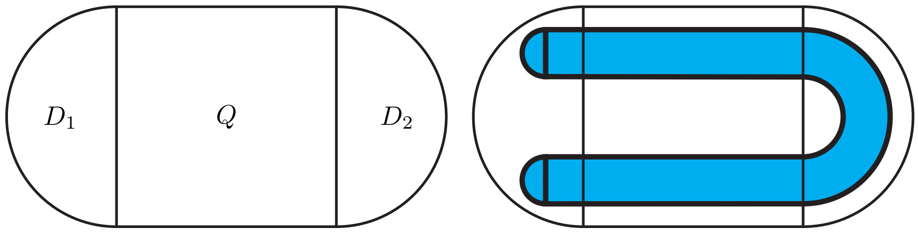

Figure 1.3. The horseshoe

Consider a continuous map $F \circlearrowleft S$ mapping S to the horseshoe shaped region shown on the right hand side part of Figure 1.3. We assume that F acts on D_1 as an affine map of the form $x \mapsto \lambda x + v_1$ for some $\lambda \in (0, 1/2)$ and $v_1 \in \mathbb{R}^2$, and on D_2 as an affine map of the form $x \mapsto -\lambda x + v_2$ for the same value of λ and for some $v_2 \in \mathbb{R}^2$.

We also assume that $F(Q) \cap Q$ is a disjoint union of two rectangles, and that the restriction of F onto the preimages of these rectangles are affine maps of the form $(x, y) \mapsto (Lx, \lambda y) + w_1$ and $(x, y) \mapsto (-Lx, -\lambda y) + w_2$ for some $w_1, w_2 \in \mathbb{R}^2$. It is not very important how F acts on the rest of S. It is not hard to see, though, that we can choose F so that it is a diffeomorphism from S to $F(S)$.

Note that $F \circlearrowleft D_1$ is a contraction, and has a unique fixed point $p = \frac{1}{1-\lambda} v_1$ such that $F^n(x) \to p$ as $n \to \infty$ for all $x \in D_1$. We also have $F(D_2) \subset D_1$, hence $F^n(x) \to p$ as $n \to \infty$ for all $x \in D_2$.

It follows that $F^n(x) \to p$ for $n \to \infty$ unless $F^n(x) \in Q$ for all $n \geqslant 0$. Consider the set $F^{-n}(Q)$ of points $x \in S$ such that $F^n(x) \in Q$. We define $F^{-0}(Q) = Q$. The set $F^{-1}(Q)$ is equal to the union of the two vertical rectangles V_1 and V_2 equal to the preimages of the two rectangles forming $F(Q) \cap Q$, see the top part of Figure 1.4. The map F acts on V_1 and V_2 by affine transformations with the linear parts $(x, y) \mapsto (Lx, \lambda y)$ and $(x, y) \mapsto (-Lx, -\lambda y)$, respectively.

The set $F^{-2}(Q)$ is equal to $F^{-1}(F^{-1}(Q) \cap F(Q))$. The horseshoe $F(Q)$ intersects with the two rectangles forming $F^{-1}(Q)$ in four rectangles (see the second row of Figure 1.4). Their preimages under F are four vertical rectangles (i.e., direct products of four segments in the horizontal side of Q with the full vertical side of Q).

Continuing this way, we conclude that $F^{-n}(Q)$ is equal to a union of 2^n vertical rectangles, and that each rectangle of $F^{-n}(Q)$ contains two rectangles of $F^{-(n-1)}(Q)$. See, for example, the third row of Figure 1.4, where $F^{-3}(Q)$ is described.

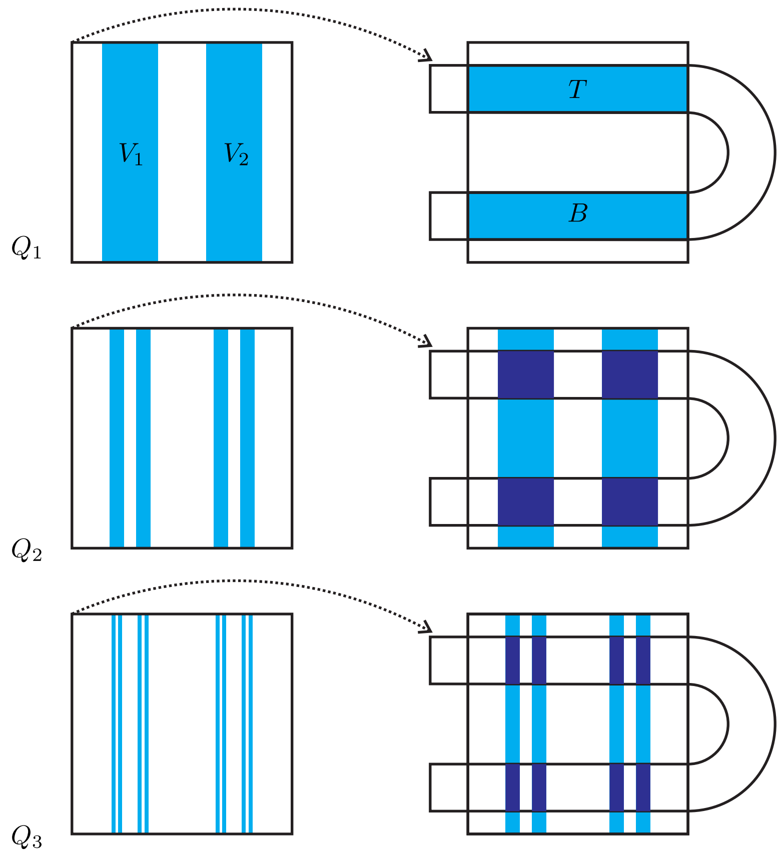

Figure 1.4. Sets $F^{-n}(Q)$

If the side of Q has length 1, then the width of the rectangles forming $F^{-n}(Q)$ is equal to λ^n. It follows that the set $\bigcap_{n\geqslant 0} F^{-n}(Q)$ of points that stay in Q under positive iterations of F is a direct product of a horizontal Cantor set with the vertical side of the square Q. Its connected components are vertical subsegments of the square Q.

We can label the rectangles of $F^{-n}(Q)$ by sequences of symbols T, B, where T and B stand for the top and the bottom rectangles of $F(Q) \cap Q$, respectively. Namely, if $x \in F^{-n}(Q)$, then the *itinerary* $X_1X_2\ldots X_n$ of x is given by the condition that $X_k = T$ (resp., $X_k = B$) if $F^k(x)$ belongs to the top (resp., bottom) rectangle of $F(Q) \cap Q$. Then the set of points with a given itinerary $X_1X_2\ldots X_n$ is one of the rectangles of $F^{-n}(Q)$. The two rectangles of $F^{-(n+1)}(Q)$ contained in the rectangle of points with itineraries $X_1X_2\ldots X_n$ are the rectangles of points with the itineraries $X_1X_2\ldots X_nT$ and $X_1X_2\ldots X_nB$. It follows that $\bigcap_{n\geqslant 0} F^{-n}(Q)$ is naturally homeomorphic to the direct product of $\{T, B\}^\omega$ with $[0, 1]$, where a set $\{X_1X_2\ldots\} \times [0, 1]$ for

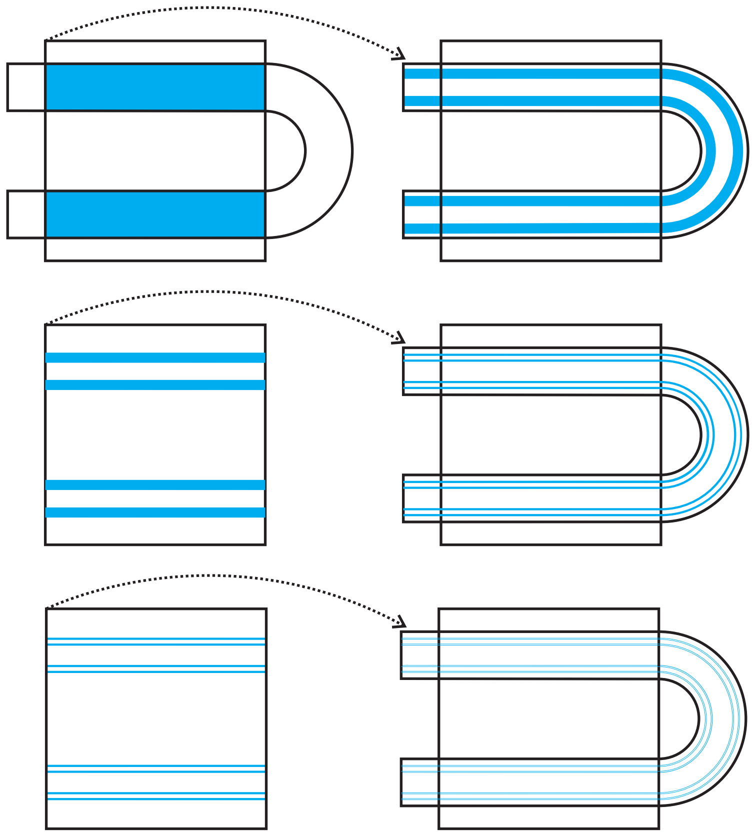

Figure 1.5. Ranges of iterations

$X_1X_2\ldots \in \{T, B\}^\omega$ is identified with the vertical subsegment of Q consisting of all points $x \in Q$ such that for every $n \geqslant 1$ the point $F^n(x)$ belongs to the rectangle X_n of $F(Q) \cap Q$.

The set $\bigcap_{n\geqslant 0} F^{-n}(Q)$ is the set of points x such that $F^n(x) \in Q$ for all $n \geqslant 0$. Note, however, that the map F is not surjective on this set, i.e., there are points that do not have preimages under F. Let us describe the sets $F^n(Q) \cap Q$. They are shown on Figure 1.5, for $n = 1, 2, 3$, inside the left-hand side squares. The right-hand side of the figure shows their images under F. We obviously have $F^n(x) \cap Q = F(F^{n-1}(x) \cap Q) \cap Q$. Similarly to the case of the sets Q_n, we see that $F^n(Q) \cap Q$ consists of 2^n horizontal rectangles of width L^{-n}. Each rectangle of $F^n(Q) \cap Q$ is labeled by itineraries $X_1X_2\ldots X_n \in \{T, B\}^n$ of its points under the map F^{-1}. Each rectangle $X_1X_2\ldots X_n$ of $F^n(Q) \cap Q$ contains two rectangles $TX_1X_2\ldots X_n$ and $BX_1X_2\ldots X_n$ of $F^{n+1}(Q) \cap Q$.

The intersection $\bigcap_{n\geqslant 1} F^n(Q) \cap Q$ is the direct product of a vertical Cantor set with the horizontal side of the square. The Cantor set is naturally identified with the space $\{T, B\}^{-\omega}$ of left-infinite sequences $\ldots X_2 X_1$ over the alphabet $\{T, B\}$.

Let us denote $W_+ = \bigcap_{n\geqslant 1} F^{-n}(Q)$, and $W_- = \bigcap_{n\geqslant 1} F^n(Q) \cap Q$. The set W_+ is a Cantor set of vertical segments in the square Q. The set W_- is a Cantor set of horizontal segments. We have $F(W_+) \subset W_+$ and $F(W_-) \supset W_-$. The set W_+ is equal to the set of points $x \in S$ such that $F^n(x) \in Q$ for all $n \geqslant 1$, i.e., precisely to the set of points such that $F^n(x)$ does not converge to the fixed point p. Note that if $x \in W_+$, then $F^n(x) \in \bigcap_{k=1}^{n} F^k(Q) \cap Q$, which implies that the distance from $F^n(x)$ to W_- converges to zero as $n \to \infty$. Note also that $F^n(x) \in W_+$ for all $n \geqslant 1$. Consequently, the distance from $F^n(x)$ to $W_+ \cap W_-$ converges to zero. The map F acts as a homeomorphism on $W = W + \cap W_-$.

We see that either $F^n(x)$ converges to the fixed point p (if $x \notin W_+$) or the distance from $F^n(x)$ to W goes to zero. It follows that W is an *attractor*.

Definition 1.1.7. Let $f \circlearrowleft \mathcal{X}$ be a continuous map, where $\mathcal{X}$ is compact. A compact set $C \subset \mathcal{X}$ is called an *attractor* if there exists an open set $U \supset C$ such that $C = \bigcap_{n\geqslant 1} f^n(U)$.

Recall that the vertical lines forming W_+ are labeled by right-infinite sequences $X_0 X_1 \ldots$ of the symbols $\{T, B\}$, while the horizontal lines forming W_- are labeled by the left-infinite sequences $\ldots X_{-2} X_{-1}$ over the same set of symbols. Every horizontal segment intersects every vertical segment in exactly one point of W; and, conversely, every point of W is the intersection point of a horizontal and a vertical segment. We see that the points of W can be labeled by bi-infinite sequence $\ldots X_{-2} X_{-1} X_0 X_1 \ldots$. The sequence $\ldots X_{-2} X_{-1} X_0 X_1 \ldots$ is the *itinerary* of the corresponding point x in the sense that $F^n(x)$ belongs to the rectangle labeled by X_n for every $n \in \mathbb{Z}$. It is easy to see that this correspondence between the points and their itineraries is a homeomorphism.

It also follows directly from the definition that this homeomorphism conjugates the action of F on W with the *two-sided shift* s given by

$$\mathsf{s}(\ldots X_{-2} X_{-1} X_0 X_1 \ldots) = \ldots Y_{-2} Y_{-1} Y_0 Y_1 \ldots, \qquad \text{where } Y_n = X_{n+1}.$$

Note that the map F acts in a neighborhood of every point of W as an affine transformation with linear part $(x, y) \mapsto (Lx, \lambda y)$ or $(x, y) \mapsto (-Lx, -\lambda y)$, i.e., it locally expands the horizontal and contracts the vertical directions of the square Q. This is an example of a system belonging to the class of *hyperbolic dynamical systems*, which is the subject of Section 1.4.

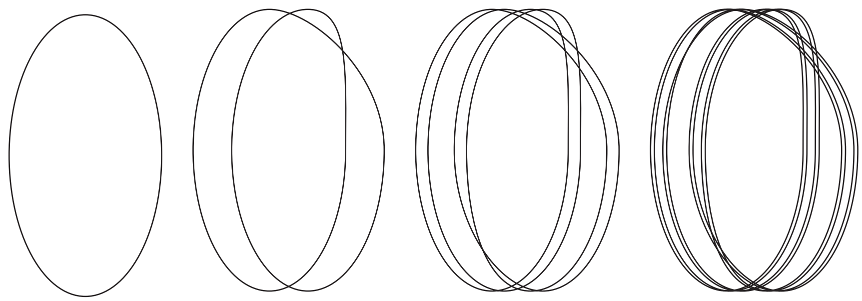

Figure 1.6. Inverse limit of angle doubling maps

For a finite alphabet X, the *two-sided full shift* is the dynamical system $\mathbb{Z} \curvearrowright X^{\mathbb{Z}}$, where the generator of $\mathbb{Z}$ acts by the homeomorphism $\mathsf{s}((x_n)_{n\in\mathbb{Z}}) = (y_n)_{n\in\mathbb{Z}}$, where $y_n = x_{n+1}$. Let us introduce on $X^{\mathbb{Z}}$ the metric

$$d((x_n), (y_n)) = 2^{-k},$$

where k is the maximal non-negative integer such that $x_{-k}x_{k+1}\ldots x_{k-1}x_k = y_{-k}y_{-k+1}\ldots y_{k-1}y_k$. Note that this metric agrees with the direct product topology on $X^{\mathbb{Z}}$.

Let $w = (x_n)$ be an arbitrary point of $X^{\mathbb{Z}}$. Denote by $W_+(w)$ the set of points (y_n) of $X^{\mathbb{Z}}$ such that $\ldots x_{-2}x_{-1} = \ldots y_{-2}y_{-1}$. The set $W_+(w)$ is naturally homeomorphic to X^ω, where the homeomorphism $W_+(w) \longrightarrow X^\omega$ erases the negative coordinates. Similarly, denote by $W_-(w)$ the set of points $(y_n) \in X^{\mathbb{Z}}$ such that the non-negative coordinates of (y_n) and (x_n) coincide. The set $W_-(w)$ is naturally homeomorphic to the space $X^{-\omega}$ of the left-infinite sequences $\ldots y_{-2}y_{-1}$. The space $X^{\mathbb{Z}}$ is naturally homeomorphic to the direct product $X^{-\omega} \times X^\omega$, where the homeomorphism $X^{-\omega} \times X^\omega \longrightarrow X^{\mathbb{Z}}$ concatenates the corresponding sequences. Consequently, we have a natural decomposition of $X^{\mathbb{Z}}$ into the direct product $W_-(w) \times W_+(w)$.

Note that if $w_1, w_2 \in W_+(w)$, then $d(\mathsf{s}^n(w_1), \mathsf{s}^n(w_2)) \to 0$ as $n \to \infty$ for any metric on $X^{\mathbb{Z}}$. Similarly, if $w_1, w_2 \in W_-(w)$, then $d(\mathsf{s}^{-n}(w_1), \mathsf{s}^{-n}(w_2)) \to 0$ as $n \to \infty$.

1.1.4. Smale-Williams solenoid and the adding machine. Consider the backwards sequence

$$\mathbb{R}/\mathbb{Z} \xleftarrow{f} \mathbb{R}/\mathbb{Z} \xleftarrow{f} \mathbb{R}/\mathbb{Z} \xleftarrow{f} \mathbb{R}/\mathbb{Z} \xleftarrow{f} \cdots$$

of the angle doubling maps $f : x \mapsto 2x \pmod 1$ from 1.1.2, see Figure 1.6.

Let $\mathcal{S}$ be the inverse limit of this sequence. By definition, it is the set of sequences $(x_1, x_2, \ldots)$ of points of $\mathbb{R}/\mathbb{Z}$ such that $f(x_{n+1}) = x_n$ for all n. The topology is induced from the direct product topology on $(\mathbb{R}/\mathbb{Z})^\omega$. Note that the map

$$\widehat{f}(x_1, x_2, \dots) = (f(x_1), f(x_2), \dots)$$

is a well defined homeomorphism of $\mathcal{S}$, where the inverse homeomorphism is the shift $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$.

Definition 1.1.8. Let $f \circlearrowleft \mathcal{X}$ be a continuous map. Its *natural extension* is the homeomorphism $\widehat{f}$ induced by f on the inverse limit of the sequence

$$\mathcal{X} \xleftarrow{f} \mathcal{X} \xleftarrow{f} \mathcal{X} \xleftarrow{f} \cdots$$

The natural extension of the angle doubling map is called the *Smale-Williams solenoid*, see [**Sma67**] page 788. Note that $\mathcal{S}$ is a compact abelian topological group, since the map $x \mapsto 2x$ is an endomorphism of the compact abelian group $\mathbb{R}/\mathbb{Z}$, and $\mathcal{S}$ is the inverse limit with respect to these endomorphisms.

Let us extend the symbolic representation of the angle doubling map described in 1.1.2 to the solenoid. Let $\xi = (x_1, x_2, \dots)$ be a point of $\mathcal{S}$. If $.b_1b_2\dots$ is a binary representation of x_n, then the binary representation of $x_{n-1} = f(x_1)$ is $.b_2b_3\dots$. It follows that the coordinates of ξ are encoded by sequences of the form

$$(.a_1a_2\dots, .a_0a_1a_2\dots, .a_{-1}a_0a_1a_2\dots, \dots).$$

It is natural then to represent ξ by the bi-infinite binary sequence

$$\dots a_{-2}a_{-1}a_0.a_1a_2\dots,$$

where $.a_{-n+2}a_{-n+3}\dots$ is the binary sequence representing $x_n \in \mathbb{R}/\mathbb{Z}$.

We have the same description of the identification of the sequences representing the same point of the solenoid as in the case of the circle. Namely, two binary sequences $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$ represent the same point of $\mathcal{S}$ if and only if they are either equal to each other, equal to $000\dots$ and $111\dots$, or are of the form $\dots a_{n-1}a_n10000\dots$ and $\dots a_{n-1}a_n01111\dots$ for some $n \in \mathbb{Z}$. One can show that the quotient of $\{0,1\}^{\mathbb{Z}}$ by this identification rule is homeomorphic to $\mathcal{S}$, and that the map on $\mathcal{S}$ induced by the two-sided shift $\mathsf{s} \circlearrowleft \{0,1\}^{\mathbb{Z}}$ is topologically conjugate to the natural extension of the angle doubling map.

Let ξ be a point of $\mathcal{S}$ represented by a binary sequence $\dots a_{-2}a_{-1}a_0.a_1a_2\dots$. Let us call the sequences $\dots a_{-2}a_{-1}a_0$ and $.a_1a_2\dots$ the *integral* and *fractional parts* of ξ, respectively. The identification rule for representation of points of $\mathcal{S}$ allows us to identify the fractional part with a point of the real interval $[0,1]$ via the natural map

$$.a_1a_2\dots \mapsto \sum_{i=0}^{\infty} a_i 2^{-i}.$$

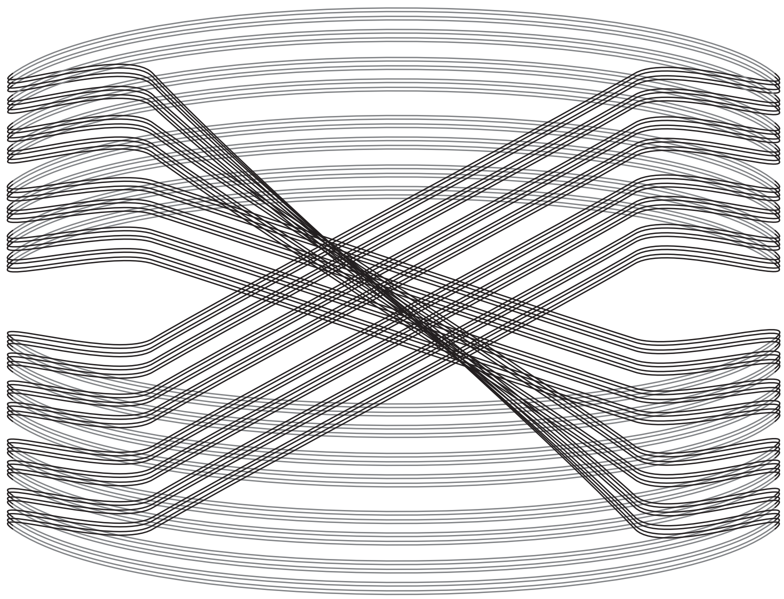

Figure 1.7. The solenoid as a mapping torus

The integral part of ξ is naturally identified with a dyadic integer via

$$\dots a_{-2}a_{-1}a_0 = \sum_{i=0}^{\infty} a_{-i}2^i \in \mathbb{Z}_2.$$

Note that the integral and the fractional parts of $\xi \in \mathcal{S}$ are uniquely defined, except for the points which can have fractional parts 0 or 1. Then a point can have fractional part 1 and integral part $a \in \mathbb{Z}_2$, or fractional part 0 and integral part $a + 1 \in \mathbb{Z}_2$.

It follows that $\mathcal{S}$ can be constructed by taking the direct product $\mathbb{Z}_2 \times [0, 1]$, and then identifying every point $(a, 1)$ with $(a + 1, 0)$. See Figure 1.7 for a schematical representation of this construction.

In other words, the solenoid $\mathcal{S}$ is the *mapping torus* of the transformation $a \mapsto a + 1$ of the ring $\mathbb{Z}_2$ of dyadic integers.

Definition 1.1.9. Let $f \circlearrowright \mathcal{X}$ be a homeomorphism. Its *mapping torus* is the quotient of the space $\mathcal{X} \times [0, 1]$ by the identification $(x, 1) \sim (f(x), 0)$.

The transformation $\tau : a \mapsto a + 1$ of $\mathbb{Z}_2$ is called the *adding machine* or *odometer*. Its action on the binary sequences is given by the classical rule of adding one to a binary integer:

$$\tau(\dots a_2a_1a_0) = \begin{cases} \dots a_2a_1 1 & \text{if } a_0 = 0; \\ \tau(\dots a_2a_1)0 & \text{if } a_0 = 1. \end{cases}$$

Proposition 1.1.10. *The odometer is a minimal homeomorphism of $\mathbb{Z}_2$.*

See Definition 1.1.2 for the definition of a minimal action.

Proof. The group $\mathbb{Z}_2$ is the inverse limit (as a topological group) of the groups $\mathbb{Z}/2^n\mathbb{Z}$ with respect to the natural epimorphisms $1+2^{n+1}\mathbb{Z} \mapsto 1+2^n\mathbb{Z}$. The odometer is the inverse limit of the maps $k \mapsto k+1$ acting on each $\mathbb{Z}/2^n\mathbb{Z}$. These maps are transitive cycles on $\mathbb{Z}/2^n\mathbb{Z}$, and every orbit of the action of the odometer on $\mathbb{Z}_2$ is mapped to these transitive cycles, which implies that every orbit is dense. □

As a corollary we get the following topological property of the solenoid.

Proposition 1.1.11. *Every path connected component of $\mathcal{S}$ is dense. In particular, $\mathcal{S}$ is connected but has uncountably many path connected components.*

Note that $\widehat{f}$ multiplies the fractional parts by 2, which is a locally expanding map on $[0,1]$. It is also multiplying the integer parts by 2, which is a contracting map on $\mathbb{Z}_2$. We see that, similarly for the two-sided shift, neighborhoods of points of the solenoid are decomposed into a direct product of an expanding and contracting directions of the dynamical system (i.e., is also a *hyperbolic dynamical system*, similarly to the two-sided shift).

The solenoid can be also realized as an attractor of a diffeomorphism, see [**Sma67**, page 788] or [**BS02**, Section 1.9.]. Consider the region R inside a torus in $\mathbb{R}^3$ obtained by rotating a disc D around a line in its plane not intersecting D. We call the images of D under the rotations *meridional discs* of R. Let $F \circlearrowleft R$ be a map extendable to a diffeomorphism on a neighborhood of R such that $F(R)$ is the region inside a torus winding twice around R, as it is shown on Figure 1.8. We assume that F maps every meridional disc of R to a smaller disc contained in a meridional disc of R. We also assume that F uniformly contracts the distances inside the meridional discs, and that it uniformly expands the distances in the direction perpendicular to the meridional discs.

More explicitly, let us introduce a local coordinate system (x, y, θ) inside the torus, where x, y, for $x^2 + y^2 < 1$ are the Cartesian coordinates in the disc D, and $\theta \in (0, 2\pi)$ is the angle of rotation of the disc D around the axis of the torus. We can define then F, for example, as $F(x, y, \theta) = \left(\frac{x}{4} + \frac{\cos t}{2}, \frac{y}{4} + \frac{\sin t}{2}, 2\theta\right)$. One can show then that the intersection of the ranges of F^n is homeomorphic to the solenoid, and that the restriction of F onto the intersection is topologically conjugate to the natural extension of the angle doubling map.

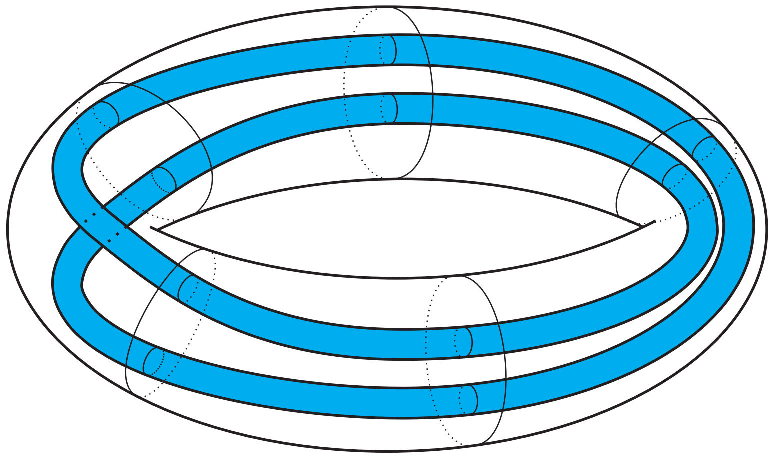

Figure 1.8. Solenoid map

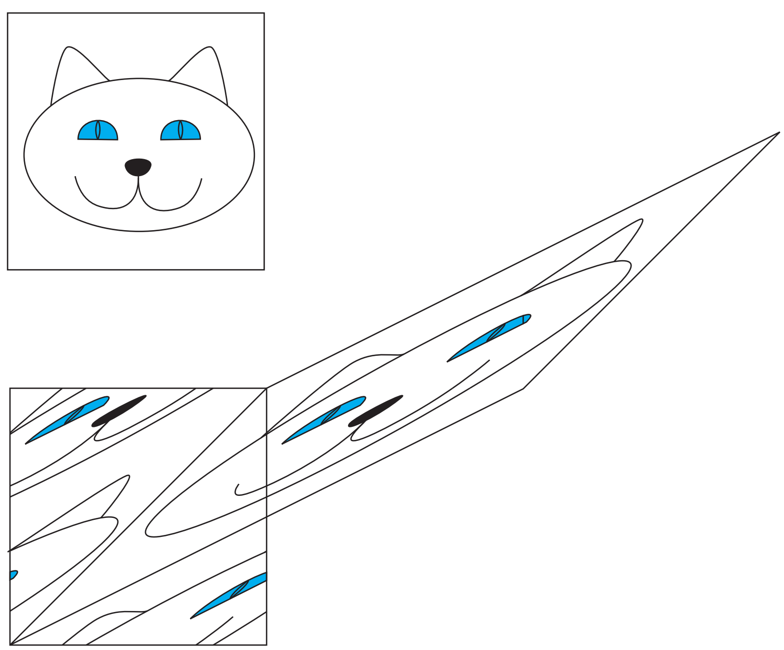

Figure 1.9. Arnold's cat

1.1.5. Anosov diffeomorphisms. Another classical example of a hyperbolic dynamical system is a hyperbolic automorphism of a torus, known in the literature as "Arnod's cat map".

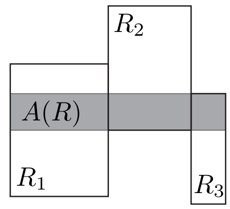

Figure 1.10. Markov condition

Consider the map $f \circlearrowleft \mathbb{R}^2/\mathbb{Z}^2$ induced by the linear transformation of $\mathbb{R}^2$ with the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Since $\det A = 1$, the map f is a homeomorphism (and an automorphism of the group $\mathbb{R}^2/\mathbb{Z}^2$).

The characteristic polynomial of A is $\lambda^2 - 3\lambda + 1 = 0$, hence its eigenvalues are $\lambda = \frac{3+\sqrt{5}}{2}$ and $\lambda^{-1} = \frac{3-\sqrt{5}}{2}$. Note that $\lambda > 1$ and $0 < \lambda^{-1} < 1$. The eigenvectors of A are $\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$. Note that they are orthogonal (A is symmetric).

An *A-rectangle* is a rectangle $R \subset \mathbb{R}^2$ with the sides parallel to the eigenvectors of A such that the quotient map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$ restricted to the interior of R is injective. The images of A-rectangles in $\mathbb{R}^2/\mathbb{Z}^2$ are also called A-rectangles. If R is an A-rectangle, and $x \in R$, then we denote by $W_s(x, R)$ the maximal segment inside R containing x and parallel to the contracting eigenspace (i.e., to the eigenspace of the eigenvalue $\lambda^{-1} < 1$), and by $W_u(x, R)$ we denote the maximal segment inside R containing x and parallel to the expanding eigenspace.

Definition 1.1.12. Let $\mathcal{R}$ be a finite set of A-rectangles $R \subset \mathbb{R}^2/\mathbb{Z}^2$ satisfying the following conditions:

(1) The interiors of the rectangles $R \in \mathcal{R}$ are disjoint, and union of their closures is the whole torus.

(2) If x belongs to the interior of $R_1 \in \mathcal{R}$ and $f(x)$ belongs to the interior of $R_2 \in \mathcal{R}$, then

$$A(W_s(x, R_1)) \subset W_s(A(x), R_2)$$

and

$$W_u(A(x), R_2) \subset A(W_u(x, R_1)).$$

Then the set $\mathcal{R}$ is called a *Markov partition* of $A \circlearrowleft \mathbb{R}^2/\mathbb{Z}^2$.

For example, we can construct a Markov partition of $A \circlearrowleft \mathbb{R}^2/\mathbb{Z}^2$ in the following way. Consider the square formed by the lines parallel to the

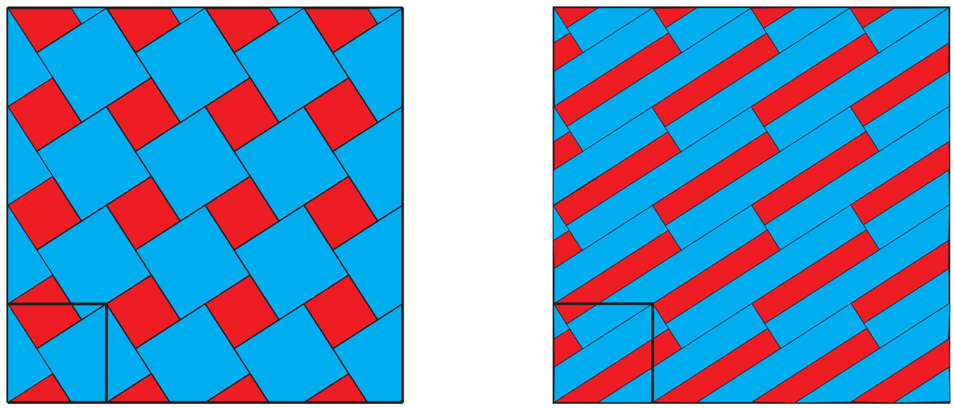

Figure 1.11. Markov partition

eigenvectors with one vertex $(0, 1)$, and one side containing $(0, 0)$. Consider all its translations by the elements of $\mathbb{Z}^2 \subset \mathbb{R}^2$. See the left-hand side of Figure 1.11, where they are shown blue. The components of the part of $\mathbb{R}^2$ not covered by these squares are also squares. Their sides are parallel to the eigenvectors of A and form one $\mathbb{Z}^2$-orbit (red on Figure 1.11). Consider the images of the two constructed $\mathbb{Z}^2$-orbits of squares in the torus. We get a partition of the torus $\mathbb{R}^2/\mathbb{Z}^2$ into two squares with sides parallel to the eigenvectors of A. Their images under the action of A are shown on the right-hand half of Figure 1.11. It is easy to check that the constructed partition of the torus satisfies the conditions of Definition 1.1.12.

Let $\mathcal{R}$ be a Markov partition of $A \circlearrowright \mathbb{R}^2/\mathbb{Z}^2$. Then for every $R \in \mathcal{R}$ the intersections of the form $A(R) \cap R_i$ for $R_i \in \mathcal{R}$ subdivide the rectangle $A(R)$ into a finite number of disjoint sub-rectangles, by cutting the expanding direction into pieces. For each of these sub-rectangles R' and $x \in R'$ we have $W_s(x, R) = W_s(x, R')$, see Figure 1.10. Note that an intersection $A(R) \cap R_i$ can consist of several sub-rectangles of $A(R)$.

Consider the oriented graph with the vertices identified with the elements of $\mathcal{R}$ and an edge from R to R_i for every rectangular piece of a non-empty intersection $A(R) \cap R_i$. Let us call this graph the *structural graph* of the Markov partition. For an edge e of the structural graph, denote by R_e the corresponding (closed) piece of an intersection $A(R_1) \cap R_2$ for $R_i \in \mathcal{R}$ (so that R_1 and R_2 correspond to the beginning and end of the edge e, respectively).

For example, the structural graph of the Markov partition from Figure 1.11 is shown on Figure 1.12.

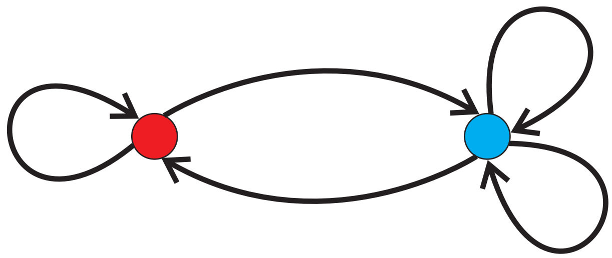

Figure 1.12. Structural graph

Denote by $\mathcal{M}$ the set of all bi-infinite paths in the structural graph. It is a *subshift*, i.e., a closed shift-invariant subset of the full shift E^ω, where E is the set of edges of the graph.

Proposition 1.1.13. *For every infinite path $w = \ldots e_{-1}e_0e_1 \ldots \in \mathcal{M}$ there exists exactly one point $\phi(w) \in \mathbb{R}^2/\mathbb{Z}^2$ such that $A^n(\phi(w)) \in R_{e_n}$ for all $n \in \mathbb{Z}$. The map $\phi : \mathcal{M} \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$ is a surjective semi-conjugacy from the subshift $\mathsf{s} \circlearrowleft \mathcal{M}$ to $A \circlearrowleft \mathbb{R}^2/\mathbb{Z}^2$.*

Proof. Let $\ldots a_{-1}a_0a_1 \ldots \in \mathcal{M}$. It follows from the definition of the map ϕ that the image of the cylindrical set $C_{a_0a_1\ldots a_n} = \{\ldots e_{-1}e_0e_1 \ldots : e_i = a_i, i = 0, 1, \ldots, n\}$ is equal to the rectangle $R_{a_0} \cap A^{-1}(R_{a_1}) \cap A^{-2}(R_{a_2}) \cap \ldots \cap A^{-n}(R_{a_n})$. It follows that $\phi(C_{a_0a_1\ldots a_n})$ is a rectangle R such that the side $W_s(x, R)$ is equal to the side $W_s(x, R_{a_0})$, and the side $W_u(x, R)$ has length not more than $K\lambda^{-n}$, where K is the maximum length of the sides $W_u(x, R_i)$ for all rectangles R_i of the Markov partition. It follows that $\phi(C_{a_0a_1\ldots a_n})$, $n \geqslant 0$, is a descending sequence of compact rectangles such that one side (parallel to the contracting direction) stays the same, while the length of the other side (parallel to the expanding direction) is exponentially decreasing.

Similarly, the set $\phi(\{\ldots e_{-1}e_0e_1 \ldots : e_i = a_i, i = 0, -1, \ldots, -n\})$ is a subrectangle R of R_{a_0} such $W_u(x, R)$ is equal to the side $W_u(x, R_{a_0})$, while the contracting sides $W_s(x, R)$ form a nested sequence of subintervals of $W_s(x, R_{a_0})$ of exponentially decreasing lengths.

It follows that for every finite path $a_{-n}a_{-n+1} \ldots a_{n-1}a_n$ in the structural graph the set $\phi(\{\ldots e_{-1}e_0e_1 \ldots : e_i = a_i, -n \leqslant i \leqslant n\})$ is a rectangle with length both sides less than $K\lambda^{-n}$ for some fixed $K > 0$ and for $\lambda > 1$. This implies that ϕ is continuous. It is easy to check that it is onto and a semi-conjugacy. $\square$

The encoding of points of the torus by bi-infinite sequences is analogous to the encodings of the systems considered in 1.1.3 and 1.1.4. In fact, they are examples of a general construction, which will be studied in 1.4.11.

1.2. Subshifts

1.2.1. Definition and examples. Let H be a semigroup, and let X be a finite alphabet. We always assume that $|\mathsf{X}| \geqslant 2$. Consider the space X^H with the topology of the direct product of discrete sets. In other terms, X^H is the set of all maps $f : H \longrightarrow \mathsf{X}$, and the topology is defined by the basis of open *cylindrical sets*:

$$C_{f_0:A\longrightarrow\mathsf{X}} = \{f : H \longrightarrow \mathsf{X} \ : \ f|_A = f_0\},$$

where A runs through the set of all finite subsets of H, and f_0 runs through the set of all maps $A \longrightarrow \mathsf{X}$.

Note that $C_{f_0:A\longrightarrow\mathsf{X}}$ is also closed, since it is equal to the complement of the union of the sets of the form $C_{f:A\longrightarrow\mathsf{X}}$ for all $f \in \mathsf{X}^A$ such that $f \neq f_0$.

If H is the semigroup $\mathbb{N}$ of non-negative integers, we we represent the elements $f \in \mathsf{X}^{\mathbb{N}}$ as sequences $x_0x_1x_2\ldots = f(0)f(1)f(2)\ldots$. Similarly, for $H = \mathbb{Z}$, we write the elements of $\mathsf{X}^{\mathbb{Z}}$ as bi-infinite sequences $\ldots x_{-2}x_{-1} \,.\, x_0x_1\ldots$, where x_n is the value of the element at n. We denote by a dot the place between the coordinates number -1 and 0.

It follows directly from the definitions that for every element $h \in H$ the map

$$f \mapsto h \cdot f,$$

where $h \cdot f$ is defined by

$$h \cdot f(x) = f(xh),$$

is a (left) action of H on X^H by continuous maps. We call the dynamical system $H \curvearrowright \mathsf{X}^H$ the *(full) H-shift*.

For example, the action of the generator 1 of $\mathbb{N}$ on $\mathsf{X}^{\mathbb{N}}$ is given in terms of sequences by

$$x_0x_1\ldots \mapsto x_1x_2\ldots,$$

while the action of the generator 1 of $\mathbb{Z}$ on $\mathsf{X}^{\mathbb{Z}}$ is given by

$$\ldots x_{-2}x_{-1} \,.\, x_0x_1\ldots \mapsto \ldots x_{-1}x_0 \,.\, x_1x_2\ldots.$$

These maps are called the (full) *one-sided* and *two-sided* shifts, respectively.

Definition 1.2.1. An *H-subshift* is a dynamical system $H \curvearrowright \mathcal{X}$, where $\mathcal{X}$ is a closed H-invariant subset of the full shift space X^H. Here a subset $\mathcal{Y} \subset \mathcal{X}$ is said to be H-invariant if $h\mathcal{Y} \subset \mathcal{Y}$ for every $h \in H$.

One way to define a subshift is to take an arbitrary closed subset $\mathcal{F} \subset \mathcal{X}$, and consider the intersection $\bigcap_{h\in H} h\mathcal{F}$. Another standard way is choose an element f of X^H and take the closure of the H-orbit $\{h \cdot f \ : \ h \in H\}$.

Definition 1.2.2. Let $\mathcal{X} \subset \mathsf{X}^{\mathbb{Z}}$ be a $\mathbb{Z}$-subshift. Its *language* is the set $W_{\mathcal{X}}$ of all finite words $v = a_0 a_1 \dots a_n$ such that there exists $(x_n)_{n\in\mathbb{Z}} \in \mathcal{X}$ such that $x_i = a_i$ for all $i = 0, 1, \dots, n$.

The following is straightforward.

Proposition 1.2.3. *Let $W_{\mathcal{X}}$ be the language of a subshift $\mathcal{X} \subset \mathsf{X}^{\mathbb{Z}}$. A sequence $(x_n)_{n\in\mathbb{Z}}$ belongs to $\mathcal{X}$ if and only if $x_n x_{n+1} \dots x_m$ belongs to $W_{\mathcal{X}}$ for all $n \leqslant m$.*

The following is a classical fact (see, for example [**MH38**, Theorem 7.2]). We say that a subshift is *minimal* if the $\mathbb{Z}$-action on it is minimal, i.e., if all orbits of the shift are dense in it, see Definition 1.1.2. Note that in this context the word "minimal" has the usual sense: a subshift is minimal if and only if it does not contain a proper non-empty subshift.

Proposition 1.2.4. *Let $w \in \mathsf{X}^{\mathbb{Z}}$. The closure of the orbit $\{\mathsf{s}^n(w) : n \in \mathbb{Z}\}$ of w in $\mathsf{X}^{\mathbb{Z}}$ is a minimal subshift if and only if for every finite subword v of w there exists $R > 0$ such that every subword of w of length R contains v as a subword.*

In other words, the closure of the orbit of w is a minimal subshift if and only if every finite subword of w appears in w infinitely often with gaps of uniformly bounded length.

Proof. Let $\mathcal{C}$ be the closure of the orbit of w. Suppose that $\mathsf{s} \circlearrowright \mathcal{C}$ is minimal, and let v be a subword of w of length m. Then for every $w' \in \mathcal{C}$ there exists $n \in \mathbb{Z}$ such that $\mathsf{s}^n(w')$ belongs to the cylindrical set $C_v = \{\dots x_{-1}x_0x_1 \dots \in \mathcal{C} : x_1x_2\dots x_m = v\}$, as the shift orbit of w' is dense in $\mathcal{C}$. In other words, the sets of the form $\mathsf{s}^n(C_v)$ for $n \in \mathbb{Z}$ cover $\mathcal{C}$. But $\mathcal{C}$ is compact, so there exists a finite cover $\mathsf{s}^{n_1}(C_v), \mathsf{s}^{n_2}(C_v), \dots, \mathsf{s}^{n_k}(C_v)$ of $\mathcal{C}$. It follows that for every $n \in \mathbb{Z}$ there exists n_i such that $\mathsf{s}^{n-n_i}(w) \in C_v$. This exactly means that the word v appears in w on a uniformly bounded distance from any place in w.

The converse statement (if every finite subword of w appears in w with uniformly bounded gaps, then the closure of the orbit of w is a minimal shift) is straightforward, and is left as an exercise. □

1.2.2. Expansive actions.

Definition 1.2.5. An action of a semigroup H on a metric space $\mathcal{X}$ is said to be *expansive* if there exists $\delta > 0$ such that if for all $g \in H$ we have $d(g(x), g(y)) < \delta$ then $x = y$.

The definition of expansivity of an action does not depend on the metric if $\mathcal{X}$ is compact. Namely, we have the following equivalent definition.

Definition 1.2.6. Let $\mathcal{X}$ be a compact space. A neighborhood $U \subset \mathcal{X} \times \mathcal{X}$ of the diagonal $\{(x, x) \; : \; x \in \mathcal{X}\}$ is called an *expansion entourage* for an action $H \curvearrowright \mathcal{X}$ if $(g(x), g(y)) \in \overline{U}$ for all $g \in H$ implies $x = y$.

Here $\overline{U}$ denotes the closure of U in $\mathcal{X} \times \mathcal{X}$. It is easy to check that an action $H \curvearrowright \mathcal{X}$ on a compact metric space $\mathcal{X}$ is expansive if and only if it has an expansion entourage.

Let U be a closed expansion entourage for an action of H on a compact space $\mathcal{X}$. For a finite subset $A \subset H$, denote by U_A the set of pairs $(x, y) \in \mathcal{X} \times \mathcal{X}$ such that $(h(x), h(y)) \in U$ for all $h \in A$. In other words,

$$U_A = \bigcap_{h \in A} h^{-1}(U),$$

where H acts on $\mathcal{X}^2$ diagonally.

Lemma 1.2.7. *For every neighborhood of the diagonal $W \subset \mathcal{X} \times \mathcal{X}$ there exists a finite set $A \subset H$ such that $U_A \subset W$.*

Proof. It is enough to prove the lemma for the case when W is open. Then $\mathcal{X}^2 \setminus W$ is compact, and for every $(x, y) \in \mathcal{X}^2 \setminus W$ there exists $h \in H$ such that $(h(x), h(y)) \notin U$. The set $\mathcal{X}^2 \setminus h^{-1}(U)$ is open, so we have an open cover of $\mathcal{X}^2 \setminus W$ by sets of the form $\mathcal{X}^2 \setminus h^{-1}(U)$. There exists a finite subcover, and its union is equal to $\mathcal{X}^2 \setminus U_A$ for some finite $A \subset H$. It follows that $U_A \subset W$. $\square$

Expansive dynamical systems are chaotic in the sense that they exhibit sensitive dependence on the "initial conditions". Namely, however small is the distance between the initial positions of two points x, y, as soon as $x \neq y$ there exists a moment $g \in H$ such that the distance between $g(x)$ and $g(y)$ is at least δ. This and similar conditions are ingredients of various definitions of chaos, see [**LY75, Dev89, AH03**].

Example 1.2.8. An action by isometries is obviously not expansive. In particular, a rotation of the circle and the odometer (see its definition before Proposition 1.1.10) are examples of non-expansive action.

Example 1.2.9. The two-sided shift, the solenoid, and the Arnold's Cat map are expansive. This follows from the corresponding local direct product decomposition into expanding and contracting directions. Namely, any two sufficiently close points x, y belong to one rectangle of such a decomposition. If $x \neq y$, then either their projections onto the expanding side or the projections onto the contracting side of the rectangle are different. Then the distance between the corresponding projections of the points $f^n(x), f^n(y)$ will grow exponentially (if it is small) for positive or negative values of n, respectively. This proves that $f^n(x)$ and $f^n(y)$ can not be close to each other for all $n \in \mathbb{Z}$. We will study this approach and this proof in more detail in 1.4.7.

The following description of expansive actions is a classical result, see....

Theorem 1.2.10. *An action $H \curvearrowright \mathcal{X}$ of a semigroup H on a compact totally disconnected space $\mathcal{X}$ is expansive if and only if the dynamical system $H \curvearrowright \mathcal{X}$ is topologically conjugate to an H-subshift.*

Proof. Let δ be as in Definition 1.2.5. Consider a finite partition $\mathcal{U}$ of $\mathcal{X}$ into clopen sets of diameters less than δ. For a point $x \in \mathcal{X}$, define its *itinerary* as the map $I_x : H \mapsto \mathcal{U}$ given by the rule

$$h(x) \in I_x(h).$$

The map $x \mapsto I_x$ is a continuous map from $\mathcal{X}$ to $H^{\mathcal{U}}$, since it is locally constant on each coordinate. It follows that its range is a compact subset of $H^{\mathcal{U}}$. We have $gh(x) \in I_x(gh)$ and $gh(x) \in I_{h(x)}(g)$. It follows that $h \cdot I_x(g) = I_x(gh) = I_{h(x)}(g)$, so that $h \cdot I_x = I_{h(x)}$, i.e., that the map $I. : \mathcal{X} \longrightarrow H^{\mathcal{U}}$ is H-equivariant.

The map $x \mapsto I_x$ is also injective, since $I_x = I_y$ implies that the distance from $h(x)$ to $h(y)$ is less than δ for every $h \in H$. It follows that $x \mapsto I_x$ is a homeomorphism from $\mathcal{X}$ to its images, since every injective continuous map from a compact space to a Hausdorff space is a homeomorphism onto the image. □

1.2.3. Shifts of finite type. Let $\mathcal{P} = \{C_{f_i : A_i \longrightarrow \mathsf{X}}\}_{i \in I}$ be a collection of cylindrical subsets of X^H. Consider now the set of all sequences $w \in \mathsf{X}^H$ such that configurations $f_i : A_i \longrightarrow \mathsf{X}$ do not appear in any shifts of w. Namely, consider the set $\mathcal{X}_{\mathcal{P}}$ of elements $w \in \mathsf{X}^H$ such that $h \cdot w|_{A_i} \neq f_i$ for all $i \in I$. We say that $\mathcal{X}_{\mathcal{P}}$ is the shift defined by the set of *prohibited configurations* $\mathcal{P}$. Note that $\mathcal{X}_{\mathcal{P}}$ is closed and H-invariant, i.e., that it is a subshift. Every subshift can be defined by some collection of prohibited configurations.

Definition 1.2.11. A subshift $\mathcal{X} \subset \mathsf{X}^H$ is a *shift of finite type* if there exists a finite set of prohibited configurations defining $\mathcal{X}$.

In particular, a $\mathbb{Z}$ or $\mathbb{N}$-shift $\mathcal{X}$ is defined by a finite collection $A \subset \mathsf{X}^*$ of finite *prohibited words*. A sequence w belongs to the corresponding subshift if and only if no subword of w belongs to A.

Note that a subset of X^H is a union of a finite number of cylindrical sets if and only if it is clopen. It follows that a subshift $\mathcal{X} \subset \mathsf{X}^H$ is of finite type if and only if there exists a clopen set U such that $\mathcal{X} = \mathsf{X}^H \setminus \bigcup_{h \in H} h^{-1}(U)$, where $h^{-1}(U)$ denotes the full preimage of U under the action of h. Replacing U by $V = \mathsf{X}^H \setminus U$, we get another definition of shifts of finite type.

Lemma 1.2.12. *A subshift $\mathcal{X} \subset \mathsf{X}^H$ is of finite type if and only if there exists a clopen set U such that $\mathcal{X} = \bigcap_{h \in H} h^{-1}(U)$.*

More generally, we adopt the following definition.

Definition 1.2.13. Let $\mathcal{X} \subset \mathsf{X}^H$ be a subfshift. A subshift $\mathcal{X}_1 \subset \mathcal{X}$ is of *relative finite type in $\mathcal{X}$* if there exists a clopen subset $U \subset \mathcal{X}$ such that $\mathcal{X}_1 = \bigcap_{h \in H} h^{-1}(U)$.

Let G be a group, and let X be a finite alphabet. For a finite subset $B \subset G$, consider the alphabet X^B and the map $\beta_B : \mathsf{X}^G \longrightarrow (\mathsf{X}^B)^G$ given by the equality

$$\beta_B(f)(g) = (g \cdot f)|_B.$$

Proposition 1.2.14. *The map β_B is a G-equivariant homeomorphic embedding.*

Proof. We have

$$\beta_B(h \cdot f)(g) = ((gh) \cdot f)|_B = \beta_B(f)(gh) = h \cdot \beta_B(f)(g),$$

hence β_B is G-equivariant.

Choose an element $g_0 \in B$. Then the value of $\beta_B(f)(g_0^{-1}g)$ as a function $B \longrightarrow G$ on the point g_0 is equal to $f(g)$. It follows that $f \in \mathsf{X}^G$ can be reconstructed from $\beta_B(f)$, i.e., that β_B is injective. It is obviously continuous and the spaces X^G and $(\mathsf{X}^B)^G$ are compact and Hausdorff, hence β_B is a homeomorphic embedding. □

Definition 1.2.15. The map $\beta_B : \mathsf{X}^G \longrightarrow (\mathsf{X}^B)^G$ is called the *block map* (or *sliding block code*) with the *window* B.

For example, in the case $G = \mathbb{Z}$, it is natural to consider a window of the form $\{0, 1, \ldots, n-1\}$. Then X^B is the set of words of length n, and the block map transforms a sequence $\ldots x_{-1}.x_0x_1\ldots$ over the alphabet X to the sequence

$$\ldots (x_{-1}x_0 \ldots x_{n-2}).(x_0x_1 \ldots x_{n-1})(x_1x_2 \ldots x_n) \ldots$$

over the alphabet X^n. It is easy to see that the block map is a shift-equivariant homeomorphic embedding.

Let G be a group generated by a finite set S. Choose for every generator $s \in S$ a set $A_s \subset \mathsf{X}^2$ of pairs of letters, and define the corresponding *topological Markov shift* as the set of all G-sequences $f \in \mathsf{X}^G$ such that

$$(f(g), f(sg)) \in A_s$$

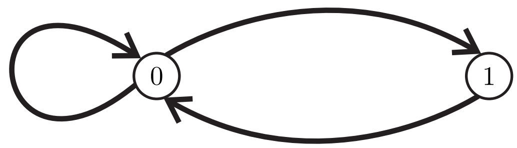

Figure 1.13. Fibonacci s.f.t.

for every $g \in G$ and $s \in S$. It is easy to see that this is a shift of finite type.

For example, if $G = \mathbb{Z}$, then it is natural to consider $S = \{1\}$, and then the corresponding topological Markov shifts are given by a set $A \subset \mathsf{X}^2$ of *allowed transitions*. A sequence $(x_n)_{n \in \mathbb{Z}}$ belongs to the corresponding shift if and only if $x_n x_{n+1} \in A$ for all $n \in \mathbb{Z}$. One can represent the set A by an $|A| \times |A|$ *transition matrix*. Its entries $a_{x,y}$ are indexed by the letters x, y of X, and we have

$$a_{x,y} = \begin{cases} 1 & \text{if } xy \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.2.16. Consider the subshift of $\{\mathsf{0}, \mathsf{1}\}^{\mathbb{Z}}$ defined by the set of allowed transitions $\{\mathsf{00}, \mathsf{01}\}$. In other words, a sequence belongs to $\mathcal{F}$ if and only if it does not contain $\mathsf{11}$ as a subword. Then transition matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Another way of representing topological Markov shifts is to use graphs with the set of vertices X, where we have an arrow from x to y if and only if $xy \in A$. Then the subshift is the set of all bi-infinite sequences that can be read along bi-infinite paths in the graph.

Example 1.2.17. The subshift from Example 1.2.16 is described by the graph shown on Figure 1.13.

Example 1.2.18. Consider the graph shown on Figure 1.12. The corresponding topological Markov chain describes the subshift encoding the Arnold's Cat map, see Proposition 1.1.13.

Proposition 1.2.19. *Let G be a group with a finite generating set $S = S^{-1}$. For every shift of finite type $\mathcal{X} \subset \mathsf{X}^G$ there exists a block map conjugating $\mathcal{X}$ with a topological Markov shift.*

Proof. Let $\{f_i : A_i \longrightarrow \mathsf{X}\}_{i=1}^m$ be a finite set of prohibited configurations defining $\mathcal{X}$. Let R be such that $\bigcup_{i=1}^m A_i \subset (S \cup \{1\})^R$. Denote $B = (S \cup \{1\})^R$.

We can define $\mathcal{X}$ by a set of prohibited configurations $f : B \longrightarrow \mathsf{X}$ defined on B. Equivalently, we can find a finite subset $\mathsf{Y} \subset \mathsf{X}^B$ of maps

$B \mapsto \mathsf{X}$ such that $f \in \mathsf{X}^G$ belongs to $\mathcal{X}$ if and only if for every $g \in G$ the restriction of $g \cdot f$ to B belongs to Y.

Consider the block map β_B with the window B. For an element $f \in \mathsf{Y}^G$ and $g \in G$ we will denote by f_g the function $f_g : gB \longrightarrow \mathsf{X}$ given by $f_g(h) = f(g)(g^{-1}h)$ (check...), where $f(g) : B \longrightarrow \mathsf{X}$ is the element of Y corresponding to $g \in G$ (i.e., the value of $f \in \mathsf{Y}^G$ at g).

An element $f \in \mathsf{Y}^G$ belongs to $\beta_B(\mathcal{X})$ if and only if for every pair $g_1, g_2 \in G$ we have $f_{g_1}(h) = f_{g_2}(h)$ for every $h \in g_1B \cap g_2B$. In other words, $f \in \mathsf{Y}^G$ belongs to $\beta_B(\mathcal{X})$ if and only if the functions f_{g_1} and f_{g_2} agree on the intersection of their domains.

Consider the topological Markov $\mathcal{S}$ shift consisting of all sequences $f \in \mathsf{Y}^G$ given by the condition that f_g and f_{sg} agree on the intersection $gB \cap sgB$ of their domains, where $g \in G$ and $s \in S$. Let us show that $\mathcal{S}$ coincides with $\beta_B(\mathcal{X})$.

The inclusion $\beta_B(\mathcal{X}) \subset \mathcal{S}$ is obvious. Suppose that $f \in \mathcal{S}$. We have to prove that for any $g_1, g_2, h \in G$ we have $f_{g_1}(h) = f_{g_2}(h)$ whenever both sides of the equality are defined. By G-equivariance, it is enough to prove this statement for the case $h = 1$. We have $1 \in g_1B \cap g_2B$ if and only if g_1 and g_2 are on distance at most R from 1. Consequently, it is enough to prove that $f_{g_1}(1) = f_{g_2}(1)$ for every $g_1, g_2 \in B$, i.e., that $f_g(1) = f_1(1)$ for every $g \in B$. Write g as a product of at most R generators $g = s_1s_2 \cdots s_n$. Then, by definition of $\mathcal{S}$, we have

$$f_1(1) = f_{s_n}(1) = f_{s_{n-1}s_n}(1) = \ldots = f_{s_1s_2\cdots s_n}(1),$$

which finishes the proof. □

1.2.4. Substitutional subshifts. Let X be a finite alphabet. Denote by X^* the free monoid generated by X, i.e., the set of all finite words $x_1x_2\ldots x_n$ over the alphabet X, including the empty word $\varnothing$. It is a semigroup (monoid) with respect to the operation of concatenation. We say that v is a *subword* of $w \in \mathsf{X}^*$ if there exist $v_1, v_2 \in \mathsf{X}^*$ such that $w = v_1vv_2$. If $x \in \mathsf{X}$ is a letter, then we sometimes say that x *appears in* w if x is a subword of w. The notion of a sub-word of an infinite word (sequence) is defined analogously.

A *substitution* is an endomorphism $\sigma : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ of the monoid. It is defined by the values $\sigma(x) \in \mathsf{X}^*$ on the elements of X. Namely, for every word $x_1x_2\ldots x_n$ we have $\sigma(x_1x_2\ldots x_n) = \sigma(x_1)\sigma(x_2)\ldots\sigma(x_n)$.

If $w = \ldots x_{-2}x_{-1} . x_0x_1 \ldots$ is an element of $\mathsf{X}^{\mathbb{Z}}$, and $\sigma : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ is a substitution, then we denote by $\sigma(w)$ the infinite word

$$\ldots\sigma(x_{-2})\sigma(x_{-1}) . \sigma(x_0)\sigma(x_1)\ldots,$$

where dot, as before, denotes the place between the coordinates number -1 and 0.

Definition 1.2.20. The *substitutional subshift* generated by $\sigma : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ is the set of all bi-infinite words $w \in \mathsf{X}^{\mathbb{Z}}$ such that for every finite subword v of w there exists $x \in \mathsf{X}$ and $n \in \mathbb{N}$ such that v is a subword of $\sigma^n(x)$.

In other words, the language of the subshift generated by σ is the set of all subwords of words of the form $\sigma^n(x)$ for $x \in \mathsf{X}$.

Note that in order for the substitutional subshift to exist, the length of the word $\phi^n(x)$ must go to infinity for some letter $x \in \mathsf{X}$. Another (probably more common in the literature) definition of the substitutional subshift is to choose a particular $x \in \mathsf{X}$ such that the length of $\sigma^n(x)$ goes to infinity and to consider the set of all bi-infinite sequences w such that every finite subword of w is a subword of $\sigma^n(x)$ for some n.

Note that if $\mathcal{F}$ is the substitutional subshift generated by $\sigma : \mathsf{X}^* \longrightarrow \mathsf{X}^*$, then $\mathcal{F}$ is σ-invariant, i.e., $\sigma(\mathcal{F}) \subset \mathcal{F}$. The range $\sigma(\mathcal{F})$ is usually not equal to $\mathcal{F}$, but if m is the maximum of the lengths of the words $\sigma(x)$, $x \in \mathsf{X}$, then $\bigcup_{k=0}^{m-1} \mathsf{s}^k(\sigma(\mathcal{F})) = \mathcal{F}$, where s is the shift.

Example 1.2.21. Let $\mathsf{X} = \{0, 1\}$, and define an endomorphism $\sigma : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ by

$$\sigma(0) = 01, \qquad \sigma(1) = 10.$$

The corresponding substitutional shift is called the *Thue-Morse subshift*. The first letter of $\sigma(0)$ is 0. It follows by induction that the word $\sigma^{n+1}(0)$ begins with $\sigma^n(0)$ for every n. Therefore, the sequence $\sigma^n(0)$ naturally converges to one right-infinite sequence

$$0110100110010110100101100110100 1\ldots.$$

This particular sequence is called the *Thue-Morse sequence*. It is easy to see that an infinite sequence belongs to the Thue-Morse subshift if and only if all its finite subwords are subwords of the Thue-Morse sequence. Thue-Morse sequence was defined independently by E. Prohuet [**Pro51**], A. Thue [**Thu12**], and M. Morse [**Mor21**]. E. Prouhet implicitly discovered the sequence as a solution of a particular case of *Prouhet-Tarry-Escott problem*, see Exercise 1.16. A. Thue considered it as an example of an infinite cube-free sequence, see Exercise 1.16. M. Morse gave it, essentially as an example of a minimal infinite subshift. See ... [literature] for more properties of the Thue-Morse sequence

Example 1.2.22. Consider now the substitution

$$\sigma(0) = 01, \qquad \sigma(1) = 0.$$

Here too the word $\sigma^n(0)$ is a beginning of $\sigma^{n+1}(0)$, and in the limit we get an infinite sequence

$$010010100100101001010010010100 1001\ldots$$

called the *Fibonacci word*. The corresponding subshift consisting of all bi-infinite sequences whose finite subwords are subwords of the Fibonacci word is called the *Fibonacci substitutional subshift* (not to be confused with the Fibonacci shift of finite type from Example 1.2.16).

Note that in general the set of finite subwords of the subshift $\mathcal{F}_\sigma$ generated by a substitution $\sigma : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ may be strictly smaller that the set of all finite subwords of the words of the form $\sigma^n(x)$ for $x \in \mathsf{X}$. For example, if $\sigma : \{0, 1\}^* \longrightarrow \{0, 1\}^*$ is given by $\sigma(0) = 10$, $\sigma(1) = 1$, then $\mathcal{F}_\sigma$ consists of one sequence $\ldots 111 \ldots$.

Proposition 1.2.23. *Let $\mathcal{F}_\sigma$ be the subshift generated by the a substitution $\sigma : \mathsf{X} \longrightarrow \mathsf{X}^*$. Then the following conditions are equivalent.*

(1) *For every $x \in \mathsf{X}$ there exists $w \in \mathcal{F}_\sigma$ such that x appears in w.*

(2) *The set of finite subwords of elements of $\mathcal{F}_\sigma$ is the same as the set of subwords of the words of the form $\sigma^n(x)$ for $x \in \mathsf{X}$ and $n \geqslant 0$.*

(3) *For every $x \in \mathsf{X}$ there exists $n \geqslant 1$ and $y \in \mathsf{X}$ such that $\sigma^n(y) = v_1 x v_2$ for some non-empty words $v_1, v_2 \in \mathsf{X}^*$.*

Proof. Let us show that (1) implies (2). Let v be a subword of $\sigma^n(x)$. There exists $w \in \mathcal{F}_\sigma$ containing x. Then $\sigma^n(w) \in \mathcal{F}_\sigma$ and it contains $\sigma^n(x)$, hence it contains v.

We obviously have that (2) implies (1), since every letter x is equal to $\sigma^0(x)$, by definition.

Let us show that (3) implies (1). Let $x_1 = x \in \mathsf{X}$ be an arbitrary letter. Then, there exists n_1 and $x_2 \in \mathsf{X}$ such that x_1 is appears strictly inside $\sigma^{n_1}(x_2)$. Inductively, there exists $x_{k+1} \in \mathsf{X}$ and $n_k \geqslant 1$ such that x_k appears strictly inside $\sigma^{n_k}(x_{k+1})$. Choosing a convergent subsequence of the sequence of the words $\sigma^{n_1+n_2+\cdots+n_k}(x_{k+1})$ we can find a bi-infinite sequence $w \in \mathcal{F}_\sigma$ containing x.

Conversely, if (1) is satisfied, then for every $x \in \mathsf{X}$ there exist letters $a, b \in \mathsf{X}$ such that axb is a subword of some sequence $w \in \mathcal{F}_\sigma$. Then, by definition of $\mathcal{F}_\sigma$ there exists $y \in \mathsf{X}$ and $n \geqslant 1$ such that axb is a subword of $\sigma^n(y)$. □

Proposition 1.2.24. *Let $\sigma : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ be a substitution, and let $(\mathcal{F}_\sigma, \mathsf{s})$ be the subshift generated by it. Then the following two conditions are equivalent.*

(1) *There exists $N \geqslant 1$ such that for every pair of letters $x, y \in \mathsf{X}$ the letter y appears in $\sigma^N(x)$.*

(2) *The dynamical system $(\mathcal{F}_\sigma, \mathsf{s})$ is minimal, for every letter $x \in \mathsf{X}$ there exists $w \in \mathcal{F}_\sigma$ containing x, and the length of $\sigma^n(x)$ goes to infinity for every $x \in \mathsf{X}$.*

Substitutions satisfying condition (1) of Proposition 1.2.24 are called *primitive*.

references...

Proof. Let us prove the implication (1)$\Longrightarrow$(2). Let N be as in (1). Then for every $x \in \mathsf{X}$ the word $\sigma^N(x)$ contains all letters of the alphabet X, hence the length of $\sigma^N(x)$ is at least $|\mathsf{X}| > 1$. It follows that the length of $\sigma^{N^k}(x)$ is at least $|\mathsf{X}|^k$ for every $k \geqslant 1$. Consequently, the length of $\sigma^k(x)$ goes to infinity. It is also easy to see that if (1) is satisfied, then the condition (3) of Proposition 1.2.23 is satisfied, hence every letter of X appears in some element of $\mathcal{F}_\sigma$.

It remains to show that $\mathcal{F}_\sigma$ is minimal. Suppose that L is larger than the length of every word of the form $\sigma^N(x)$ for $x \in \mathsf{X}$, and let $w \in \mathcal{F}_\sigma$ be arbitrary. A shift of w belongs to $\sigma^N(\mathcal{F}_\sigma)$. It follows that every subword of w of length L contains a subword of the form $\sigma^N(x)$ for some $x \in \mathsf{X}$. Consequently, every subword of w of length L contains every letter of X. It follows that for all $n \geqslant 1$, $x \in \mathsf{X}$, $w \in \mathcal{F}_\sigma$ the word $\sigma^n(x)$ is a subword of w, which implies that the subshift $\mathcal{F}_\sigma$ is minimal by Proposition 1.2.4.

Let us show that (2) implies (1). Assume that (2) is satisfied. Then for every sequence $w \in \mathcal{F}_\sigma$, every $x \in \mathsf{X}$, and every $k \in \mathbb{Z}$ there exists $n \in \mathbb{Z}$ such that $|k-n| \leqslant M$ and $w(n) = x$. In other words, every letter appears in every word $w \in \mathcal{F}_\sigma$ with bounded gaps. By the conditions of the proposition, there exists N such that the lengths of $\sigma^N(x)$ are bigger than M for all $x \in \mathsf{X}$. Then $\sigma^N(x)$ will contain every letter $y \in \mathsf{X}$ for every $x \in \mathsf{X}$. $\square$

If $\mathcal{F}_\sigma$ is minimal, but a letter $x \in \mathsf{X}$ does not appear in a sequence $w \in \mathcal{F}_\sigma$, then the letter x does not appear in any sequence $w \in \mathcal{F}_\sigma$. It follows that there exists n such that x does not appear in any word $\sigma^n(y)$, $y \in \mathsf{X}$. Then $\mathcal{F}_\sigma$ is generated by the restriction of σ to the monoid $(\mathsf{X} \setminus \{x\})^*$. In other words, the condition that every letter appears in some element of the subshift is not restrictive if we want to describe all minimal substitutional subshifts.

Here are some examples of minimial substitutional shifts not satisfying the other condition of the Proposition 1.2.24 (that the length of $\sigma^n(x)$ always goes to infinity).

Example 1.2.25. Let $\mathsf{X} = \{0, 1, 2\}$, and consider the substitution

$$\sigma(0) = 021, \quad \sigma(1) = 10, \quad \sigma(2) = 2.$$

Then the length of $\sigma^n(x)$ goes to infinity if and only if $x \neq 2$. It is easy to see that the subshift generated by σ is the set of all bi-infinite sequences obtained from the sequence belonging to the Thue-Morse subshift by inserting a 2 after every 0.

Example 1.2.26. Consider the *Chacon substitution*

$$\sigma(0) = 0010, \qquad \sigma(1) = 1.$$

Denote $b_n = \sigma^n(0)$. In particular, $b_0 = 0$ and $b_1 = 0010 = b_0 b_0 1 b_0$. It follows by induction that $b_{n+1} = \sigma(b_n) = \sigma(b_{n-1}b_{n-1}1b_{n-1}) = b_n b_n 1 b_n$. The recurrent formula $b_{n+1} = b_n b_n 1 b_n$ implies that the subshift generated by the Chacon substitution is minimal.

The last example is actually conjugate to a subshift generated by a primitive substitution. Namely, consider the substitution

$$\varphi(0) = 0012, \quad \varphi(1) = 12, \quad \varphi(2) = 012.$$

It is primitive, so it generates a minimal subshift. Define $v_n = \varphi^n(0)$, and let v'_n be the word v_n in which the first symbol 0 is replaced by 2.

Let us show by induction that $v_{n+1} = v_n v_n 1 v'_n$. It is true for $n = 0$. Suppose that it is true for n, let us show it for $n + 1$. We have $v_{n+2} = \varphi(v_n)\varphi(v_n)12\varphi(v'_n)$. The word $\varphi(v'_n)$ is obtained from $\varphi(v_n)$ by replacing the initial 0012 by 012. It follows that $2\varphi(v'_n)$ is obtained from $\varphi(v_n)$ by replacing the initial 0012 by 2012, i.e., it is obtained from $\varphi(v_n)$ by replacing the initial 0 by 2. Consequently, $2\varphi(v'_n) = v'_{n+1}$, and $v_{n+2} = v_{n+1}v_{n+1}1v'_{n+1}$.

We see that if σ is the Chacon substitution from Example 1.2.26, then $\sigma^n(0)$ is obtained from $\varphi^n(0)$ by replacing all symbols 2 by 0. In the other direction, it is easy to prove by induction that $\varphi^n(0)$ is obtained from $\sigma^n(0)$ by replacing every subword 10 of $\sigma^n(0)$ by 12. It follows that the subshifts generated by σ and φ are topologically conjugate. For more on the Chacon substitution, see [**Fer02**].

On the other hand, up to topological conjugacy, the class of minimal substitutional subshifts coincides with the class of subshifts generated by primitive substitutions.

Proposition 1.2.27. *Let $\mathcal{X} \subset \mathsf{X}^{\mathbb{Z}}$ be a substitutional subshift. If it is minimal, then there exists a finite alphabet Y and a primitive substitution $\phi : \mathsf{Y} \longrightarrow \mathsf{Y}^*$ such that the subshift $\mathcal{F}_\phi$ generated by ϕ and the subshift $\mathcal{X}$ are topologically conjugate.*

Proof. We may assume that every letter $x \in \mathsf{X}$ belongs to the language of $\mathcal{X}$, i.e., appears in some element of $\mathcal{X}$. Otherwise, we can remove all such letters from the alphabet, and replace σ by an iterate, so that the removed letters do not appear in the values of σ on the remaining letters.

Since $\mathcal{X}$ is minimal, every letter $x \in \mathsf{X}$ appears in every element of $\mathcal{X}$ with uniformly bounded gaps between consecutive appearances, see Proposition 1.2.4. Let $x_0 \in \mathsf{X}$ be such that the length of $\sigma^n(x_0)$ goes to infinity. Then there exists $N > 0$ such that every word of length N in the language

$W_{\mathcal{X}}$ of $\mathcal{X}$ contains x_0. It follows that for every word $v \in W_{\mathcal{X}}$ the length of $\sigma^n(v)$ goes to infinity. Let Y be the set of elements of $W_{\mathcal{X}}$ of length N. For every $x_1 x_2 \ldots x_N \in \mathsf{Y} \subset \mathsf{X}^*$, compute $\sigma(x_1 x_2 \ldots x_N) = a_1 a_2 \ldots a_m \in \mathsf{X}^*$, and let $\sigma(x_1) = a_1 a_2 \ldots a_k$. Consider the word

$$\phi(v) = (a_1 a_2 \ldots a_N)(a_2 a_3 \ldots a_{N+1}) \ldots (a_k a_{k+1} \ldots a_{k+N-1}) \in \mathsf{Y}^*.$$

Note that $k + n - 1 \leqslant N$, since $a_1 a_2 \ldots a_m = a_1 a_2 \ldots a_k \sigma(x_2 x_3 \ldots x_N)$. It is good to imagine $x_1 x_2 \ldots x_n \in \mathsf{Y}$ as a copy of the letter $x_1 \in \mathsf{X}$ decorated by the "future" word $x_2 \ldots x_n$. Then ϕ becomes the substitution induced by σ on the decorated version of the alphabet. It is easy to prove by induction that $\phi^n(v)$ for $v \in \mathsf{Y}$ is defined using σ^n by the same rule as ϕ was defined using σ. It follows that $\mathcal{F}_\phi$ is obtained from $\mathcal{X}$ by conjugating by the block map with window on width N. Note that the length of $\phi^n(y)$ goes to infinity for every $y \in \mathsf{Y}$, and every letter $y \in \mathsf{Y}$ is contained in every element of $\mathcal{F}_\phi$. Proposition 1.2.24 shows that ϕ is primitive. □

1.2.5. Complexity and entropy.

Definition 1.2.28. Let $\mathcal{F}$ be a subshift, and let $W_{\mathcal{F}}$ be its language. The *complexity function* of $\mathcal{F}$ is

$$p_{\mathcal{F}}(n) = |\{v \in W_{\mathcal{F}} \ : \ |v| = n\}|,$$

where $|v|$ denotes the length of v.

The complexity functions is sometimes called the *factor complexity*, *block growth*, or *subword complexity*. It was introduced by G. Hedlund and M. Morse in [**MH38**]. See [**CN10**] for an overview of the main results on this subject.

Proposition 1.2.29. *The function $p_{\mathcal{F}}(n)$ is non-decreasing. Moreover, if $p_{\mathcal{F}}(n) = p_{\mathcal{F}}(n+1)$ for some n, then $p_{\mathcal{F}}(n)$ is bounded, and $\mathcal{F}$ is finite.*

We have $p_{\mathcal{F}}(n+m) \leqslant p_{\mathcal{F}}(n) p_{\mathcal{F}}(m)$ for all $n, m \geqslant 1$. The limit of the sequence $\frac{\log p_{\mathcal{F}}(n)}{n}$ exists and is equal to its infimum.

Proof. Every word $v \in W_{\mathcal{F}}$ of length n can be continued, by adding one letter to its end, to a word $v' \in W_{\mathcal{F}}$ of length $n+1$. If $v_1 \neq v_2$, then their continuations v_1', v_2' are also different. It follows that $v \mapsto v'$ (for any choices of the continuations v') is a one-to-one map from the set of words of length n to the set of words of length $n+1$ of the language $W_{\mathcal{F}}$. If $p_{\mathcal{F}}(n) = p_{\mathcal{F}}(n+1)$, then this map is a bijection. It follows that every word $x_1 x_2 \ldots x_n \in W_{\mathcal{F}}$ has a unique continuation $x_1 x_2 \ldots x_n x_{n+1}$ to a word of length $n+1$ belonging to $W_{\mathcal{F}}$. Then the word $x_2 x_3 \ldots x_{n+1}$ also has a unique continuation, which implies that every word of length n has a unique continuation to a word of length $n+2$. By induction, we get that for every $m > n$ every word

$v \in W_{\mathcal{F}}$ is a prefix of a unique word $w \in W_{\mathcal{F}}$ of length m. Consequently, $p_{\mathcal{F}}(m) = p_{\mathcal{F}}(n)$ for every $m > n$, and $p_{\mathcal{F}}$ is bounded.

Every word $v \in W_{\mathcal{F}}$ of length $n + m$ can be split into a prefix of length n and a suffix of length m, so $p_{\mathcal{F}}(n + m) \leqslant p_{\mathcal{F}}(n)p_{\mathcal{F}}(m)$. The statement about the limit follows from the classical Fekete's lemma [**Fek23**] and [**PS72**, Problem 98] applied to $\log p_{\mathcal{F}}(n)$. □

Proposition 1.2.30. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be topologically conjugate subshifts. Then there exists $c > 1$ such that we have*

$$p_{\mathcal{F}_1}(n - c) \leqslant p_{\mathcal{F}_2}(n) \leqslant p_{\mathcal{F}_1}(n + c)$$

for all $n > c$.

Proof. Let X_i be the alphabet such that $\mathcal{F}_i \subset \mathsf{X}_i^{\mathbb{Z}}$. Suppose that $v \in \mathsf{X}_i^{2n+1}$ is a word of odd length $2n + 1$, and denote by $C_{v,i}$ the set all sequences $(x_n)_{n\in\mathbb{Z}} \in \mathcal{F}_i$ such that $x_{-n}x_{-n+1} \ldots x_{n-1}x_n = v$. Let $\Phi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ be a topological conjugacy. Then for every sequence $w = (x_n)_{n\in\mathbb{Z}} \in \mathcal{F}_i$ the intersection $\bigcap_{n\geqslant 0} C_{x_{-n}x_{-n+1}\ldots x_{n-1}x_n,i}$ is equal to $\{w\}$. Consequently, there exists k such that $\Phi(C_{x_{-k}x_{-k+1}\ldots x_{k-1}x_k,1}) \subset C_{y_0,2}$ for some letter $y_0 \in \mathsf{X}_2$. It follows from compactness of $\mathcal{F}_1$ that there exists $K \geqslant 0$ such that for every word $v \in \mathsf{X}_1^{2K+1}$ there exists $y_v \in \mathsf{X}_2$ such that $\Phi(C_{v,1}) \subset C_{y_v,2}$. Then we have that Φ is given by the "centered" block map:

$$\Phi((x_n)_{n\in\mathbb{Z}})(n) = y_{x_{n-K}x_{n-K+1}\ldots x_{n+K-1}x_{n+K}}.$$

For every $n > 2K + 1$, the constructed block map defines a surjective map from the set of words of length n of the language of $\mathcal{F}_1$ to the set of words of length $n - 2K$ of the language of $\mathcal{F}_2$:

$$x_1x_2 \ldots x_n \mapsto y_{x_1x_2\ldots x_{2K+1}}y_{x_2x_3\ldots x_{2K+2}} \cdots y_{x_{n-2K}x_{n-2K+1}\ldots x_n}.$$

It follows that $p_{\mathcal{F}_2}(n - 2K) \leqslant p_{\mathcal{F}_{1(n)}}$. □

As a corollary of Proposition 1.2.30 we get that the limit $\lim_{n\to\infty} \frac{\log p_{\mathcal{F}}(n)}{n}$ (which exists by Proposition 1.2.29) depends only on the topological conjugacy class of the subshift. It is called the *entropy* of the subshift.

Example 1.2.31. Let $\mathcal{F} \subset \mathsf{X}^{\mathbb{Z}}$ be a topological Markov shift with the transition matrix A. Then, for all $n \geqslant 2$, the complexity $p_{\mathcal{F}}(n)$ is equal to the sum of the entries of the matrix A^{n-1}. The entropy of $\mathcal{F}$ is equal therefore to the logarithm of the spectral radius of A.

Example 1.2.32. If $\mathcal{F}$ is the Fibonacci shift of finite type from Example 1.2.16, then $p_{\mathcal{F}}(n)$ is the Fibonacci sequence $2, 3, 5, 8, \ldots$. It follows that the entropy is equal to the logarithm of the golden mean.

Entropy is an important invariant of dynamical systems and a classical subject originating from the work of Shannon on information theory. The notion defined above is a particular case of *topological entropy* .... more on history ... literature...

The following theorem was proved in [**Gri73**]. See a short proof in [**CN10**, Theorem 4.4.4].

Theorem 1.2.33. *For every $h > 0$ and every integer k such that $h < \log k$ there exists a minimal subshift $\mathcal{F} \subset \{1, 2, \ldots, k\}^{\mathbb{Z}}$ of entropy h.*

Substitutional dynamical systems, on the other hand, are examples of subshifts of zero entropy.

Theorem 1.2.34. *Let $\mathcal{F}$ be a minimal substitutional subshift. Then there exists $C > 1$ such that $p_{\mathcal{F}_\sigma}(n) \leqslant Cn$ for all $n \geqslant 1$.*

Proof. It follows from Propositions 1.2.27 and 1.2.30 that we may assume that $\mathcal{F} = \mathcal{F}_\sigma$ is generated by a primitive substitution $\sigma : \mathsf{X} \longrightarrow \mathsf{X}^*$.

Moreover, we may assume that for every $x \in \mathsf{X}$ the word $\sigma(x)$ contains all letters of the alphabet (otherwise, replace σ by a high enough iterate). Then for every $x, y \in \mathsf{X}$ and $k \geqslant 1$ we have

$$|\sigma^k(x)| \leqslant |\sigma^{k+1}(y)|,$$

since $\sigma^k(x)$ is a subword of $\sigma^{k+1}(y)$.

Let $M = \max_{x \in \mathsf{X}} |\sigma(x)|$. We have then $|\sigma^{k+1}(x)| \leqslant M|\sigma^k(x)|$ for all $x \in \mathsf{X}$ and $k \geqslant 1$. It follows that

$$|\sigma^k(x)| \leqslant M|\sigma^k(y)|$$

for all $k \geqslant 1$ and $x, y \in \mathsf{X}$.

Let $n \geqslant 1$, and let k be the smallest integer such that $n \leqslant |\sigma^k(x)|$ for all $x \in \mathsf{X}$. Then there exists $x_1 \in \mathsf{X}$ such that $|\sigma^{k-1}(x_1)| < n$. Then, for every $x \in \mathsf{X}$, we have

$$|\sigma^k(x)| \leqslant M|\sigma^{k-1}(x)| \leqslant M^2|\sigma^{k-1}(x_1)| < M^2 n.$$

Suppose that $v \in W_{\mathcal{F}_\sigma}$ has length n. Let $(x_n)_{n \in \mathbb{Z}} \in \mathcal{F}_\sigma$ be such that v is a subword of $\ldots \sigma^k(x_{-2})\sigma^k(x_{-1}).\sigma^k(x_0)\sigma^k(x_1)\ldots$. Since all words $\sigma^k(x)$ are of length at least n, there exists a word $x_i x_{i+1}$ of length 2 such that v is a subword of $\sigma^k(x_i x_{i+1})$. The length of $\sigma^k(x_i x_{i+1})$ is strictly less than $2M^2 n$, hence $\sigma^k(x_i x_{i+1})$ has not more than $2M^2 n - n$ subwords of length n. It follows that $p_{\mathcal{F}_\sigma}(n) \leqslant p_{\mathcal{F}_\sigma}(2)(2M^2 - 1)n$, so the Theorem is valid for $C = p_{\mathcal{F}_\sigma}(2)(2M^2 - 1)$. □

See a similar proof of Theorem 1.2.34 (for the case of a primitive substitution) using Perron-Frobenius theorem in [**Que87**, Proposition V.19].

Minimal systems of various subexponential complexity have been constructed in the literature. For example, the following is a result of J. Goyon, see [**Fer99**, p. 149].

Theorem 1.2.35. *Let $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ be such that there exist $r, h > 1$ such that for all $k \geqslant 1$ we have $\phi(r^{k+1}) \leqslant h\phi(r^k)$. Then there exists a minimal subshift $\mathcal{F}$ and positive constants C_1, C_2 such that*

$$C_1 n\phi(n) \leqslant p_{\mathcal{F}}(n) \leqslant C_2 n\phi(n)$$

for all $n \geqslant 1$.

For example, there exist minimal subshifts whose complexities are of the form $n^{\alpha_0}(\log n)^{\alpha_1}(\log\log n)^{\alpha_2}\cdots(\log\log\ldots\log n)^{\alpha_k}$ for any $\alpha_0 > 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$.

The sequences used in the proofs of Theorems 1.2.35 and 1.2.33 belong to the class of *Toeplitz sequences.* A sequence $w \in \mathsf{X}^{\mathbb{Z}}$ is Toeplitz if for every $n \in \mathbb{Z}$ there exists $q \in \mathbb{N}$ such that $w(n + kq)$ is constant for all $k \in \mathbb{Z}$. Note that it follows directly from Proposition 1.2.4 that the closure of the orbit of any Toeplitz sequence is a minimal subshift.

It was also shown, see [**Fer99**, p. 149], that for any $1 < \alpha < \beta$ there exists a minimal subshift $\mathcal{F}$ such that $\liminf_{n\to\infty} \frac{p_{\mathcal{F}}(n)}{n^\alpha} = 0$ and $\limsup_{n\to\infty} \frac{p_{\mathcal{F}}(n)}{n^\beta} = +\infty$. More on different complexity functions of minimal subshifts, see [**Fer99**] and [**CN10**].

Proposition 1.2.36. *Let $\mathcal{F} \subset \mathsf{X}^{\mathbb{Z}}$ be a subshift. If $p_{\mathcal{F}}(n) \leqslant n$ for some n, then $p_{\mathcal{F}}$ is bounded.*

Proof. Suppose that $p_{\mathcal{F}}(n) \leqslant n$. By Proposition 1.2.29, the function $p_{\mathcal{F}}$ is non-degreasing. We have $p_{\mathcal{F}}(1) > 1$. Consequently, there exists $k \leqslant n - 1$ such that $p_{\mathcal{F}}(k) = p_{\mathcal{F}}(k + 1)$. But this implies, also according to Proposition 1.2.29, that $p_{\mathcal{F}}$ is bounded. $\square$

It follows that the smallest possible unbounded complexity is $p(n) = n + 1$. It is realized by the subshifts from the following class.

Proposition 1.2.37. *Let $\theta \in (0, 1)$ be an irrational number, and consider the rotation $R_\theta : x \mapsto x + \theta$ of $\mathbb{R}/\mathbb{Z}$. For $x \in \mathbb{R}/\mathbb{Z}$, denote by $I_x \in \{0, 1\}^{\mathbb{Z}}$ the sequence given by the condition*

$$I_x(n) = \begin{cases} \mathsf{0} & \text{if } R^n(x) \in [0, 1 - \theta); \\ \mathsf{1} & \text{if } R^n(x) \in [1 - \theta, 1), \end{cases}$$

where we identify the circle $\mathbb{R}/\mathbb{Z}$ with $[0, 1)$ in the natural way. Let $\mathcal{X}_\theta$ be the closures of the set of sequences I_x for $x \in [0, 1)$.

Then the complexity of $\mathcal{X}_\theta$ satisfies $p_{\mathcal{X}_\theta}(n) = n + 1$.

The sequences I_x are precisely the sequences considered in 1.1.1, where $\mathtt{0}$ corresponds to the symbol v, and $\mathtt{1}$ corresponds to d.

Proof. A word $v = a_1a_2\dots a_n$ belongs to $W_{\mathcal{F}_\theta}$ if and only if there exists a number $x \in [0, 1)$ such that $I_x(i) = a_i$ for $i = 1, 2, \dots, n$. The set of such points x (for a given v) is an arc of $\mathbb{R}/\mathbb{Z}$ into which the points $0, 1 - \theta = R^{-1}(0), R^{-2}(0), \dots, R^{-n}(0)$ subdivide $\mathbb{R}/\mathbb{Z}$. The number of such arcs is equal to $n + 1$. □

In fact, the converse to Proposition 1.2.37 is also true, see Theorem 1.3.38.

1.3. Minimal Cantor systems

We describe in this section theory of minimal $\mathbb{Z}$-actions on the Cantor set via Vershik-Bratteli diagrams. This theory was developed in [**HPS92**] and found important applications for the theory of orbit equivalence in [**GPS95**]. For a more detailed exposition, see the book [...]. See also ... Our interest in the subject comes primarily from the fact that minimal Cantor systems provide an interesting class of amenable groups, see... and the literature therein. Moreover, one of the main classes of groups that will be studied in this book can be seen as a direct generalizations of the model of minimal $\mathbb{Z}$-actions via Vershik-Bratteli diagrams, see... A special class of *stationary Vershik-Bratteli diagrams* provide interesting examples of groups generated by finite automata and are closely related to hyperbolic dynamical systems.

1.3.1. Examples of minimal homeomorphisms of the Cantor set.

1.3.1.1. *Odometers.* We have seen (Proposition 1.1.10) that the transformations $a \mapsto a + 1$ of the ring of dyadic integers $\mathbb{Z}_2$ is a minimal homeomorphism. A straightforward generalization of this construction is the transformation $x \mapsto x + 1$ for an aribtrary profinite completion of $\mathbb{Z}$. More explicitly, consider a sequence of integers greater than one

$$d_1, d_2, d_3, \dots,$$

and the inverse limit $\widehat{\mathbb{Z}}_{d_1d_2\dots}$ of the cyclic groups

$$\mathbb{Z}/d_1\mathbb{Z} \longleftarrow \mathbb{Z}/d_1d_2\mathbb{Z} \longleftarrow \mathbb{Z}/d_1d_2d_3\mathbb{Z} \longleftarrow \cdots$$

with respect to the natural epimorphisms $r+d_1d_2\cdots d_n\mathbb{Z} \mapsto r+d_1d_2\cdots d_{n-1}\mathbb{Z}$.

The elements of $\widehat{\mathbb{Z}}_{d_1d_2\dots}$ are uniquely represented as formal expressions

$$r_1 + r_2 \cdot d_1 + r_3 \cdot d_1d_2 + r_4 \cdot d_1d_2d_3 + \cdots, \tag{1.2}$$

where $r_i \in \{0, 1, \dots, d_i - 1\}$. Namely, the formal expression (1.2) corresponds to the element of the inverse limit given by the sequence

$$(r_1, r_1 + r_2 \cdot d_1, r_1 + r_2 \cdot d_1 + r_3 \cdot d_1d_2, \dots) \in (\mathbb{Z}/d_1\mathbb{Z}, \mathbb{Z}/d_1d_2\mathbb{Z}, \mathbb{Z}/d_1d_2d_3\mathbb{Z}, \dots).$$

The transformation $x \mapsto x+1$ can be interpreted as a procedure of adding one to the series (1.2), and then rewriting it in the same form. Namely, if $r_1 \in \{0, 1, \ldots, d_1 - 2\}$, then we just replace r_1 by $r_1 + 1$. Otherwise, if $r_1 = d_1$, we replace r_1 by 0, and then change the sequence $r_2, r_3, \ldots$, by adding one to the formal series

$$r_2 + r_3 \cdot d_2 + r_4 \cdot d_2 d_3 + \cdots$$

using the same rule (i.e., replacing r_2 by $r_2 + 1$ if $r_2 \in \{0, 1, \ldots, d_2 - 2\}$, etc.).

It is easy to see that the same arguments as in the proof of Proposition 1.1.10 show that the transformation $x \mapsto x + 1$ of $\widehat{\mathbb{Z}}_{d_1 d_2 \ldots}$ is a minimal homeomorphism.

1.3.1.2. *Irrational rotation.* Let us show how one can construct a minimal homeomorphism of a Cantor set from an irrational rotation of the circle. This construction is called sometimes the *Denjoy homeomorphism*, see...

Let $R_\theta : x \mapsto x + \theta$ for $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational rotation of $\mathbb{R}/\mathbb{Z}$, see 1.1.1. Consider the orbit $O = \{\mathrm{frac}(n\theta) \ : \ n \in \mathbb{Z}\}$ of 0 under R_θ. We represent the points of $\mathbb{R}/\mathbb{Z}$ by points of $[0, 1)$ (by their fractional parts $\mathrm{frac}(x)$). Let us replace every point $\alpha \in O \setminus \{0\}$ by two copies $\alpha + 0$ and $\alpha - 0$. We replace $0 \in O$ also by two copies: 0 and 1 (playing the role of $0 + 0$ and $0 - 0$, respectively). Consider the obtained set with the natural order: $\alpha - 0 < \alpha + 0$; and if $\alpha, \beta \in [0, 1]$ are such that $\alpha < \beta$, then every copy of α is less than every copy of β. Denote by $\mathcal{X}_\theta$ the obtained ordered set.

Consider the *order topology* on $\mathcal{X}_\theta$: a basis of topology is the set of all intervals of the form $(\alpha, \beta), [0, \alpha)$, or $(\alpha, 1]$. We have a natural continuous surjective map $\Phi : \mathcal{X}_\theta \mapsto \mathbb{R}/\mathbb{Z}$ mapping each copy of $\alpha \in [0, 1)$ to its image in $\mathbb{R}/\mathbb{Z}$. The map Φ is at most 2-to-1.

One can show that $\mathcal{X}_\theta$ is homeomorphic to the Cantor set (Exercise 1.19). It is easy to see that $\tilde{R}_\theta(\alpha + 0) = R_\theta(\alpha) + 0$, $\tilde{R}_\theta(\alpha - 0) = R_\theta(\alpha) - 0$, and $\tilde{R}_\theta(\alpha) = R_\theta(\alpha)$ defines a minimal homeomorphism $\tilde{R}_\theta$ of $\mathcal{X}_\theta$. It acts on $\mathcal{X}_\theta$ as an *interval exchange transformation*: it moves $[0, 1 - \theta - 0]$ to $[\theta + 0, 1]$ and $[1 - \theta + 0, 1]$ to $[0, \theta - 0]$ by parallel translations. The map Φ is a semiconjugacy from $\tilde{R}_\theta \curvearrowright \mathcal{X}_\theta$ to $R_\theta \curvearrowright \mathbb{R}/\mathbb{Z}$.

Proposition 1.3.1. *The homeomorphism $\tilde{R}_\theta$ generates an expansive action of $\mathbb{Z}$.*

Recall that the homeomorphism $R_\theta \circlearrowright \mathbb{R}/\mathbb{Z}$ is not expansive, as it is an isometry.

Proof. It is enough to show that for any two points $x, y \in \mathcal{X}_\theta$ such that $x < y$ and $y - x < 1/2$ there exists n such that $\tilde{R}_\theta(x)$ and $\tilde{R}_\theta(y)$ belong to different intervals $[0, \theta - 0]$ and $[\theta + 0, 1]$. But this follows from minimality of $R_\theta \curvearrowright \mathbb{R}/\mathbb{Z}$. Namely, if $\Phi(x) \neq \Phi(y)$, then there exists $n \in \mathbb{Z}$ such that $n\theta \in (\Phi(x), \Phi(y))$, and then θ belongs to the shorter arc of the circle $\mathbb{R}/\mathbb{Z}$ with the endpoints $R_\theta^{-n+1}(\Phi(x))$ and $R_\theta^{-n+1}(\Phi(y))$, which implies that θ separates the points $R_\theta^{-n+1}(x)$ and $R_\theta^{-n+1}(\Phi(y))$. If $\Phi(x) = \Phi(y)$, then $\Phi(x)$ and $\Phi(y)$ belong to the R_θ-orbit of 0, so there exists n such that $\tilde{R}_\theta(x) = \theta - 0$ and $\tilde{R}_\theta(y) = \theta + 0$. $\square$

In fact, it is easy to check that the system $\tilde{R}_\theta \circlearrowright \mathcal{X}_\theta$ is topologically conjugate to the subshift $\mathcal{X}_\theta$ described in Proposition 1.2.37.

1.3.1.3. *Minimal subshifts.* We have seen many example of minimal Cantor dynamical systems in Section 1.2, in particular, the ones generated by primitive substitutions.

Another important class of minimal subshifts are the *Toeplitz* subshifts, i.e., subshifts equal to the closure of the set of shifts of a Toeplitz sequence. A sequence $(x_n)_{n\in\mathbb{Z}} \in \mathsf{X}^{\mathbb{Z}}$ is Toeplitz if for every $n \in \mathbb{Z}$ there exists a positive integer q such that $x_{n+qk} = x_n$ for all $k \in \mathbb{Z}$. The closure of the $\mathbb{Z}$-orbit of a Toeplitz sequence under the shift is called a Toeplitz subshift. It easily follows from Proposition 1.2.4 that every Toeplitz subshift is minimal.

Any Toeplitz $(x_n)_{n\in\mathbb{Z}} \in \mathsf{X}^{\mathbb{Z}}$ sequence can be constructed in the following way. Let $*$ be a symbol not contained in X, and denote by $\mathsf{X}_* = \mathsf{X} \cup \{*\}$. Suppose that $w_1, w_2 \in \mathsf{X}_*^{\mathbb{Z}}$ are periodic sequences, and suppose that symbols $*$ appear in w_1. Denote then by $T_{w_1}(w_2)$ the sequence obtained from w_1 by replacing consecutive symbols $*$ of w_1 by the sequence w_2 (so that, for example, the coordinate number 0 of w_2 is placed in the first non-negative coordinate of w_1 equal to $*$). If p_1 and p_2 are periods of w_1 and w_2, then $T_{w_1}(w_2)$ is a periodic sequence of period $p_1 p_2$.

Let $w_1, w_2, \ldots \in \mathsf{X}_*^{\mathbb{Z}}$ be non-constant periodic sequences with periods $p_1, p_2, \ldots$ such that $*$ appears in every sequence w_i. Suppose that for infinitely many values of n the symbol $*$ does not appear on the zeroth coordinate of w_n. Then the sequence $T_{w_1}(T_{w_2}(T_{w_3}(\ldots T_{w_{n-1}}(w_n)\ldots)))$ converges to a Toeplitz sequence $w \in \mathsf{X}^{\mathbb{Z}}$.

Example 1.3.2. Consider the *Feigenbaum substitution* $\phi : \mathsf{0} \mapsto \mathsf{11}, \mathsf{1} \mapsto \mathsf{10}$, and let $\mathcal{X}$ be the subshift generated by it. Since the words $\phi(x)$ for $x \in \{\mathsf{0}, \mathsf{1}\}$ are of length two and both start with $\mathsf{1}$, for every sequence $(x_n)_{n\in\mathbb{Z}} \in \mathcal{X}$ we either have $x_{2n} = \mathsf{1}$ or $x_{2n+1} = \mathsf{1}$ for all $n \in \mathbb{Z}$. If we eraze this constant $\mathsf{1}$ subsequence, the sequence that remains is obtained from a sequence $(y_n)_{n\in\mathbb{Z}} \in \mathcal{X}$ by changing $\mathsf{0}$ to $\mathsf{1}$ and $\mathsf{1}$ to $\mathsf{0}$. It follows that every sequence in $\mathcal{X}$ is the limit of the sequence $T_{w_1}(T_{w_2}(\ldots T_{w_{n-1}}(T_{w_n})\ldots))$, where w_n for odd n is one of the two sequences of the form $\ldots * \mathsf{1} * \mathsf{1} * \mathsf{1} * \ldots$, and for even n is one of the two sequences of the form $\ldots * \mathsf{0} * \mathsf{0} * \mathsf{0} * \ldots$.

The following characterization of Toeplitz shifts shows a relation between Toeplitz subshifts and odometers similar to the relation between Denjoy systems and irrational rotations of the circle.

Theorem 1.3.3 (Downarowicz, Lacroix)**.** *A minimal subshift $f \circlearrowleft \mathcal{X}$ is Toeplitz if and only if there exists a generalized odometer $f' \circlearrowleft \mathcal{X}'$ and a surjective semiconjugacy $\pi : \mathcal{X} \longrightarrow \mathcal{X}'$ such that $|\pi^{-1}(x)| = 1$ for some $x \in \mathcal{X}'$.*

1.3.2. Rokhlin-Kakutani towers.

Lemma 1.3.4. *Let $f \circlearrowleft \mathcal{X}$ be a minimal homeomorphism of a compact topological space. For every open set $U \subset \mathcal{X}$ and every $x \in \mathcal{X}$ there exist integers $n_+ > 0$ and $n_- \leqslant 0$ such that $\{f^{n_+}(x), f^{n_-}(x)\} \subset U$.*

Proof. By minimality, the sets $f^n(U)$, for $n \in \mathbb{Z}$, cover $\mathcal{X}$. By compactness, there exists a finite set $n_1, n_2, \ldots, n_k \in \mathbb{Z}$ such that the sets $f^{n_i}(U)$ cover $\mathcal{X}$. Applying $f^{-\max n_i}$ to this cover, we get a cover $\{f^{m_i}(U)\}_{i=1,\ldots,k}$ where all the numbers m_i are non-positive. Similarly, applying $f^{-\min n_i+1}$, we get a cover $\{f^{m_i}(U)\}_{i=1,\ldots,k}$ where all the numbers m_i are positive. $\square$

Let $\tau \circlearrowleft \mathcal{X}$ be a minimal homeomorphism of a compact totally disconnected metrizable space.

A *Rokhlin-Kakutani partition* is a collection of finite sequences

$$\begin{array}{ccccc}
(C_1, & \tau(C_1), & \tau(C_1), & \ldots, & \tau^{k_1-1}(C_1)),\\
(C_2, & \tau(C_2), & \tau(C_2), & \ldots, & \tau^{k_2-1}(C_2)),\\
 & & \vdots & & \\
(C_m, & \tau(C_m), & \tau(C_m), & \ldots, & \tau^{k_m-1}(C_m))
\end{array}$$

of clopen subsets of $\mathcal{X}$ such that $\{\tau^i(C_j) \;:\; 0 \leqslant i < k_i\}$ is a parition of $\mathcal{X}$, and $\tau^{k_i}(C_i) \subset \bigcup_{j=1}^m C_j$. Each sequence

$$(C_i, \quad \tau(C_i), \quad \tau^2(C_i), \quad \ldots, \quad \tau^{k_i-1}(C_i))$$

is called a *tower* of the partition. The set $\bigcup_{i=1}^m C_i$ is called the *base* of the partition, and the set C_i is called the *base* of the corresponding tower.

See Figure 1.14 where the action of τ on a Rokhlin-Kakutani partition is shown schematically.

Proposition 1.3.5. *For every finite clopen partition $\mathcal{P}$ of $\mathcal{X}$ and every clopen subset $C \subset \mathcal{X}$ there exists a Rokhlin-Kakutani parition subordinate to $\mathcal{P}$ with the base equal to C.*

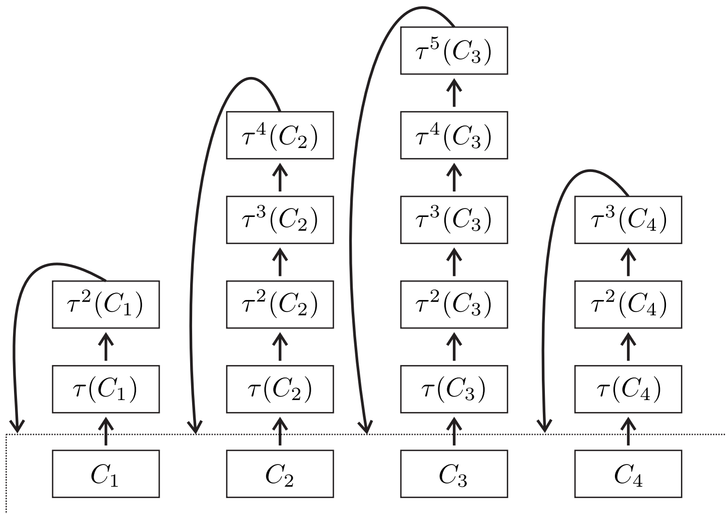

Figure 1.14. A Rokhlin-Kakutani partition

Proof. By Lemma 1.3.4, for every $x \in \mathcal{X}$ there exists a positive integer n_+ and a non-positive integer n_- such that $\tau^{n_+}(x), \tau^{n_-} \in C$. Let $n_+(x)$ and $n_-(x)$ be the smallest and the largest of such numbers, respectively. They are the *first return times* to C of the dynamical system.

Note that for every $k \in \mathbb{N}$ the set of points $\{x \in \mathcal{X} \,:\, n_+(x) = k\}$ is equal to

$$\tau^{-k}(C) \smallsetminus \bigcup_{1 \leqslant i < k} \tau^{-i}(C).$$

Similarly, the set $\{x \in \mathcal{X} \,:\, n_-(x) = -k\}$ is equal to

$$\tau^{k}(C) \smallsetminus \bigcup_{0 \leqslant i < k} \tau^{i}(C).$$

Note that since C is clopen and τ is a homeomorphism, these sets are clopen. It follows that the functions $n_+(x)$ and $n_-(x)$ are locally constant, i.e., continuous. Consequently, the map $x \mapsto (n_-(x), n_+(x))$ is locally constant, hence its set of values is finite (as $\mathcal{X}$ is compact).

Denote by

$$C_{i,j} = \{x \in \mathcal{X} \,:\, n_-(x) = -i, n_+(x) = j\}$$

its level sets. They form a finite clopen partition of $\mathcal{X}$.

Note that C is equal to the set of points $x \in \mathcal{X}$ such that $n_-(x) = 0$. It follows that C is partitioned into a finite collection of disjoint sets

$$C_{0,k_1}, \quad C_{0,k_2}, \quad , \ldots, \quad C_{0,k_m}$$

for some positive integers k_i.

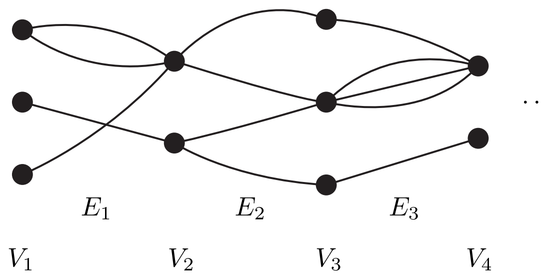

Figure 1.15. A Bratteli diagram

For every k_i and $1 \leqslant l < k_i$ we obviously have

$$\tau^l(C_{0,k_i}) = C_{l,k_i-l},$$

since the first moment in the future when points of $\tau^l(C_{0,k_i})$ come back to C, by definition, is $k_i - l$ (as the first positive return time to C of points of C_{0,k_i} is k_i); and the first time in the past when points of $\tau^l(C_{0,k_i})$ are in C is $-l$. It follows that the sets $C_{i,j}$ form a Rokhlin-Kakutani partition with base C and towers

$$C_{0,k_i}, \quad C_{1,k_i-1}, \quad C_{2,k_i-2}, \quad ,\ldots, \quad C_{k_i-1,1}.$$

For a point $x \in C_{0,k_i}$ consider the *itinerary* $(P_0, P_1, \ldots, P_{k_i-1})$ of x with respect to $\mathcal{P}$, i.e., a sequence of elements of $\mathcal{P}$ such that $\tau^l(x) \in P_l$ for $0 \leqslant l \leqslant k_i - 1$. Since $\mathcal{P}$ is finite, the set of all itineraries $(P_0, P_1, \ldots, P_{k_i-1})$ is finite, hence C_{0,k_i} is partitioned into a finite set of clopen subsets $A_1, A_2, \ldots, A_{M_i}$ such that all points one set A_j have the same itineraries. Let us split the tower into M_i towers

$$A_j, \quad \tau(A_j), \quad \tau^2(A_j), \quad \ldots, \quad \tau^{k_i-1}(A_j).$$

Then each element of the tower is a subset of an element of $\mathcal{P}$. The union of all such towers (for all towers of the partition $\{C_{i,j}\}$) will be a Rokhlin-Kakutani partition subordinate to $\mathcal{P}$ and with the base equal to C. $\square$

1.3.3. Bratteli diagrams. A *Bratteli diagram* B consists of sequences $(V_1, V_2, \ldots)$ and $(E_1, E_2, \ldots)$ of finite sets and sequences of maps $\mathbf{s}_n : E_n \longrightarrow V_n$, $\mathbf{r}_n : E_n \longrightarrow V_{n+1}$. The sets $V = \bigsqcup_{n\geqslant 1} V_n$ and $E = \bigsqcup_{n\geqslant 1} E_n$ are the sets of *vertices* and *edges* of the diagram, respectively. An edge $e \in E_n$ connects the vertices $\mathbf{s}_n(e) \in V_n$ and $\mathbf{r}_n(e) \in V_{n+1}$. See Figure 1.15 where a beginning of a Bratteli diagram is shown.

We assume that the maps $\mathbf{s}_n$ and $\mathbf{r}_n$ are surjective (though more general Bratteli diagrams are also sometimes considered in the literature). We will sometimes denote $\mathbf{s} = \mathbf{s}_n$ and $\mathbf{r} = \mathbf{r}_n$ when the domains of the corresponding maps are clear from the context.

Let us enumerate each set V_n, i.e., identify it with $\{1, 2, \ldots, |V_n|\}$. The the Bratteli diagram is determined then by the sequence of matrices $A_n = (m_{ij})_{1 \leqslant i \leqslant |V_{n+1}|, 1 \leqslant j \leqslant |V_n|}$, where m_{ij} is the number of edges connecting $i \in V_{n+1}$ to $j \in V_n$. For example, if we enumerate the vertices of the levels of the diagram shown on Figure 1.15 from left to right, then we have

$$A_1 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A (finite or infinite) path in the diagram B is a (finite or infinite) sequence $(e_1, e_2, \ldots)$, where $e_n \in E_n$ and $\mathbf{r}_n(e_n) = \mathbf{s}_{n+1}(e_{n+1})$ for all n. If $(e_1, e_2, \ldots, e_n)$ is a finite path, then its *length* is n. Additionally, a *path of length* 0 is a vertex of V_1. We will also sometimes consider paths from V_i to V_j for arbitrary $i < j$, but then we explicitly specify that the path begins in a vertex of V_i.

It is easy to see that the number of paths from $i \in V_1$ to $j \in V_n$ is equal to the entry in the ith column and jth row of the matrix $A_1 A_2 \cdots A_{n-1}$.

We denote by $\mathcal{P}_n(\mathsf{B})$ the set of paths of length n (from V_1 to V_n). The set of all infinite paths $\mathcal{P}(\mathsf{B})$ is a closed subset of the direct product $E_1 \times E_2 \times \cdots$, and we consider it as a topological space with the induced topology. The space $\mathcal{P}(\mathsf{B})$ is compact, totally disconnected, and metrizable.

We say that a path $(e_1, e_2, \ldots, e_n) \in \mathcal{P}_n(\mathsf{B})$ *starts* in $\mathbf{s}_1(e_1)$ and *ends* in $\mathbf{r}_n(e_n)$. The *multiplicity* of a vertex $v \in V_n$ is the number of paths $\gamma \in \mathcal{P}_n(\mathsf{B})$ ending in v. We denote it $m(v)$.

If $\mathsf{B} = ((V_n)_{n=1}^{\infty}, (E_n)_{n=1}^{\infty}, \mathbf{s}, \mathbf{r})$ is a Bratteli diagram, then we denote by $\dot{\mathsf{B}}$ the diagram $((V_n)_{n=0}^{\infty}, (E_n)_{n=0}^{\infty}, \mathbf{s}, \mathbf{r})$ obtained by adding one level of vertices V_0 consisting of one vertex and one level of edges E_0 such that $\mathbf{r} : E_0 \longrightarrow V_1$ is a bijection.

Definition 1.3.6. Let $\mathsf{B} = ((V_n)_{n=0}^{\infty}, (E_n)_{n=0}^{\infty}, \mathbf{s}, \mathbf{r})$ be a Bratteli diagram. Let $k_0 = 0 < k_1 < k_2 < \ldots$ be an increasing sequence of integers. The *telescoping* of B defined by the sequence is the diagram $((V'_n)_{n=0}^{\infty}, (E'_n)_{n=0}^{\infty}, \mathbf{s}, \mathbf{r})$, where $V'_n = V_{k_n}$, E'_n is the set of paths in B from V_{k_n} to $V_{k_{n+1}}$, and $\mathbf{s}$, $\mathbf{r}$ are the beginning and the end of the paths, respectively.

We write $\mathsf{B}_1 \sim \mathsf{B}_2$ (and call the diagrams B_1 and B_2 *equivalent*) if one can transform $\dot{\mathsf{B}}_1$ by a sequence of telescopings and operations inverse to telescoping to a diagram isomorphic to $\dot{\mathsf{B}}_2$. We will see later that $\mathsf{B}_1 \sim \mathsf{B}_2$ if and only if there exists a diagram B such that a telescoping of $\dot{\mathsf{B}}$ is a telescoping of $\dot{\mathsf{B}}_1$ and another telescoping of $\dot{\mathsf{B}}$ is a telescoping of $\dot{\mathsf{B}}_2$.

Example 1.3.7. Let B_1 be the diagram with one vertex and two edges on every level. Let B_2 be the diagram with two vertices on each level and

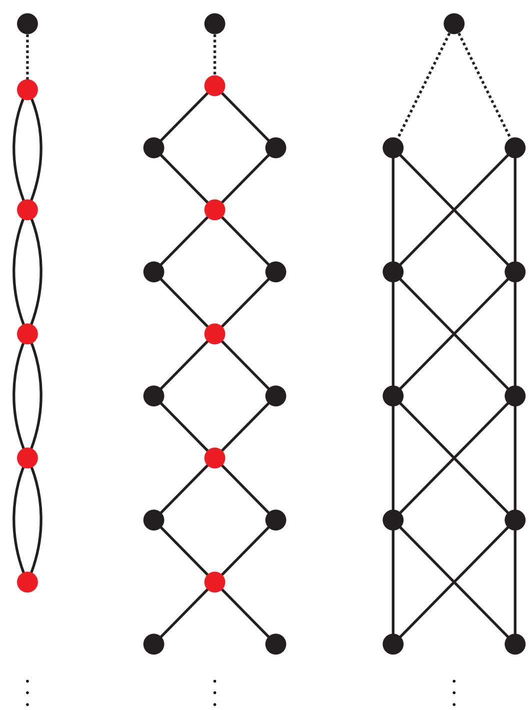

Figure 1.16. Equivalence of diagrams

complete bipartite graphs of edges on each level. Then they are equivalent, see Figure 1.16, where telescopings of the diagram B shown in the middle are isomorphic to the diagram $\dot{\mathsf{B}}_1$ shown on the left and to the diagram $\dot{\mathsf{B}}_2$ shown on the right.

Bratteli diagrams were introduced by O. Bratteli in [**Bra72**] to describe approximately finite C^*-algebras (i.e., direct limits of finitely dimensional C^*-algebras). They can be used to describe inductive limits of direct products of various algebraic structures with respect to *block-diagonal* embeddings.

Example 1.3.8. Consider for every level n of the diagram B the direct sum $A_n = \bigoplus_{v \in V_n} M_{m(v)\times m(v)}(\Bbbk)$ of the algebras of $m(v) \times m(v)$-matrices over a field $\Bbbk$. For every vertex $v \in V_n$ consider the set of edges $e \in E_{n-1}$ ending in v. Then the algebra $M_{m(v)\times m(v)}(\Bbbk)$ contains a sub-algebra of block-diagonal matrices isomorphic to $\bigoplus_{e \in \mathbf{r}^{-1}(v)} M_{m(\mathbf{s}(e))\times m(\mathbf{s}(e))}(\Bbbk)$. We get hence a *block-diagonal embeddings* $A_{n-1} \hookrightarrow A_n$, and the corresponding inductive limit $M_{\mathsf{B}}(\Bbbk)$, defined by the diagram. We can also consider the inductive limit in the category of C^*-algebras, in the case $\Bbbk = \mathbb{C}$, which was the original motivation of [**Bra72**].

Example 1.3.9. Let G be a group. For every $n \geqslant 1$, let G_n be the direct product $\prod_{v \in V_n} G^{m(v)}$. For every $v \in V_n$ the group $G^{m(v)}$ is isomorphic to $\prod_{e \in \mathbf{r}^{-1}(v)} G^{m(\mathbf{s}(e))}$, where the direct factors are labeled by paths of length n. We have natural *block-diagonal embeddings* $G_{n-1} \hookrightarrow G_n$ mapping the factor corresponding to a path γ diagonally to the factors corresponding to its continuations. The inductive limit of these maps is naturally isomorphic to the group of all continuous (i.e., locally constant) maps $\mathcal{P}(\mathsf{B}) \longrightarrow G$ with pointwise multiplication, where G has discrete topology.

Example 1.3.10. Consider the direct sums G_n of the symmetric groups $\mathsf{S}_{m(v)}$ for $v \in V_n$. Each group $\mathsf{S}_{m(v)}$ acts by permutations on the set of all paths $\gamma \in \mathcal{P}_n(\mathsf{B})$ ending in v. Then the permutation group $\mathsf{S}_{m(v)}$ naturally contains an isomorphic copy of the direct sum of the permutation groups $\mathsf{S}_{m(\mathbf{s}(e))}$ for all $e \in \mathbf{r}^{-1}(v)$. We get hence block-diagonal embeddings $G_{n-1} \hookrightarrow G_n$ and the direct limit of finite groups, defined by the diagram. Similarly, one can take direct sums of the alternating groups $\mathsf{A}_{m(v)}$ and the corresponding embeddings and direct limit. See 5.2.3 for more about these constructions.

Example 1.3.11. Let B be a Bratteli diagram defined by the sequence of matrices $A_1, A_2, \ldots$. Consider the sequence of abelian groups

$$\mathbb{Z}^{|V_1|} \xrightarrow{A_1} \mathbb{Z}^{|V_2|} \xrightarrow{A_2} \mathbb{Z}^{|V_3|} \xrightarrow{A_3} \cdots .$$

Its direct limit is called the *dimension group* of the diagram. Its *positive cone* is the union of the images of the subsemigroups $\mathbb{Z}_+^{|V_n|} \subset \mathbb{Z}^{|V_n|}$ in the direct limit (where $\mathbb{Z}_+$ is the semigroup of non-negative integers).

Note that in all the above examples $\mathsf{B}_1 \sim \mathsf{B}_2$ implies for each of Examples 1.3.8–1.3.11 that the direct limits defined by the diagrams B_1 and B_2 are isomorphic.

1.3.4. Vershik-Bratteli diagrams.

Definition 1.3.12. A *Vershik-Bratteli diagram* (or an *ordered diagram*) is a Bratteli diagram together with a linear order on each set $\mathbf{r}^{-1}(v)$. An edge e is called *minimal* (resp. *maximal*) if it is minimal (resp. maximal) in the set $\mathbf{r}^{-1}(\mathbf{r}(e))$. A path is said to be *minimal* (resp. *maximal*) if it consists of minimal (resp. maximal) edges only.

An ordering of the edges of a Vershik-Bratteli diagram $\mathsf{B} = ((V_i), (E_i), (\mathbf{s}_i), (\mathbf{r}_i))$ defines a natural *lexicographic order* on the sets of path between V_i and V_j for every pair $i < j$ and on the set $\mathcal{P}(\mathsf{B})$ of infinite paths in B.

Namely, two finite paths $(e_1, e_2, \ldots, e_n), (f_1, f_2, \ldots, f_n)$ are comparable if and only if $\mathbf{r}(e_n) = \mathbf{r}(f_n)$. Then $(e_1, e_2, \ldots, e_n) < (f_1, f_2, \ldots, f_n)$ if $e_k < f_k$

for the largest index k such that $e_k \neq f_k$. Note that then $\mathbf{r}(e_k) = \mathbf{r}(f_k)$, so e_k and f_k are comparable. In particular, a telescoping of a Vershik-Bratteli diagram is again a Vershik-Bratteli diagram with respect to the lexicographic order on the contracted paths (which become edges of the telescoping, see Definition 1.3.6).

Similarly, two infinite paths $(e_1, e_2, \ldots)$ are comparable if and only if they are co-final, i.e., if $e_n = f_n$ for all n big enough. Then we have $(e_1, e_2, \ldots) < (f_1, f_2, \ldots)$ if $e_k < f_k$ for the largest index k such that $e_k \neq f_k$.

The *adic transformation* (or the *Vershik map*) is defined on the set of non-maximal paths in $\mathcal{P}(\mathsf{B})$ and maps a path $\gamma = (e_1, e_2, \ldots)$ to the next path in the lexicographic order (i.e., to the smallest path among the paths bigger that γ). It is computed in the following way. Find the first index n such that the edge e_n is non-maximal. Let e'_n be the next edge in $\mathbf{r}_n^{-1}(\mathbf{r}_n(e_n))$, and let $(e'_1, e'_2, \ldots, e'_{n-1})$ be the minimal path such that $\mathbf{r}_{n-1}(e'_{n-1}) = \mathbf{s}_n(e'_n)$ (it exists and is unique, since for every vertex v there exists a unique minimal edge $e \in \mathbf{r}^{-1}(v)$). It is easy to see that the path $(e'_1, e'_2, \ldots, e'_{n-1}, e'_n, e_{n+1}, e_{n+2}, \ldots)$ is the minimal path among the paths bigger than γ in the lexicographic order. See Figure 1.17, where the arrows show the ordering of the edges, and point from a smaller edge to the bigger one. The map τ changes the highlighted red beginning of a path to the black path. (check the colors...)

The adic transformation is defined on the set of non-maximal paths, and its set of values is the set of non-minimal paths. Note that the inverse of the adic transformation is the adic transformation of the Vershik-Bratteli diagram obtained from B by reverting the ordering of the edges.

The adic transformation is continuous on the set of non-maximal paths, since it changes in every non-maximal path $(e_1, e_2, \ldots)$ only the finite beginning $(e_1, e_2, \ldots, e_n)$, where e_n is the first non-maximal edge in the path. However, it may not have a continuous extension to the whole space $\mathcal{P}(\mathsf{B})$.

Consider, for example, the Vershik-Bratteli diagram shown on the right-hand side of Figure 1.25. Let us label the vertices of each level by $\mathsf{0}$ and $\mathsf{1}$ so that $\mathsf{0}$ is on the left-hand side, and denote paths in the diagram by the sequence of vertices through which it passes (in this case this notation is non-ambiguous). It has two minimal paths $\mathsf{010101}\ldots$ and $\mathsf{101010}\ldots$, and one maximal path $\mathsf{1111}\ldots$. The adic transformation can not be extended continuously to the whole space of paths. Namely, it maps a sequence $w = \mathsf{111}\ldots\mathsf{101}v$ either to $\mathsf{0101}\ldots\mathsf{011}v$ or to $\mathsf{1010}\ldots\mathsf{011}v$ depending on the parity of the number of the leading ones in w, so it can not be continuously defined at $\mathsf{111}\ldots$.

Another example is given by the diagram shown on the right-hands side half of Figure 1.19. In this case there are two maximal and two minimal

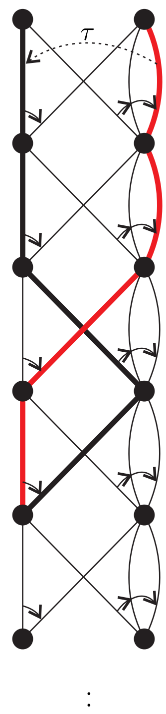

Figure 1.17. Adic transformation

infinite paths, but there is no continuous extension of the adic transformation to the whole space of paths.

Definition 1.3.13. We say that a Vershik-Bratteli diagram B is *properly ordered* if it has a unique maximal and a unique minimal paths in $\mathcal{P}(\mathsf{B})$.

Proposition 1.3.14. *Suppose that a Vershik-Bratteli diagram* B *is properly ordered. Let* $\gamma_{\max}$ *and* $\gamma_{\min}$ *be the maximal and the minimal paths, respectively. Then the extension of the adic transformation* τ *given by* $\tau(\gamma_{\max}) = \gamma_{\min}$ *is a homeomorphism.*

We will also call the extension of the adic transformation τ given in Proposition 1.3.14 the *adic homeomorphism*.

Proof. Since changing the ordering to the opposite one changes the adic transformation to the inverse, it is enough to prove that the defined extension of the adic transformation is continuous. The transformation is continuous at non-maximal paths, hence it is enough to prove that τ is continuous at $\gamma_{\max}$.

Let γ_i be a sequence of non-maximal infinite paths converging to $\gamma_{\max}$. It is enough to prove that we always have $\lim_{i\to\infty} \tau(\gamma_i) = \gamma_{\min}$. Let α_i be the longest common prefix of γ_i and $\gamma_{\max}$. Let n_i be the length of α_i. We have $n_i \to \infty$ as $i \to \infty$. By the definition of τ, the beginning of $\tau(\gamma_i)$ of length n_i is a finite minimal path (i.e., a finite path consisting of minimal edges only). It follows that the limit $\lim_{i\to\infty} \gamma_i$ is a minimal infinite path, hence it is equal to $\gamma_{\min}$. □

Definition 1.3.15. We say that a Bratteli diagram B is *simple* if for every level V_n there exists $m > n$ such that for every pair $v \in V_n$ and $u \in V_m$ there exists a path $(e_n, e_{n+1}, \dots, e_{m-1})$ in B starting in v and ending in u.

Proposition 1.3.16. *Let B be a properly ordered Vershik-Bratteli diagram, and let $\tau \circlearrowright \mathcal{P}(\mathsf{B})$ be the corresponding adic homeomorphism. There exists a unique closed non-empty subset $\mathcal{Y} \subset \mathcal{P}(\mathsf{B})$ such that $\tau \circlearrowright \mathcal{Y}$ is a minimal homeomorphism. If B is simple, then $\tau \circlearrowright \mathcal{P}(\mathsf{B})$ is minimal, i.e., $\mathcal{Y} = \mathcal{P}(\mathsf{B})$.*

Proof. Let $\gamma \in \mathcal{P}(\mathsf{B})$ be an arbitrary path, and consider a finite beginning α of γ. By the definition of the adic transformation, there exists $k \geqslant 0$ such that $\tau^k(\gamma)$ begins by the maximal path in the lexicographic order among the paths ending in the same vertex as α. It follows that the forward orbit $\tau^n(\gamma)$ for $n \geqslant 0$ contains arbitrarily long prefixes that are maximal paths (it is possible that one of such prefixes is the whole path $\gamma_{\max}$). It follows from the uniqueness of the maximal path that $\gamma_{\max}$ belongs in the closure of the τ-orbit of γ. Consequently, closure of the τ-orbit of $\gamma_{\max}$ is the unique closed non-empty subset $\mathcal{Y} \subset \mathcal{P}(\mathsf{B})$ such that $\tau \circlearrowright \mathcal{Y}$ is minimal.

If B is simple, then for any infinite path γ and for any finite path α there exists a finite path β such that $\alpha\beta$ ends in a vertex of γ. It follows that there exists a path $\gamma' \in \mathcal{P}(\mathsf{B})$ starting with α that has a common infinite suffix with γ. Then γ' and γ belong to the same τ-orbit. This shows that the τ-orbit of γ intersects every open subset of $\mathcal{P}(\mathsf{B})$, i.e., that every τ-orbit is dense. □

Example 1.3.17. Let $d_1, d_2, \dots$ be a sequence of integers greater than 1. Consider the associated odometer $x \mapsto x+1$ acting on the inverse limit of the cyclic groups $\mathbb{Z}/d_1 d_2 \cdots d_n\mathbb{Z}$, see 1.3.1.1. It follows from the description of the action of the odometer on the set of infinite formal series (1.2) that it is topologically conjugate to the adic homeomorphism defined by the Vershik-Bratteli diagram B such that $|V_n| = 1$ and $|E_n| = d_n$. A particular case, for $2 = d_1 = d_2 = \dots$, was considered in 1.1.4.

1.3.5. Diagrams associated with sequences of Rokhlin-Kakutani partitions. Let $\tau \circlearrowright \mathcal{X}$ be a minimal homeomorphism of a Cantor set. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are Rokhlin-Kakutani partitions, then we write $\mathcal{R}_1 \prec \mathcal{R}_2$ if the base of $\mathcal{R}_1$ is contained in the base of $\mathcal{R}_2$, and $\mathcal{R}_2$ is a refinement of $\mathcal{R}_1$ (i.e., every element of $\mathcal{R}_2$ is a subset of an element of $\mathcal{R}_1$).

Consider a sequence of Rokhlin-Kakutani paritions $\mathcal{R}_1, \mathcal{R}_2, \ldots$ with bases $B_1, B_2, \ldots$ such that $\mathcal{R}_1 \prec \mathcal{R}_2 \prec \ldots$. We assume that the base B_1 is equal to $\mathcal{X}$, i.e., that all towers of $\mathcal{R}_1$ are of length 1.

We are going to associate a Vershik-Bratteli diagram B with the sequence $\mathcal{R}_n$. Take the nth level set of vertices V_n equal to the set of towers of $\mathcal{R}_n$. (In particular, V_1 is the set of elements of $\mathcal{R}_1$.)

Let $(C, \tau(C), \ldots, \tau^{k-1}(C))$ be a tower of $\mathcal{R}_{n+1}$. Then $C \subset B_{n+1}$ is a subset of an element C' of $\mathcal{R}_n$, and $C' \subset B_n$. Each of the elements of the tower is also contained in an element of $\mathcal{R}_n$. It follows that the tower is naturally split into segments

$$\begin{array}{llll}
(C_0 = CC, & \tau(C), & \ldots, & \tau^{l_1-1}(C)), \\
(C_1 = \tau^{l_1}(C), & \tau^{l_1+1}(C), & \ldots, & \tau^{l_1+l_2-1}(C)), \\
(C_2 = \tau^{l_1+l_2}(C), & \tau^{l_1+l_2+1}(C), & \ldots, & \tau^{l_1+l_2+l_3-1}(C)), \\
\vdots & & & \\
(C_{s-1} = \tau^{l_1+l_2+\cdots+l_{s-1}}(C), & \tau^{l_1+l_2+\cdots+l_{s-1}+1)}(C), & \ldots, & \tau^{l_1+l_2+\cdots l_s-1}(C)),
\end{array}$$

such that for each segment

$$(C_i, \tau(C_i), \ldots \tau^{l_{i+1}-1}(C_i))$$

there exists a (necessarily unique) tower $v_i = (C_i, \tau(C_i), \ldots, \tau_{l_i-1}(C_i))$ of $\mathcal{R}_n$ such that

$$\tau^{l_1+l_2+\cdots+l_{i-1}+j}(C) \subset \tau^j(C_i).$$

In other words, the towers of $\mathcal{R}_{n+1}$ are split into disjoint unions of restrictions of towers of $\mathcal{R}_n$. Let us connect the vertex of V_{n+1} corresponding to the tower $(C, \tau(C), \ldots, \tau^{k-1}(C))$ of $\mathcal{R}_{n+1}$ to the vertices corresponding to the towers v_i in the natural order of their appearance in the above list of segments. Note that v_i are not necessarily pairwise different, so we may get multiple edges. Then each edge connecting a tower $(A, \tau(A), \ldots, \tau^k(A))$ of $\mathcal{R}_{n+1}$ to a tower $(B, \tau(B), \ldots, \tau^l(B))$ of $\mathcal{R}_n$ is in a bijective correspondence with a segment $(\tau^i(A), \tau^{i+1}(A), \ldots, \tau^{i+l}(A))$ of the first tower such that $\tau^i(A) \subset B$. We say that this segment *corresponds* to the edge.

We call the constructed Vershik-Bratteli diagram B the *diagram associated with the sequence* $(\mathcal{R}_n)_{n=1,2,\ldots}$.

See Figure 1.18, where an example of a pair $\mathcal{R}_n \prec \mathcal{R}_{n+1}$ of Rokhlin-Kakutani partitions and the corresponding level of the Vershik-Bratteli diagram are shown. Bigger boxes represent elements of $\mathcal{R}_n$. Black and gray rectangles depict elements of the partitions $\mathcal{R}_{n+1}$ and $\mathcal{R}_n$, respectively, belonging to their bases. The towers and the corresponding vertices of the Vershik-Bratteli diagram are labeled by letters v_i and u_i are placed near the elements belonging to the bases.

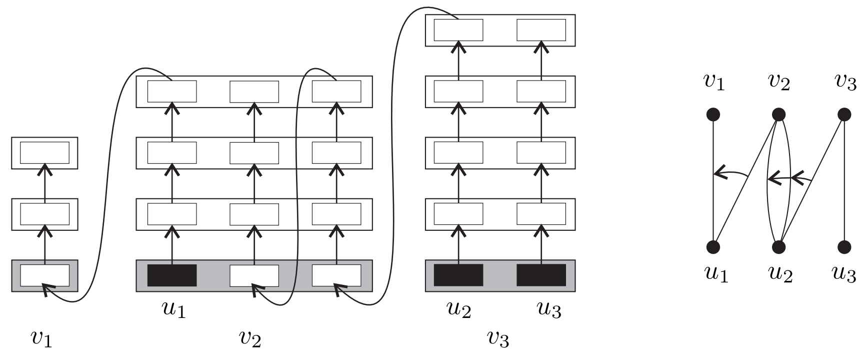

Figure 1.18. Refinement of a Rokhlin-Kakutani partition

Proposition 1.3.18. *Let* B *be the Vershik-Bratteli diagram associated with a sequence* $\mathcal{R}_1 \prec \mathcal{R}_2 \prec \ldots$. *Then there is a unique sequence of bijections* $\phi_n : \mathcal{P}_n(\mathsf{B}) \longrightarrow \mathcal{R}_n$ *such that* $\phi_n(e_1, e_2, \ldots, e_n) \in \mathcal{R}_n$ *is a subset of* $\phi_{n-1}(e_1, e_2, \ldots, e_{n-1})$ *(of* $\phi_0(\mathbf{s}(e_1))$ *for* $n = 1$*), and an element of the segment* e_n *of the tower* $\mathbf{r}(e_n)$.

Proof. Let $e \in E_n$ be an edge of B. Then $\mathbf{s}(e)$ is a tower of $\mathcal{R}_n$, and $\mathbf{r}(e)$ is a tower of $\mathcal{R}_{n+1}$. The tower $\mathbf{r}(e)$ contains a segment (determined by e) $A, \tau(A), \ldots, \tau^{l-1}(A)$ such that the tower $\mathbf{s}(e)$ is of the form $A', \tau(A'), \ldots, \tau^{-1}(A')$ for some $A' \in \mathcal{R}_n$ such that $A \subset A'$. We see that for every element C' of the tower $\mathbf{s}(e)$ there is a unique element C of the tower $\mathbf{r}(e)$ such that $C \subset C'$ and C belongs to the segment fo $\mathbf{r}(e)$ corresponding to the edge e. The proof of the proposition now follows by induction. □

The following proposition is also proved by induction on n. We leave it as an exercise.

Proposition 1.3.19. *Two paths* $\gamma_1, \gamma_2 \in \mathcal{P}_n(\mathsf{B})$ *are comparable if and only if* $\phi_n(\gamma_1)$ *and* $\phi_n(\gamma_2)$ *belong to the same tower of* $\mathcal{R}_n$. *If* γ_2 *is the smallest path bigger than* γ_1, *then* $\tau(\phi_n(\gamma_1)) = \phi_n(\gamma_2)$.

In particular, a path $\gamma \in \mathcal{P}_n(\mathsf{B}))$ *is minimal (resp. maximal) if and only if* $\phi_n(\gamma)$ *(resp.* $\tau(\phi_n(\gamma))$*) is contained in the base of* $\mathcal{R}_n$.

The following theorem is a result of [**HPS92**]...

Theorem 1.3.20 (R.H. Herman, I.F. Putnam, C.F. Skau)**.** *Let* $\tau \circlearrowright \mathcal{X}$ *be a minimal homeomorphism of a Cantor set. Then it is topologically conjugate to the adic homeomorphism of a simple properly ordered Vershik-Bratteli diagram.*

Proof. Let $f \circlearrowright \mathcal{X}$ be a minimal homeomorphism of a Cantor set. Choose a metric d on $\mathcal{X}$ compatible with the topology. Take an arbitrary point $x_0 \in \mathcal{X}$.

Let $\mathcal{R}_0$ be an arbitrary finite partition of $\mathcal{X}$ into non-empty clopen subsets (e.g., $\{\mathcal{X}\}$), seen as a Rokhlin-Kakutani partition with the base $\mathcal{X}$. Let $\mathcal{P}_n$ be a sequence of finite clopen partitions such that maximum D_n of diameters of elements of $\mathcal{P}_n$ converges to zero. Let $C_n \in \mathcal{P}_n$ be the element containing x_0. Construct recursively $\mathcal{R}_n$ as a Rokhlin-Kakutani partition subordinate to $\mathcal{R}_{n-1}$ and $\mathcal{P}_n$ with the base a subset of C_n, see Proposition 1.3.5. We get a sequence $\mathcal{R}_0 \prec \mathcal{R}_1 \prec \ldots$. Let B be the associated Vershik-Bratteli diagram. Let $\phi_n : \mathcal{P}_n(\mathsf{B}) \longrightarrow \mathcal{R}_n$ be the bijections described in Proposition 1.3.18. Then for every infinite path $\gamma = (e_1, e_2, \ldots) \in \mathcal{P}(\mathsf{B})$ we get a decreasing sequence of sets $\phi_n(e_1, e_2, \ldots, e_n) \in \mathcal{R}_n$. Their intersection is non-empty by compactness, and of diameter zero, since the diameter of every element of $\mathcal{R}_n$ is less than D_n. It follows that they intersect in one point, which we will denote $\phi(\gamma)$. If γ_1 and γ_2 agree on a beginning of length n, then $d(\phi(\gamma_1), \phi(\gamma_2)) \leqslant D_n$, which implies that $\phi : \mathcal{P}(\mathsf{B}) \longrightarrow \mathcal{X}$ is continuous. For every $x \in \mathcal{X}$ and n the point x belongs to a single element of $\mathcal{R}_n$, and, by Proposition 1.3.18, this element is equal to $\phi_n(e_1, e_2, \ldots, e_n)$ for some $\gamma = (e_1, e_2, \ldots) \in \mathcal{P}(\mathsf{B})$ not depending on n. It follows that ϕ is onto. It is also easy to see that it is one-to-one, hence a homeomorphism.

The intersection of the bases of $\mathcal{R}_n$ is non-empty (since the base of $\mathcal{R}_{n+1}$ is non-empty and contained in the base of $\mathcal{R}_n$) and contained in C_n. It follows that the intersection of the bases is x_0. It follows that the minimal path in B is unique and its image under ϕ is x_0 (see the characterization of the minimal path in Proposition 1.3.18). Similarly, the intersection of the images of the bases under f^{-1} is equal to $\{f^{-1}(x_0)\}$. In particular, the minimal and the maximal paths in B are unique. Proposition 1.3.18 shows now that ϕ conjugates the adic homeomorphism with f. $\square$

In fact, the proof of Theorem 1.3.20 shows if $\tau \circlearrowright \mathcal{X}$ is a minimal system, and $x \in \mathcal{X}$, then there exists a Vershik-Bratteli diagram B and a homeomorphism $\phi : \mathcal{P}(\mathsf{B}) \longrightarrow \mathcal{X}$ satisfying the conditions of the theorem and such that ϕ maps the minimal path to x.

In fact, we have the following characterization of the Vershik-Bratteli models of minimal Cantor systems.

Theorem 1.3.21. *Let B_1 and B_2 be properly ordered Vershik-Bratteli diagrams, and let τ_i be the adic homeomorphism of $\mathcal{P}(\mathsf{B}_i)$. Then the Vershik-Bratteli diagrams B_i are equivalent if and only if there exists a homeomorphism $\phi : \mathcal{P}(\mathsf{B}_1) \longrightarrow \mathcal{P}(\mathsf{B}_2)$ mapping the minimal, resp. maximal, path of B_1 to the minimal, resp. maximal, path of B_2 and conjugating the corresponding adic transformations.*

reference...

Proof. (A sketch.) Telescoping a Vershik-Bratteli diagram does not change the space of its paths and the lexicographic order on them, therefore it does not change the corresponding adic transformation. It follows that equivalent Vershik-Bratteli diagrams have topologically conjugate adic transformations.

In the other direction, it is easy to see that if $\mathcal{R}_0 \prec \mathcal{R}_1 \prec \mathcal{R}_2 \prec \ldots$ is an increasing sequence of Rokhlin-Kakutani partitions with the associated Vershik-Bratteli diagram B, then the diagram associated with a subsequence $\mathcal{R}_0 \prec \mathcal{R}_{n_1} \prec \mathcal{R}_{n_2} \prec \ldots$ is a telescoping of the diagram B. Suppose that $\mathcal{R}_0 \prec \mathcal{R}_1 \prec \mathcal{R}_2 \prec \ldots$ and $\mathcal{R}_0 \prec \mathcal{R}'_1 \prec \mathcal{R}'_2 \prec$ are two sequences of Rokhlin-Kakutani partitions for a given minimal system $f \circlearrowleft \mathcal{X}$ such that the maximal diameters of the elements of the partitions go to zero in both sequences, and $x_0 \in \mathcal{X}$ is contained in the bases of all partitions $\mathcal{R}_n$ and $\mathcal{R}'_n$.

Then it is easy to prove (using Lebesgue lemma) that for every n there exists m such that $\mathcal{R}_n \prec \mathcal{R}'_m$ and $\mathcal{R}'_n \prec \mathcal{R}_m$. It follows that there exists a sequence $n_1 < n_2 < n_3 < \ldots$ such that $\mathcal{R}_0 \prec \mathcal{R}'_{n_1} \prec \mathcal{R}_{n_2} \prec \mathcal{R}'_{n_3} \prec \ldots$. This shows that the associated Vershik-Bratteli diagrams of the sequences $(\mathcal{R}_n)$ and $(\mathcal{R}_n)'$ are equivalent. □

1.3.6. Kakutani equivalence. Let $f \circlearrowleft \mathcal{X}$ be a minimal homeomorphism of a Cantor set, and let $\mathcal{Y} \subset \mathcal{X}$ be a non-empty clopen subset. Then for every $x \in \mathcal{Y}$ there exists $n > 0$ such that $f^n(x) \in \mathcal{Y}$, see Lemma 1.3.4. Let n_x be the smallest such number. Then $f_{\mathcal{Y}} : x \mapsto f^{n_x}(x)$ is called the *first return map* to $\mathcal{Y}$. Since $x \mapsto n_x$ is locally constant, it is continuous. The inverse map is the map $x \mapsto f^{m_x}(x)$, where m_x is the smallest positive integer such that $f^{-m_x}(x) \in \mathcal{Y}$. Since the orbits of the first return map are intersections of the f-orbits with $\mathcal{Y}$, the first return map $f_{\mathcal{Y}} \circlearrowleft \mathcal{Y}$ is a minimal homeomorphism.

Definition 1.3.22. Two minimal Cantor dynamical systems $f_1 \circlearrowleft \mathcal{X}_1$ and $f_2 \circlearrowleft \mathcal{X}_2$ are said to be *Kakutani equivalent* if there exist non-empty clopen subsets $\mathcal{Y}_i \subset \mathcal{X}_i$ such that first return maps to $(f_1)_{\mathcal{Y}_1} \circlearrowleft \mathcal{Y}_1$ and $(f_2)_{\mathcal{Y}_2} \circlearrowleft \mathcal{Y}_2$ are topologically conjugate.

Proposition 1.3.23. *The Kakutani equivalence is an equivalence relation.*

Proof. The Kakutani equivalence is obviously reflexive and symmetric. Let us show that it is transitive. Suppose that $f_1 \circlearrowleft \mathcal{X}_1$ is Kakutani equivalent to $f_2 \circlearrowleft \mathcal{X}_2$, and $f_2 \circlearrowleft \mathcal{X}_2$ is equivalent to $f_3 \circlearrowleft \mathcal{X}_3$. Then there exist non-empty clopen subsets $\mathcal{Y}_1 \subset \mathcal{X}_1$, $\mathcal{Y}_2 \subset \mathcal{X}_2$, $\mathcal{Z}_2 \subset \mathcal{X}_2$, $\mathcal{Z}_3 \subset \mathcal{X}_3$ such that $(f_1)_{\mathcal{Y}_1} \circlearrowleft \mathcal{Y}_1$ is topologically conjugate to $(f_2)_{\mathcal{Y}_2} \circlearrowleft \mathcal{Y}_2$ and $(f_2)_{\mathcal{Z}_2} \circlearrowleft \mathcal{Z}_2$ is topologically conjugate to $(f_3)_{\mathcal{Z}_3} \circlearrowleft \mathcal{Z}_3$.

There exists m such that $U = f^m(\mathcal{Z}_2) \cap \mathcal{Y}_2$ is non-empty. Then the conjugacy $\phi_1 : \mathcal{Y}_1 \longrightarrow \mathcal{Y}_2$ between $(f_1)_{\mathcal{Y}_1} \circlearrowleft (f_2)_{\mathcal{Y}_2}$ restricts to a topological conjugacy of $(f_1)_{\phi^{-1}(U)} \circlearrowleft \phi^{-1}(U)$ with $(f_2)_U \circlearrowleft U$. The map $f^{-m} : \mathcal{X}_2 \longrightarrow \mathcal{X}_2$ is a conjugacy of $f_2 \circlearrowleft \mathcal{X}_2$ with itself, which induces a conjugacy of $(f_2)_U \circlearrowleft U$ with $(f_2)_{f^{-m}(U)} \circlearrowleft f^{-m}(U)$. Note that $f^{-m}(U) \subset \mathcal{Z}_2$, and then the conjugacy $\phi_2 : \mathcal{Z}_2 \mapsto \mathcal{Z}_3$ of $(f_2)_{\mathcal{Z}_2} \circlearrowleft \mathcal{Z}_2$ with $(f_3)_{\mathcal{Z}_3} \circlearrowleft \mathcal{Z}_3$ will induce a conjugacy of $(f_2)_{f^{-m}(U)} \circlearrowleft f^{-m}(U)$ with $(f_3)_{\phi_2(f^{-m}(U))} \circlearrowleft \phi_2(f^{-m}(U))$. It follows that $(f_1)_U \circlearrowleft U$ and $(f_3)_{\phi_2(f^{-m}(U))} \circlearrowleft \phi_2(f^{-m}(U))$ are topologically conjugate, hence f_1 and f_3 are Kakutani equivalent. $\square$

Example 1.3.24. Consider the Denjoy homeomorphism $\tilde{R}_\theta \circlearrowleft \mathcal{X}_\theta$ described in 1.3.1.2. Let $a, b \in [0, 1]$ be elements of the orbit of 0 under the rotation R_θ. Suppose that $a < b$, and let $[a + 0, b - 0]$ be the corresponding clopen subset of $\mathcal{X}_theta$. We will sometimes denote such subsets just $[a, b]$, when it does not lead to confusion. Let n be the smallest positive integer such that $c = \mathrm{frac}(b + n\theta) \in (a, b)$. Then the first return map $(\tilde{R}_\theta)_{[a,b]}$ maps the interval $[b-c+a, b]$ to $[a, c]$ and the interval $[a, b-c+a]$ to $[c, b]$ by parallel translations. If we identify $[a, b]$ with $[0, 1]$ by the affine transformation $x \mapsto \frac{x-a}{b-a}$, then we get the transformation swapping $\left[\frac{b-c}{b-a}, 1\right]$ with $\left[0, \frac{b-c}{b-a}\right]$, i.e., the rotation by the angle $\frac{b-c}{b-a}$. Note that $\frac{b-c}{b-a}$ is of the form $\frac{k\theta+l}{m\theta+n}$ for some $k, l, m, n \in \mathbb{Z}$, since a, b, c belong to the orbit of 0 under the rotation by θ. We will give a complete description of the Kakutani equivalence classes of the homeomorphisms $\tilde{R}_\theta$ later.

Proposition 1.3.25. *Let* B_1 *and* B_2 *are properly ordered Vershik-Bratteli diagrams. The associated adic transformations are Kakutani equivalent if and only if the diagrams* B_1 *and* B_2 *are equivalent to Vershik-Bratteli diagrams* B_1' *and* B_2' *that differ from each other on a finite number of levels.*

Proof. Let $\mathcal{R}_1 \prec \mathcal{R}_2 \prec \dots$ be a sequence of Rokhlin-Kakutani partitions, and let B be the associated Vershik-Bratteli diagram, as in in 1.3.2. Let $\mathcal{R}_n'$ for $n \geqslant 2$ be the partition of the base of $\mathcal{R}_2$ consisting of the elements of $\mathcal{R}_n$ contain in it. Then $\mathcal{R}_2' \prec \mathcal{R}_3' \prec \dots$ is a sequence of Rokhlin-Kakutani partitions of the first return map to the base of $\mathcal{R}_2$. It is also easy to check that the Vershik-Bratteli diagram associated with the sequence $(\mathcal{R}_n')_{n \geqslant 2}$ is obtained by deleting the first level of the diagram B. It follows that removing a finite number of initial levels of a Vershik-Bratteli diagram does not change the Kakutani class of the associated adic homeomorphism. Consequenlty, any finite change in the Vershik-Bratteli diagram does not change the Kakutani class of the adic transformation, since any such a change can be erased by deleting a finite number of levels. This proves the “if” direction of the proposition.

Let $f \circlearrowleft \mathcal{X}$ be a minimal homeomorphism, let $\mathcal{Y} \subset \mathcal{X}$ be a non-empty clopen subset, and let $\mathcal{R}_0 \prec \mathcal{R}_1 \prec \mathcal{R}_2 \prec \ldots$ be a sequence of Rokhlin-Kakutani partitions of $f \circlearrowleft \mathcal{X}$ separating the points of $\mathcal{X}$, so that the adic transformation on the associated Vershik-Bratteli diagram is naturally conjugate to $f \circlearrowleft \mathcal{X}$. We assume that $\mathcal{R}_0 = \{\mathcal{X}\}$. We may also assume that the base of $\mathcal{R}_1$ is contained in $\mathcal{Y}$ and every element of $\mathcal{R}_1$ is either contained in $\mathcal{Y}$ or disjoint with it. Then the same conditions will be satisfied for all $\mathcal{R}_n$, $n \geqslant 1$ (by the definition of the relation "$\prec$" on Rokhlin-Kakutani partitions). Denote by $\mathcal{R}'_n$, for $n \geqslant 1$, the partition of $\mathcal{Y}$ equal to set of the elements of $\mathcal{R}_n$ that are contained in $\mathcal{Y}$. It is checked directly that $\mathcal{R}'_n$ is a Rokhlin-Kakutani partition of the first return map $f_{\mathcal{Y}} \circlearrowleft \mathcal{Y}$, so that we get sequence of Rokhlin-Kakutani partitions

$$\{\mathcal{X}\} \prec \mathcal{R}_1 \prec \mathcal{R}_2 \prec \mathcal{R}_3 \prec \ldots$$

and

$$\{\mathcal{Y}\} \prec \mathcal{R}'_1 \prec \mathcal{R}'_2 \prec \mathcal{R}'_3 \prec \ldots$$

of the sysetms $f \circlearrowleft \mathcal{X}$ and $f_{\mathcal{Y}} \circlearrowleft \mathcal{Y}$, respectively. Both sequences separates the points of the corresponding spaces, so the associated Vershik-Bratteli diagrams model the corresponding dynamical systems. These Vershik-Bratteli diagrams differ only on the first level. This proves the "only if" direction of the proposition. □

1.3.7. Vershik-Bratteli diagrams as sequences of substitutions. Every level (V_n, E_n, V_{n+1}) of a Vershik-Bratteli diagram naturally defines a homomorphism of monoids $\phi_n : V_{n+1}^* \longrightarrow V_n^*$. Namely, if $v \in V_{n+1}$, and if $(e_1, e_2, \ldots, e_m)$ is the set $\mathbf{r}^{-1}(v)$ listed according to the ordering of E_n, then we set $\phi_n(v) = \mathbf{s}(e_1)\mathbf{s}(e_2)\ldots\mathbf{s}(e_m)$.

Conversely, every sequence of monoid homomorphisms

$$V_1^* \xleftarrow{\phi_1} V_2^* \xleftarrow{\phi_2} V_3^* \xleftarrow{\phi_3} \cdots \tag{1.3}$$

is naturally encoded by the Vershik-Bratteli diagram with the sets of vertices $V_1, V_2, \ldots$, where for every vertex $v \in V_{n+1}$ the set $\mathbf{r}^{-1}(v)$ of edges ending in v has $|\phi(v)|$ elements $e_1 < e_2 < \ldots < e_{|\phi(v)|}$, and $\phi(v) = \mathbf{s}(e_1)\mathbf{s}(e_2)\ldots\mathbf{s}(e_{|\phi(v)|})$. Telescoping of a Vershik-Bratteli diagram corresponds in this interpretation to compositions of the homomorphisms, i.e., replacing the sequence (1.3) of monoids by a subsequence (containing V_1^*) connected by the corresponding compositions of morphisms.

Definition 1.3.26. Consider a sequence

$$\mathsf{X}_1^* \xleftarrow{\phi_1} \mathsf{X}_2^* \xleftarrow{\phi_2} \mathsf{X}_3^* \xleftarrow{\phi_3} \cdots$$

of substitutions. The *subshift generated by it* is the subshift $\mathcal{F} \subset \mathsf{X}_1^{\mathbb{Z}}$ of all sequences $(x_n)_{n\in\mathbb{Z}}$ such that for every finite subword $v = x_n x_{n+1} \ldots x_m$ there exists k and $x \in \mathsf{X}_k$ such that v is a subword of $\phi_1 \circ \phi_2 \circ \cdots \circ \phi_k(x)$.

If the diagram is *stationary*, i.e., if all sequences $V_n, E_n, \mathbf{s}_n, \mathbf{r}_n$ and the ordering are constant, then the sequence ϕ_n is also constant. Conversely, the Vershik-Bratteli diagram associated with a constant sequence $(\phi, \phi, \ldots)$ of endomorphisms $\phi : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ is stationary.

For example, the stationary Vershik-Bratteli diagram on the left-hand side part of Figure 1.25 is associated with the substitution

$$\sigma(\mathsf{0}) = \mathsf{01}, \qquad \sigma(\mathsf{1}) = \mathsf{011},$$

if we label the vertices of a level of the diagram by $\mathsf{0}$ and $\mathsf{1}$ from left to right.

Let B be a Vershik-Bratteli diagram, and consider the corresponding sequence $\phi_n : V_{n+1}^* \longrightarrow V_n^*$ of monoid homomorphisms. For every $v \in V_n$ the set of paths ending in v is in a natural bijective order-preserving correspondence with the letters of the word $\phi_1 \circ \phi_2 \circ \cdots \circ \phi_{n-1}(v)$. Let $\gamma = (e_1, e_2, \ldots)$ be an infinite path in B. Denote $w_n(\gamma) = x_{k_1} x_{k_1+1} \ldots x_{k_2} = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n(\mathbf{r}(e_n))$, where the numbering of the letters is by consecutive integers such that x_0 is the letter corresponding to the path $(e_1, e_2, \ldots, e_n)$. Then $w_{n+1}(\gamma)$ is obtained from $w_n(\gamma)$ by appending letters to the beginning and/or to the end of $w_n(\gamma)$. It follows that in the limit we get a word $w_\infty(\gamma)$ associated with the path γ. The adic transformation acts as the shift on the associated words. Note that if γ is a minimal path, then $w_\infty(\gamma)$ is of the form $x_0 x_1 \ldots$. If γ is maximal, then $w_\infty(\gamma)$ is of the form $\ldots x_{-1} x_0$. In general, the word $w_\infty(\gamma)$ may be finite.

Different paths γ may produce the same words. For example, if $V_n = \{x\}$ and $|E_n| = 2$ for every n, then infinitely many paths in the diagram will be associated with the word $\ldots xxxx \ldots$. The remaining paths will be associated with one-sided sequences $xxxx\ldots$ and $\ldots xxxx$. In order to recover the path γ from the word $w_\infty(\gamma)$, we have to remember the "production process" of the words w_n and position of the zeroth coordinate. For example, in the last example, we can put brackets around subwords equal to the images of the single letters $x \in V_n$ under compositions of the homomorphisms. Such a bracketing may be still not enough for some Vershik-Bratteli diagrams, for example, if images of the different letters are equal. (Take, for example, the Vershik-Bratteli diagram of the constant sequence of substitutions $\sigma(a) = ab, \sigma(b) = ab$.)

The problem of uniqueness of the path γ for a word $w_\infty(\gamma)$ is called *recognizability*, see [**Ku03**, Section 4.3].

Example 1.3.27. Let $\sigma : \mathsf{0} \mapsto \mathsf{01}$, $\sigma : \mathsf{1} \mapsto \mathsf{10}$ be the Thue-Morse substitution. Note that if $w = (x_n)_{n\in\mathbb{Z}}$ is an element of the subshift generated by it, and $x_n x_{n+1} = \mathsf{00}$ or $x_n x_{n+1} = \mathsf{11}$, then we know that $x_{n-1}x_n$ and $x_{n+1}x_{n+2}$ are images of letters under σ. After that the preimage of w is uniquely determined. Note that we always can find such an index n, since otherwise the word w is of the form $\ldots abababab \ldots$, which does not belong to the Thue-Morse subshift. Consequently, the word w uniquely determines the corresponding path in the Vershik-Bratteli diagram of the substitution.

Example 1.3.28. Similarly, consider the Fibonacci substitution $\sigma(\mathsf{0}) = \mathsf{01}$, $\sigma(\mathsf{1}) = \mathsf{0}$, and let $\mathcal{S}_\sigma$ be the subshift generated by it. Then in every word $w \in \mathcal{S}_\sigma$ every letter $\mathsf{1}$ is preceded by $\mathsf{0}$, and the corresponding subword $\mathsf{01}$ is the image of a letter $\mathsf{0}$. The remaining letters $\mathsf{0}$ are the images of $\mathsf{1}$, so that the sequence w is uniquely decomposed into a concatenation of σ-images of letters of a sequence $w' \in \mathcal{S}_\sigma$.

In fact, the above examples are fairly typical. Namely, we have the following result of Mossé, see [**Ku03**, Theorem 4.36].

Theorem 1.3.29. *Let $\sigma : \mathsf{X} \longrightarrow \mathsf{X}^*$ be a primitive aperiodic substitution. Then it is recognizalbe, i.e., every sequence w in the subshift $\mathcal{F}_\sigma$ generated by σ can be uniquely decomposed into a concatenation of subwords $\ldots \sigma(y_{-1})\sigma(y_0)\sigma(y_1)\ldots$ for a word $\ldots y_{-1}y_0y_1 \ldots \in \mathcal{F}_\sigma$.*

Here we call a substitution $\sigma : \mathsf{X} \longrightarrow \mathsf{X}^*$ is *periodic* if it generates a finite subshift, i.e., if every element of the subshift generated by σ is a periodic sequence. Otherwise, it is called *aperiodic*.

Let $\sigma : \mathsf{X} \longrightarrow \mathsf{X}^*$ be a primitive substitution generating a subshift $\mathcal{F}_\sigma$, and let B_σ be the *stationary* Vershik-Bratteli diagram defined by the constant sequence σ. As described above, every path γ of B_σ defines an infinite sequence $w_\infty(\gamma)$. Let us assume for a moment that $\mathcal{F}_\sigma$ is infinite, i.e., that σ is aperiodic. Then $\mathcal{F}_\sigma$ is a minimal subshift, hence it can be defined by a properly ordered Vershik-Bratteli diagram. Note that B_σ is not properly ordered in general. Moreover, minimal and maximal paths of B_σ define one-sided sequences, and it is possible that they can be extended to elements of $\mathcal{F}_\sigma$ in many ways, so some paths of B_σ will correspond to several elements of $\mathcal{F}_\sigma$.

Example 1.3.30. The left-hand side part of Figure 1.19 shows the Vershik-Bratteli diagrams associated with the Fibonacci substitution $\mathsf{0} \mapsto \mathsf{01}, \mathsf{1} \mapsto \mathsf{0}$ (see Example 1.2.22)

Note that the diagram has one minimal path (passing through the vertices $\mathsf{0}, \mathsf{0}, \mathsf{0}, \mathsf{0}, \ldots$) and two maximal paths (passing through the vertices $\mathsf{1}, \mathsf{0}, \mathsf{1}, \mathsf{0}, \ldots$ and $\mathsf{0}, \mathsf{1}, \mathsf{0}, \mathsf{1}, \ldots$). The corresponding words $w_\infty(\gamma)$ are the right-infinite limit $\mathsf{01001010}\ldots$ of $\sigma^n(\mathsf{0})$, the left-infinite limit $\ldots\mathsf{01001010}$ of $\sigma^{2n}(\mathsf{0})$, and the left-infinite limit $\ldots\mathsf{01001001}$ of $\sigma^{2n}(\mathsf{1})$, respectively. Note

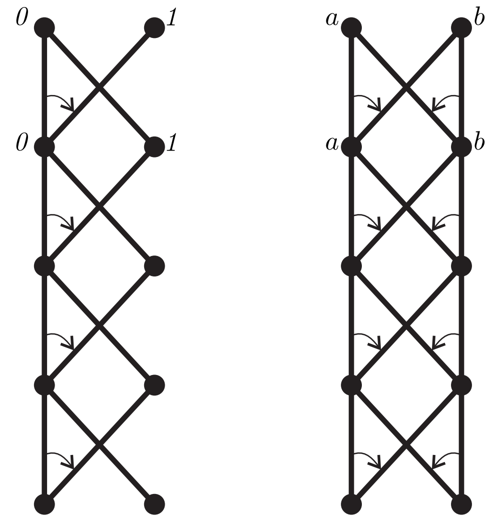

Figure 1.19. Vershik-Bratteli diagrams of Fibonacci and Thue-Morse substitutions

that concatenations of the right-infinite word with both left-infinite words belong to the subshift $\mathcal{F}_\sigma$ generated by σ. It follows that the minimal path in the Vershik-Bratteli diagram represents two different points of $\mathcal{F}_\sigma$.

Proposition 1.3.31. *The Vershik-Bratteli diagram* B_σ *associated with a substitution* $\sigma : \mathsf{X} \longrightarrow \mathsf{X}^*$ *is properly ordered if and only if there exists* $k \geqslant 1$ *and letters* $x_0, x_1 \in \mathsf{X}$ *such that for every* $x \in \mathsf{X}$ *the words* $\sigma^k(x)$ *starts with* x_0 *and ends with* x_1*.*

Proof. Consider the map $\alpha : \mathsf{X} \longrightarrow \mathsf{X}$ mapping x to the first letter of $\sigma(x)$. Note that $\alpha^k(x)$ is equal to the first letter of $\sigma^k(x)$.

The orbit of every letter $x \in \mathsf{X}$ under the iterations of α is eventually periodic, i.e., there exist m, n such that $\alpha^m(x) = \alpha^{m+n}(x)$. Note that if x belongs to a cycle, i.e., if $x = \alpha^n(x)$ for some $n \geqslant 1$, then there exists a minimal path starting in x and in every letter of the sequence $\alpha^i(x)$, $i \geqslant 1$. Since there exists only one minimal path, we must have $x = \alpha(x)$. Similarly, there can be only one point of X belonging to an α-cycle, and it is an α-fixed point. Denote it by x_0. Then for every $x \in \mathsf{X}$ there exists n such that $\alpha^n(x) = x_0$. Denoting by k the maximal value of such numbers n, we will get that the first letter of $\sigma^k(x)$ is x_0 for all $x \in \mathsf{X}$. The same argument proves the statement for the last letter.

Conversely, if σ satisfies the conditions of the proposition, then it is easy to see that there exist unique minimal (resp. maximal) path corresponding to the first (resp. last) letters of the words $\sigma^n(x_0)$ (resp. $\sigma^n(x_1)$). The same argument shows that there exists k such that the *last* letters of $\sigma^k(x)$ are equal for all $x \in \mathsf{X}$. $\square$

There is a simple algorithm described in [**DHS99**] producing a properly ordered stationary Vershik-Bratteli diagram B starting from a proper substitution σ, such that the adic transformation on B is topologically conjugate with the subshift generated by σ. Here we extend the notion of a stationary Vershik-Bratteli diagram by allowing the first level to be different from the subsequent levels. Namely, we adopt the following definition.

Definition 1.3.32. A *stationary* Vershik-Bratteli diagram is the diagram associated with an eventually constant sequence of substitutions.

Let $\sigma : \mathsf{X} \longrightarrow \mathsf{X}^*$ be a primitive aperiodic substitution, and let $\mathcal{F}_\sigma$ be the subshift generated by σ. After replacing σ by some iterate σ^n, we may assume that there exists a word $\xi = \dots x_{-2}x_{-1}.x_0x_1 \dots \in \mathcal{F}_\sigma$ such that

$$\dots x_{-2}x_{-1}.x_0x_1 \dots = \dots \sigma(x_{-2})\sigma(x_{-1}).\sigma(x_0)\sigma(x_1) \dots .$$

Since the subshift $\mathcal{F}_\sigma$ is minimal, the subword $x_{-1}x_0$ appears in ξ infinitely many times with uniformly bounded gaps between consecutive occurrences. More formally, we say that $k \in \mathbb{Z}$ is an *occurrence* of $x_{-1}x_0$ in a word $(y_n)_{n\in\mathbb{Z}}$ if $y_{k-1} = x_{-1}$ and $y_k = x_0$. Two occurrences $k_1 < k_2$ are *consecutive* if there is no occurrence k such that $k_1 < k < k_2$. It follows from Proposition 1.2.4 that there exists a uniform upper bound on $k_2 - k_1$ for any consecutive occurrences of $x_{-1}x_0$ in any element of $\mathcal{F}_\sigma$.

If $k_1 < k_2$ are consecutive occurrences of $x_{-1}x_0$ in $(x_n)_{n\in\mathbb{Z}}$, then we call the word $x_{k_1}x_{k_1+1}\dots x_{k_2-1}$ the *return word* for $x_{-1}x_0$. Let $\mathcal{R}_{x_{-1}x_0}$ be the set of all return words. It is finite, since the length of the return words is uniformly bounded. By minimality of $\mathcal{F}_\sigma$, every sequence $(y_n)_{n\in\mathbb{Z}} \in \mathcal{F}_\sigma$ can be uniquely decomposed into a concatenation of elements of $\mathcal{R}_{x_{-1}x_0}$: just cut $(y_n)_{n\in\mathbb{Z}}$ inside every subword $y_{k-1}y_k = x_{-1}x_0$.

Suppose that $w \in \mathcal{R}_{x_{-1}x_0}$. Then $x_{-1}wx_0$ belongs to the language of $\mathcal{F}_\sigma$, the first letter of w is x_0, the last letter of w is x_{-1}. It follows that $\sigma(x_{-1}wx_0) = \sigma(x_{-1})\sigma(w)\sigma(x_0)$ also belongs to the language of $\mathcal{F}_\sigma$. The first letter of $\sigma(w)$ and $\sigma(x_0)$ is x_0, the last letter of $\sigma(w)$ and $\sigma(x_{-1})$ is x_{-1}. It follows that if we cut $\sigma(w)$ at every occurrence of $x_{-1}x_0$, we will decompose $\sigma(w)$ into a product of elements of $\mathcal{R}_{x_{-1}x_0}$.

We have proved that the sub-semigroup of X^* generated by $\mathcal{R}_{x_{-1}x_0}$ is σ-invariant. It follows that the restriction of σ to this semigroup is a substitution, which we will denote by $\phi : \mathcal{R}_{x_{-1}x_0} \longrightarrow \mathcal{R}^*_{x_{-1}x_0}$.

Let $w_0, w_1 \in \mathcal{R}_{x_{-1}x_0}$ be the prefix of $x_0x_1\dots$ and a the suffix of $\dots x_{-2}x_{-1}$, respectively. The words $\dots x_{-2}x_{-1}$ and $x_0x_1\dots$ are σ-invariants and are limits of the words $\sigma^n(x_{-1})$ and $\sigma^n(x_0)$ for $n \to \infty$. Since every word $w \in \mathcal{R}_{x_{-1}x_0}$

starts with x_0 and ends with x_{-1}, there exists k such that $\sigma^k(w)$ starts with w_0 and ends with w_1. It follows by Proposition 1.3.31 that the Vershik-Bratteli diagram of ϕ is properly ordered. Let us add on top of this diagram one more level associated with the substitution $\phi_0 : \mathcal{R}_{x_{-1}x_0} \longrightarrow \mathsf{X}^*$ mapping every word $w \in \mathcal{R}_{x_{-1}x_0}$ to itself (but as an element of X^*). We get a stationary Vershik-Bratteli diagram such that its adic transformation is topologically conjugate to $\mathcal{F}_\sigma$. This diagram is equivalent to the Vershik-Bratteli diagram obtained from the diagram associated with $\phi, \phi, \phi, \ldots$ by adding one vertex on the top level and connecting it to each vertex $w \in \mathcal{R}_{x_{-1}x_0}$ by $|w|$ edges.

Example 1.3.33. Let us consider the Fibonacci substitution

$$\sigma(0) = 01, \qquad \sigma(1) = 0.$$

Let us pass to the second iterate $\sigma^2(0) = 010$, $\sigma^2(1) = 01$, and consider the corresponding σ^2-invariant sequence

$$\ldots 01001001.01001010 \ldots$$

We have $\mathcal{R}_{10} = \{w_0 = 01, w_1 = 001\}$, and

$$\sigma^2(01) = 01|001 = w_0 w_1$$
$$\sigma^2(001) = 01|001|001 = w_0 w_1 w_1.$$

We get the substitution

$$\phi : w_0 \mapsto w_0 w_1, \qquad w_1 \mapsto w_0 w_1 w_1.$$

It follows that the subshift generated by the Fibonacci substitution is conjugate to the adic transformation shown on the left-hand side part of Figure 1.20. Note that it is equivalent to the diagram shown on the right-hands side part of the same figure. Both diagrams are properly ordered.

1.3.8. Thue-Morse subshift. The Thue-Morse subshift $\tau \circlearrowright \mathcal{T}$ is generated by the substitution

$$\sigma(0) = 01, \qquad \sigma(1) = 10,$$

see Example 1.2.21. The associated Vershik-Bratteli diagram is shown on the right-hand side of Figure 1.19. Note that this diagram has two minimal paths, corresponding to two infinite to the right limits of the words $\sigma^n(0)$ and $\sigma^n(1)$, respectively:

$$w_0 = 0110100110010110\ldots, \qquad w_1 = 1001011001101001\ldots.$$

They are represented on the Vershik-Bratteli diagram by the paths consisting of vertical edges only.

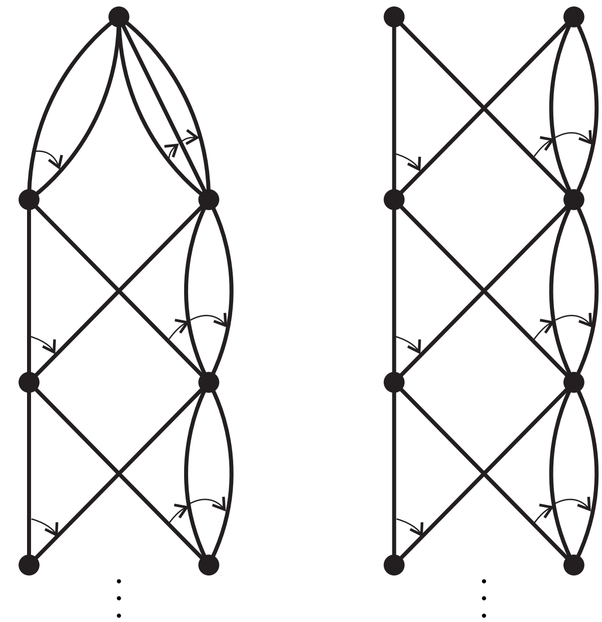

Figure 1.20. Vershik-Bratteli diagrams of the Fibonacci subshift

Similarly, it has two maximal paths, coresponding to two infinite to the left limits:

$$w_0^- = \ldots 0110100110010110, \qquad w_1^- = \ldots 1001011001101001.$$

They are represented by the paths consisting of diagonal edges only. Note that $\sigma(w_0) = w_0$, $\sigma(w_1) = w_1$, $\sigma(w_0^-) = w_1^-$, and $\sigma(w_1^-) = w_0^-$.

For every word $w \in \mathcal{T}$ there is a unique path in the Vershik-Bratteli diagram producing w (or a one-sided infinite subword of w containing $w(0)$) as an inductive limit of words $\sigma^n(x)$, $x \in \{a, b\}$, in the way described in 1.3.7, see Example 1.3.27.

All four concatenations $w_0^- w_0, w_0^- w_1, w_1^- w_0, w_1^- w_1$ belong to the subshift generated by σ. It follows that each of the maximal and minimal paths in the Vershik-Bratteli diagram represents two points of the subshift. Moreover, the adic transformation from the set of non-maximal paths to the set of non-minimal paths does not admit a continuous extension to the space of all paths.

Non-existence of a continuous extension can be easily corrected by "*collaring*", i.e., applying a sliding block map (see Definition 1.2.15) to the substitution. Let us use the block map

$$\ldots x_{-1} \,.\, x_0 x_1 \ldots \mapsto \ldots (x_{-2} x_{-1}) \,.\, (x_{-1} x_0)(x_0 x_1) \ldots.$$

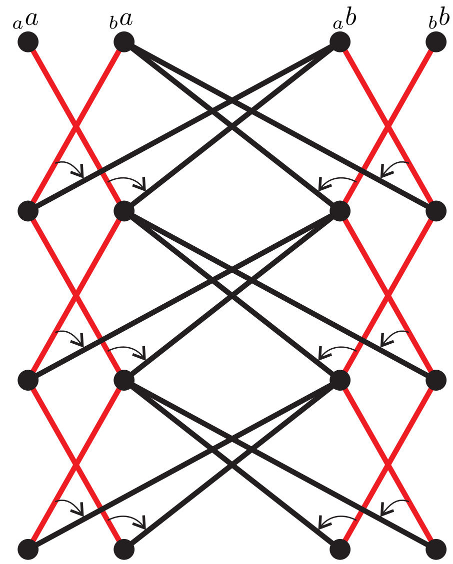

Figure 1.21. A continuous diagram of the Thue-Morse subshift

We will write the symbol xy of the block code as $_xy$ in order to stress that it replaces letter y in the original sequence. Applying σ to two-letter words:

$$\sigma(00) = 01 \cdot 01, \qquad \sigma(11) = 10 \cdot 10,$$
$$\sigma(10) = 10 \cdot 01, \qquad \sigma(01) = 01 \cdot 10$$

we see that the substitution induced on the block code is

$$\sigma({}_00) = {}_10\ {}_10, \qquad \sigma({}_11) = {}_01\ {}_10,$$
$$\sigma({}_10) = {}_00\ {}_01, \qquad \sigma({}_01) = {}_11\ {}_10.$$

The Vershik-Bratteli diagram of this substitution is shown on Figure 1.21, where minimal edges are red and maximal edges are black. Note that we have four minimal and four maximal edges, but this time the corresponding one-sided infinite paths are matched to each other in a unique way, since each letter x_i of the code remembers the previous letter of the original sequence $w \in \mathcal{T} \subset \{0, 1\}^{\mathbb{Z}}$, and the sequence w_0, w_1 start with different letters. Every path in this diagram corresponds to a unique point of the subshift.

Let us show how to construct a properly ordered Vershik-Bratteli diagram for the Thue-Morse subshift, using the return words, as it is described in the previous subsection. Since the substitution σ does not have fixed points on $\mathcal{T}$, we will need to consider its second interation

$$\sigma^2(0) = 0110, \qquad \sigma^2(1) = 1001.$$

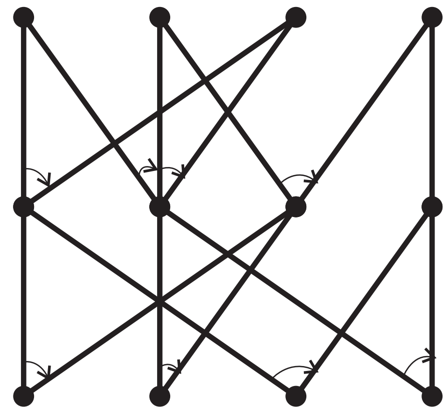

Figure 1.22. A well ordered Bratteli-Vershik diagram of the Thue-Morse subshift

Let us use the sequence $w_0^-.w_0$ as our fixed point, and consider the set of return words $\mathcal{R}_{00}$. We have

$$w_0 = \mathsf{011010|0110|01011010|010110|011010|0110|}\ldots$$

We get $\mathcal{R}_{00} = \{\mathsf{011010}, \mathsf{0110}, \mathsf{01011010}, \mathsf{010110}\}$. Let us index them in the order of their first appearance in w_a:

$$v_1 = \mathsf{011010}, \quad v_2 = \mathsf{0110}, \quad v_3 = \mathsf{01011010}, \quad v_4 = \mathsf{010110}.$$

We have

$$\begin{aligned}
\sigma^2(v_1) &= \mathsf{011010|0110|01011010|010110} = v_1v_2v_3v_4,\\
\sigma^2(v_2) &= \mathsf{011010|0110|010110} = v_1v_2v_4,\\
\sigma^2(v_3) &= \mathsf{011010|01011010|0110|01011010|010110} = v_1v_3v_2v_3v_4,\\
\sigma^2(v_4) &= \mathsf{011010|01011010|0110|010110} = v_1v_3v_2v_4.
\end{aligned}$$

The associated Vershik-Bratteli diagram is rather messy to draw. Instead, let us decompose the substitution $\phi = \sigma^2|_{\mathcal{R}_{00}^*}$ into a composition $\phi = \phi_2 \circ \phi_1$ of two substitutions

$$\begin{aligned}
\phi_1(v_1) &= x_1y_1, \qquad & \phi_1(v_2) &= x_1y_2,\\
\phi_1(v_3) &= x_2y_1, \qquad & \phi_1(v_4) &= x_2y_2,
\end{aligned}$$

and

$$\begin{aligned}
\phi_2(x_1) &= v_1v_2, \qquad & \phi_2(x_2) &= v_1v_3v_2,\\
\phi_2(y_1) &= v_3v_4, \qquad & \phi_2(y_2) &= v_4.
\end{aligned}$$

We get then the diagram shown on Figure 1.22 (except for the first level...).

Recall that the Vershik-Bratteli diagram of the odometer is the diagram consisting of one vertex and two edges on each level. Let us denote the edges by $\mathtt{0}$ and $\mathtt{1}$ with the ordering $\mathtt{0} < \mathtt{1}$. Then the adic transformation will be the usual binary adding machine action.

We have a natural semiconjugacy $\pi : \mathcal{T} \longrightarrow \{\mathtt{0}, \mathtt{1}\}^\omega$ from the Thue-Morse subshift to the binary odometer. Namely, let $w \in \mathcal{T}$, and let $\gamma = (e_1, e_2, \ldots)$ be the corresponding path in the Vershik-Bratteli diagram of the substitution. Then $\pi(e_1, e_2, \ldots) = x_1 x_2 \ldots$, where $x_i = \mathtt{0}$ if e_i is minimal, and $x_i = \mathtt{1}$ if e_i is maximal. It follows directly from the definition of the adic transformation that π is a semiconjugacy.

Proposition 1.3.34. *If the sequence $w \in \{\mathtt{0}, \mathtt{1}\}^\omega$ is not eventually constant, then $\pi^{-1}(w)$ consists of two elements. Otherwise it consists of four elements.*

Proof. It is easy to see from the structure of the Vershik-Bratteli diagram of the substitution σ that for every $w \in \{\mathtt{0}, \mathtt{1}\}^\omega$ there are exactly two paths in the diagram (one starting in a and one starting in b) such that every $w \in \mathcal{T}$ corresponding to these paths is mapped by π to w. The statement about the size of $\pi^{-1}(w)$ follows now from our analysis of the relation of the diagram with $\mathcal{T}$. $\square$

1.3.9. Vershik-Bratteli diagrams of irrational rotations. As an example of application of Theorem 1.3.20, let us find Vershik-Bratteli diagrams realizing the minimal homeomorphisms $\tilde{R}_\theta \circlearrowleft \mathcal{X}_\theta$ of the Cantor set defined in 1.3.1.2.

Let $\theta \in (0, 1)$ be an irrational number. Consider the continued fraction expansion

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}. \tag{1.4}$$

The positive integers a_i are found by the following recurrent rule. Set $\theta_1 = \theta$, and then

$$a_n = \lfloor \theta_n^{-1} \rfloor, \qquad \theta_{n+1} = \theta_n^{-1} - a_n.$$

Note that the last equality is equivalent to $\theta_n = \frac{1}{a_n + \theta_{n+1}}$.

Let us change the recurrent rule by setting

$$b_n = \lfloor \theta_n^{-1} \rfloor, \qquad \theta_{n+1} = 1 - (\theta_n^{-1} - b_n), \tag{1.5}$$

so that we have $\theta_n = \frac{1}{b_n+1-\theta_{n+1}}$, and

$$\theta = \cfrac{1}{b_1 + 1 + \cfrac{-1}{b_2 + 1 + \cfrac{-1}{b_3 + 1 + \cfrac{-1}{\ddots}}}}.$$

The expansions (1.5) are called *negative-regular continued fractions.*

We could also make different choices between $\theta_{n+1} = \theta_n^{-1} - a_n$ and $\theta_{n+1} = 1 - (\theta_n^{-1} - a_n)$. Such generalized continued fractions are called *semi-regular continued fractions*, see [**Per54**, Chapter V], where they are called *halbregelmäßige.* An algorithm transforming semi-regular continued fractions to the classical expansion (1.4) is described in [**Per54**, V.40]. In the particular case of the sequence (b_n) from (1.5) we get the following.

Proposition 1.3.35. *The sequence* $(b_1, b_2, \ldots)$ *is equal to*

$$a_1, \underbrace{1, 1, \ldots, 1}_{a_2 - 1 \text{ times}}, \ a_3 + 1, \ \underbrace{1, 1, \ldots, 1}_{a_4 - 1 \text{ times}}, \ a_5 + 1, \ \underbrace{1, 1, \ldots, 1}_{a_6 - 1 \text{ times}}, \ a_7 + 1, \ldots.$$

In particular, infinitely many entries of the sequence b_n greater than 1.

Note that the negative-regular continued fraction (1.5) with the constant sequence $b_n = 1$ is equal to 1.

Proof. If $\cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}} = \xi$, then $\cfrac{1}{a_3 + 1 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}} = \frac{1}{1+\frac{1}{\xi}} = \frac{\xi}{\xi+1}$. Consequently, the proposition will follow from the identity

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \xi}} = \cfrac{1}{a_1 + 1 - \cfrac{1}{2 - \cfrac{1}{2 - \ldots \cfrac{1}{2 - \cfrac{1}{1 + \frac{1}{\xi}}}}}}, \tag{1.6}$$

where 2 appears $a_2 - 1$ times on the right-hand side.

It is easy to prove by induction on n that

$$\cfrac{1}{2-\cfrac{1}{2-\dots\cfrac{1}{2-x}}} = \frac{n-(n-1)x}{(n+1)-nx},$$

where 2 appears n times on the left-hand side.

Consequently, if $x = \frac{1}{1+\frac{1}{\xi}}$, the right-hand side of (1.6) is equal to

$$\frac{1}{a_1+1-\frac{a_2-1-(a_2-2)x}{a_2-(a_2-1)x}} = \frac{1}{a_1+\frac{1-x}{a_2-(a_2-1)x}}.$$

Substitution $x = \frac{\xi}{\xi+1}$ into $\frac{1-x}{a_2-(a_2-1)x}$ gives

$$\frac{1-\frac{\xi}{\xi+1}}{a_2-(a_2-1)\frac{\xi}{\xi+1}} = \frac{1}{a_2(\xi+1)-(a_2-1)\xi} = \frac{1}{a_2+\xi},$$

which finishes the proof. □

Theorem 1.3.36. *Let $\theta \in (0,1)$ be an irrational number, and let $(b_1, b_2, \dots)$ be the sequences of positive integers such that*

$$\theta = \cfrac{1}{b_1+1-\cfrac{1}{b_2+1-\cfrac{1}{b_3+1-\cfrac{1}{\ddots}}}}.$$

Define the substitutions

$$\psi_k(0) = 0^k 1, \qquad \psi_k(1) = 0^{k-1}1.$$

Then the dynamical system $\tilde{R}_\theta \circlearrowleft \mathcal{X}_\theta$ is topologically conjugate to the substitutional system generated by the sequence

$$\psi_{b_1}, \psi_{b_2}, \psi_{b_3}, \dots$$

and to the adic transformation on the associated Vershik-Bratteli diagram.

The Vershik-Bratteli diagram of the substitution ψ_3 is shown on the left-hand side of Figure 1.23.

Proof. Let $\mathcal{R}_1$ be the partition $[0, 1-\theta], [1-\theta, 1]$ of the Cantor set $\mathcal{X}_\theta$ seen as Rokhlin-Kakutani partitions with base equal to the whole space $\mathcal{X}_\theta$. (We will drop $+0$ and -0 from the interval notation, since it will be always clear what are the intervals.)

Figure 1.23. Vershik-Bratteli diagrams of a rotation

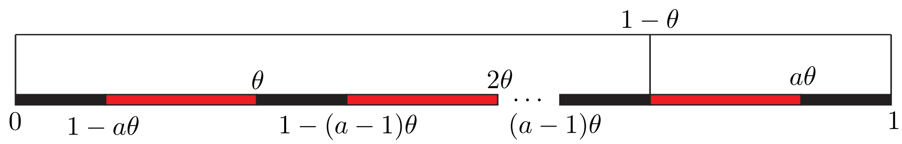

Figure 1.24. Rokhlin-Kakutani partition of a rotation

Let us construct a Rokhlin-Kakutani partition subordinate to $\mathcal{R}_1$ with the base $[0, \theta]$. Let $a = b_1 = \lfloor \frac{1}{\theta} \rfloor$. If $x \in [0, 1-a\theta]$, then the smallest positive integer n such that $R^n(x) \in [0, \theta]$ is equal to $a+1$, while for $x \in [1-a\theta, \theta]$ it is a. The first return map maps $[0, 1-a\theta]$ to $[(a+1)\theta-1, 1-a\theta+(a+1)\theta-1] = [(a+1)\theta - 1, \theta]$, and maps $[1-a\theta, \theta]$ to $[0, (a+1)\theta - 1]$. Hence the first return map swaps the intervals $[0, 1-a\theta]$ and $[1-a\theta, \theta]$. We define therefore $\mathcal{R}_2$ as the Rokhlin-Kakutani partition with the base $[0, \theta]$ and two towers:

$$T_{[0,1-a\theta]} = \{[0, 1-a\theta], R([0, 1-a\theta]), R^2([0, 1-a\theta], \ldots R^a([0, 1-a\theta])\}$$

and

$$T_{[1-a\theta,\theta]} = \{[1-a\theta, \theta], R([1-a\theta, \theta]), \ldots, R^{a-1}([1-a\theta, \theta])\},$$

see Figure 1.24, where the intervals of the first tower are black, and the intervals from the second tower are red.

The interval $[0, 1-\theta] \in \mathcal{R}_1$ is equal to the union of the following $2a - 1$ elements of $\mathcal{R}_2$:

$$\begin{aligned}
&[0, 1-a\theta] \cup R([0, 1-a\theta]) \cup \cdots \cup R^{a-1}([0, 1-a\theta]) \cup \\
&\qquad [1-a\theta, \theta] \cup R([1-a\theta, \theta]) \cup \cdots R^{a-2}([1-a\theta, \theta]) = \\
&\qquad [0, 1-a\theta] \cup [\theta, 1-(a-1)\theta] \cup \cdots \cup [(a-1)\theta, 1-\theta] \cup \\
&\qquad\qquad [1-a\theta, \theta] \cup [1-(a-1)\theta, 2\theta] \cup \cdots [1-2\theta, (a-1)\theta].
\end{aligned}$$

The interval $[1-\theta, 1] \in \mathcal{R}_1$ is equal to the union of the following two elements of $\mathcal{R}_2$:

$$R^{a-1}([1-a\theta, \theta]) \cup R^a([0, 1-a\theta]) = [1-\theta, a\theta] \cup [a\theta, 1].$$

In other words, $[1-\theta, 1]$ is the union of the two last elements of the towers, while $[0, 1-\theta]$ is the union of all the remaining elements. See Figure 1.24 where these decompositions are shown.

We see that the level of the Vershik-Bratteli diagram associated with the pair $\mathcal{R}_1$ and $\mathcal{R}_2$ has a edges $T_{[0,1-a\theta]} \in V_2$ to $\{[0, 1-\theta]\} \in V_1$, one edge connecting $T_{[0,1-a\theta]}$ to $\{[1-\theta, 1]\} \in V_1$, $a-1$ edges connecting $T_{[1-a\theta,\theta]} \in V_2$ to $\{[0, 1-\theta]\}$, and one edge connecting $T_{[1-a\theta,\theta]}$ to $\{[1-\theta, 1]\}$. The edge connecting a vertex of V_2 to $\{[1-\theta, 1]\}$ is greater (comes later in the ordering of the diagram) than the edges connecting it to $\{[0, 1-\theta]\}$.

In other words, it is the diagram of the substitution

$$\psi_a : \mathsf{0} \mapsto \mathsf{0}^a\mathsf{1}, \qquad \mathsf{1} \mapsto \mathsf{0}^{a-1}\mathsf{1},$$

if we label the towers $[0, 1-\theta]$ and $T_{[0,1-a\theta]}$ by $\mathsf{0}$, and the towers $[1-\theta, 1]$ and $T_{[1-a\theta,\theta]}$ by $\mathsf{1}$.

We can identify the base $[0, \theta]$ of $\mathcal{R}_2$ with the interval $[0, 1]$ by the transformation $x \mapsto x/\theta$. Then the partition $[0, 1-a\theta], [1-a\theta, \theta]$ is transformed to the partition $[0, \theta^{-1}-a], [\theta^{-1}-a, 1]$, and the first return map swaps these intervals, i.e., is a rotation by $\theta' = 1-(\theta^{-1}-a)$.

(Note that if we identify the base $[0, \theta]$ with the interval $[0, 1]$ by the transformation $x \mapsto 1-x/\theta$, then the first return map is a rotation by $\theta' = \theta^{-1}-a$.)

We apply our procedure again to the rotation by θ'. We will get a sequence $\mathcal{R}_n$ of Rokhlin-Kakutani partitions such that its Vershik-Bratteli diagram is the diagram associated with the sequence

$$\psi_{b_1}, \psi_{b_2}, \psi_{b_3}, \ldots.$$

The bases of the partitions $(\mathcal{R}_n)$ are the segments $[0, \theta_n]$, where θ_n are defined by the condition $\theta_n = \frac{1}{a_n+1-\frac{1}{\theta_{n+1}}}$. Consequently, the intersection of the bases consists of 0 only. As the length of the towers grow to infinity, for every m there exists n such that images of $[0, 1-a_n\theta_n]$ under $\tilde{R}_\theta^k$ belong to $\mathcal{R}_n$ for all $k = 0, 1, \ldots, m$. It follows that any two points $t-0, t+0$, where t is in the forward orbit of 0 under R_θ are separated by some partition $\mathcal{R}_n$.

The union of the top elements of the towers of $\mathcal{R}_n$ is the interval $[1-\theta_n, 1]$, and intersection of these intervals is $\{1\}$. By the same argument as above, this shows that any two points $t-0, t+0$, where t is in the backward orbit of 0 under R_θ are separated by some partition $\mathcal{R}_n$.

It follows that adic transformation of the Vershi-Bratteli diagram of $(\mathcal{R}_n)$ is topologically conjugate to $\tilde{R}_\theta$. □

Definition 1.3.37. We say that a subshift $\mathcal{F} \subset \mathsf{X}^{\mathbb{Z}}$ is *Sturmian* if $p_{\mathcal{F}}(n) = n+1$ for all $n \geqslant 0$, where $p_{\mathcal{F}}$ is the complexity function, see...

Note that since $p_{\mathcal{F}}(1) = 2$, we may assume that $|\mathsf{X}| = 2$.

Theorem 1.3.38 (Hedlund-Morse)**.** *Suppose that a subshift $\mathcal{F} \subset \{0,1\}^{\mathbb{Z}}$ is minimal and satisfies $p_{\mathcal{F}}(n) = n + 1$ for all $n \geqslant 1$. Then there exists an irrational number θ such that $\mathcal{F} = \mathcal{F}_\theta$.*

Reference from Hedlund-Morse...

For the countable case, see...

Proof. Denote by W_n the set of words of length n belonging to $W_{\mathcal{F}}$. Consider the map $vx \mapsto v$ from the W_{n+1} to W_n. It is surjective, and since $|W_{n+1}| = n + 2$ and $|W_n| = n + 1$, there exists one word $w_n \in W_n$ (which we will call *special*) that has two preimages. All the other words $v \in W_n$ have one preimage. Conversely, if for every $n \geqslant 1$ there exists exactly one word $w_n \in W_n$ with two preimages and all the other words have only one preimage, then $|W_{n+1}| = |W_n| + 1$, and $|W_n| = n + 1$ for all n.

If $w_n \in W_{\mathcal{F}}$ is the special word, i.e., if $w_n\mathsf{0}, w_n\mathsf{1} \in W_{\mathcal{F}}$, then every its suffix of w_n is also special. Since there is only one special word of every length, it follows that special words are suffixes of one left-infinite word $w_\infty = \dots x_2x_1x_0 \in \mathsf{X}^{-\omega}$. We will call it the *left-infinite special word*.

Let $k \geqslant 0$ be the smallest number such that $\mathsf{1}\mathsf{0}^k\mathsf{1}$ belongs to $W_{\mathcal{F}}$. If the word $\mathsf{0}^{k+1}$ does not belong to $W_{\mathcal{F}}$, then there are exactly k letters $\mathsf{0}$ between any two consecutive $\mathsf{1}$s. But then $\mathcal{F}$ is finite. Note also that no element of $\mathcal{F}$ contains an infinite string of $\mathsf{0}$s, since this would contradict minimality. It follows that the words $\mathsf{0}^{k+1}\mathsf{1}\mathsf{0}^k$ and $\mathsf{0}^k\mathsf{1}\mathsf{0}^{k+1}$ belong to $W_{\mathcal{F}}$. Consequently, all the words of the form $\mathsf{0}^i\mathsf{1}\mathsf{0}^{k+1-i}$ for $i = 0, 1, \dots, k+1$ belong to $W_{\mathcal{F}}$. We also have $\mathsf{1}\mathsf{0}^k\mathsf{1} \in W_{\mathcal{F}}$. This is already a list of $k+3$ words in $W_{\mathcal{F}}$ of length $k+2$. It follows that there are no other words of length $k+2$ in $W_{\mathcal{F}}$. In particular $\mathsf{0}^{k+2} \notin W_{\mathcal{F}}$, i.e., the number of zeros between any two consecutive ones is either k or $k+1$.

Consequently, every element of $\mathcal{F}$ can be written as a concatenation of the words $\mathsf{0}^k\mathsf{1}$ and $\mathsf{0}^{k+1}\mathsf{1}$. Consider now the subshift $\mathcal{F}'$ over the alphabet $\{\mathsf{0}^k\mathsf{1}, \mathsf{0}^{k+1}\mathsf{1}\}$ consisting of sequences obtained by factoring sequences $w \in \mathcal{F}$ into concatenations of subwords $\mathsf{1}\mathsf{0}^k$ and $\mathsf{1}\mathsf{0}^{k+1}$.

Let us show that the obtained subshift $\mathcal{F}'$ is also Sturmian. It is enough to show that for every n there exists a unique word $v \in W_{\mathcal{F}'}$ of length n such that $v \cdot (\mathsf{0}^k\mathsf{1}), v \cdot (\mathsf{0}^{k+1}\mathsf{1}) \in \mathcal{F}'$. Let us show at first the existence. Let w_∞ be the left-infinite special word for $\mathcal{F}$. Then $w_\infty\mathsf{0}$ and $w_\infty\mathsf{1}$ are subwords of elements of $\mathcal{F}$. It follows that $\mathsf{1}\mathsf{0}^k$ is a suffix of w_∞, and w_∞ can be factored into a concatenation $\dots a_3a_2\mathsf{0}^k$, where $a_i \in \{\mathsf{0}^k\mathsf{1}, \mathsf{0}^{k+1}\mathsf{1}\}$. But then both $\dots a_3a_2\mathsf{0}^k\mathsf{1}$ and $\dots a_3a_2\mathsf{0}^{k+1}\mathsf{1}$ are subwords of elements of $\mathcal{F}'$, hence $\dots a_3a_2$

is a left-infinite special word of $\mathcal{F}'$. For every n the suffix of length n of this word will be special for $\mathcal{F}'$.

In order to prove the uniqueness, it is enough to show that if $v_1, v_2 \in \mathcal{F}'$ are such that $\{v_1(0^k1), v_1(0^{k+1}1), v_2(0^k1), v_2(0^{k+1}1)\} \subset W_{\mathcal{F}'}$, then one of the words v_1, v_2 is a suffix of the other. Note that $v_i(0^{k+1}1) \in W_{\mathcal{F}'}$ implies that $v_i0^{k+1} \in W_{\mathcal{F}}$, so that we have that $\{v_10^k1, v_110^{k+1}, v_20^k1, v_20^{k+1}\} \subset W_{\mathcal{F}}$, i.e., that v_10^k and v_20^k are special words for $\mathcal{F}$. But we know that all special words of $\mathcal{F}$ are suffixes of the unique left-infinite special word. It follows that one of the words v_1, v_2 is a suffix of the other.

We see that $\mathcal{F}'$ is also a Sturmian minimal subshift (minimality of $\mathcal{F}'$ follows directly from the minimality of $\mathcal{F}$ by Proposition 1.2.4). Let us relabel the letters of the alphabet $\{0^{k+1}1, 0^k1\}$ by the letters $0, 1$ using the substitution

$$\psi_{k+1} : 0 \mapsto 0^{k+1}1, \qquad 1 \mapsto 0^k1.$$

Note that this is the same substitution as in Theorem 1.3.36. We will identify then $\mathcal{F}'$ with a Sturmian subshift of $\{0, 1\}^{\mathbb{Z}}$.

Continue the above construction with $\mathcal{F}$ replaced by the new $\mathcal{F}' \subset \{0, 1\}^{\mathbb{Z}}$. We will get a sequence of positive integers $k_1, k_2, \ldots$, a sequence of subshifts $\mathcal{F}_n$, and a sequence of substitutions

$$\psi_{k_n+1} : 0 \mapsto 0^{k_n+1}1, \qquad 1 \mapsto 0^{k_n}1.$$

The elements of $\mathcal{F}_n$ are obtained from the elements of $\mathcal{F}_{n+1}$ by applying ψ_{k_n} (and then taking all shifts).

Let B be the Vershik-Bratteli diagram associated with the obtained sequence ψ_{k_n+1}, and let us show that every path in B corresponds to exactly one element of $\mathcal{F}$. It is enough to check the minimal and the maximal paths.

But ψ_1 is the substitution $0 \mapsto 01$, $1 \mapsto 1$ generating the subshift consisting of the constant 1 sequence. Consequently, the sequence k_n is not eventually equal to zero, since then $\mathcal{F}$ would be finite.

It follows that the only minimal path is the unique path passing through the vertices $0 \in V_n$. The corresponding right-infinite sequence is limit of the words $\psi_{k_1+1} \circ \psi_{k_2+1} \circ \cdots \psi_{k_n+1}(0)$. Note $\psi^{k_n+1}(0)$ that it begins with $0^{k_n+1}1$, which has a unique allowed extension to the left $10^{k_n+1}1$, since $k_n + 1$ is the maximal allowed number of zeros between two consecutive ones in $\mathcal{F}_n$. It follows that in every element of $\mathcal{F}$ the word $\psi_{k_1+1} \circ \psi_{k_2+1} \circ \cdots \psi_{k_n+1}(0)$ is preceded by $\psi_{k_1+1} \circ \psi_{k_2+1} \circ \cdots \psi_{k_{n-1}+1}(1)$. The length of this word goes to infinity, since k_n is not eventually zero. Consequently, the right-infinite sequence corresponding to the minimal path has a unique extension to an element of $\mathcal{F}$.

Let us show that the maximal path corresponds to a unique element of $\mathcal{F}$. It is enough to prove this statement for any $\mathcal{F}_n$. It follows that we may assume that $k_1 \neq 0$. The only maximal path is the path passing through the vertices $1 \in V_n$. The corresponding left-infinite sequence is the left-infinite limit of the words $\psi_{k_1+1} \circ \psi_{k_2+1} \circ \cdots \psi_{k_n+1}(1)$. Then 1 is the last letter of this word, and then the only possible one-letter continuation is 10, as $k_1 > 0$. The proof is finished in the same way as for the case of the minimal path. This shows that the diagram B models the subshift $\mathcal{F}$. Theorem 1.3.36 shows (together with its proof, as we have to check that the encodings by 0 and 1 agree) that $\mathcal{F}$ coincides with the system $\tilde{R}_\theta \circlearrowleft \mathcal{X}_\theta$ for

$$\theta = \cfrac{1}{k_1 + 2 - \cfrac{1}{k_2 + 2 - \cfrac{1}{k_3 + 2 - \cfrac{1}{\ddots}}}}.$$

□

We want to consider now the classical continued fractions...

Let

$$\psi_{k+1} : 0 \mapsto 0^{k+1}1, \qquad 1 \mapsto 0^k 1,$$

be the substitution from Theorem 1.3.36. Define also the substitutions

$$\phi_{k+1} : 0 \mapsto 0^k 1, \qquad 1 \mapsto 0^{k+1}1.$$

Proposition 1.3.39. *The subshift generated by the sequence*

$$\phi_{k_1+1}, \phi_{k_2+1}, \phi_{k_3+1}, \ldots$$

coincides with the subshift generated by the sequence

$$\psi_{k_1+1}, \psi_1^{k_2}, \psi_{k_3+2}, \psi_1^{k_4}, \psi_{k_5+2}, \psi_1^{k_6}, \psi_{k_7+2}, \ldots$$

See Definition 1.3.26 of subshifts generated by sequences of substitutions.

Proof. Denote

$$\eta : 0 \mapsto 0, \qquad 1 \mapsto 01.$$

Then $\psi_{k+2} = \eta \circ \psi_{k+1}$. Let us compare now $\phi_{k_1+1} \circ \phi_{k_2+1}$ with $\psi_{k_1+1} \circ \psi_1^{k_2} \circ \eta$. We have

$$\phi_{k_1+1} \circ \phi_{k_2+1}(0) = \phi_{k_1+1}(0^{k_2}1) = (0^{k_1}1)^{k_2} 0^{k_1+1}1$$

and

$$\psi_{k_1+1} \circ \psi_1^{k_2} \circ \eta(0) = \psi_{k_1+1}(01^{k_2}) = 0^{k_1+1}1(0^{k_1}1)^{k_2}.$$

We have

$$\phi_{k_1+1} \circ \phi_{k_2+1}(1) = \phi_{k_1+1}(0^{k_1+1}1) = (0^{k_1}1)^{k_1+1} 0^{k_1+1}1$$

and

$$\psi_{k_1+1} \circ \psi^{k_2} \circ \eta(1) = \psi_{k_1+1} \circ \psi^{k_2}(01) = \psi_{k_1+1}(01^{k_2+1}) = 0^{k_1+1}1(0^{k_1}1)^{k_2+1}.$$

It follows that application of $\phi_{k_1+1} \circ \phi_{k_2+1}$ to a bi-infinite word produces, up to a shift, the same word as application of $\psi_{k_1+1} \circ \psi^{k_2} \circ \eta(u) = 0^{k_1+1}v$. It follows that the subshift generated by the sequence

$$\phi_{k_1+1}, \phi_{k_2+1}, \phi_{k_3+1}, \ldots$$

is the same as the subshift generated by the sequence

$$\psi_{k_1+1}, \psi^{k_2}, \eta, \psi_{k_3+1}, \psi^{k_4}, \eta, \ldots.$$

The last sequence is equivalent to

$$\psi_{k_1+1}, \psi^{k_2}, \psi_{k_3+2}, \psi^{k_4}, \psi_{k_5+2}, \ldots.$$

□

Theorem 1.3.40. *Let $\theta \in (0, 1)$ be an irrational number, and let $(a_1, a_2, \ldots)$ be the sequence of positive integers such that*

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}.$$

Define the substitutions

$$\phi_k(0) = 0^{k-1}1, \qquad \phi_k(1) = 0^k 1.$$

Then the dynamical system $\tilde{R}_\theta \circlearrowleft \mathcal{X}_\theta$ is topologically conjugate to the system generated by the sequence

$$\phi_{a_1}, \phi_{a_2}, \phi_{a_3}, \ldots.$$

If infinitely many of the values of a_i are greater than 1, then it is also topologically conjugate to the adic transformation of the Vershiki-Bratteli diagram associated with the above sequence.

Example 1.3.41. Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the *golden mean*, and consider the rotation by $\varphi \cong \varphi - 1\frac{-1+\sqrt{5}}{2}$ (mod 1).

The inverse rotation is by $\theta = 1 - \varphi = \frac{3-\sqrt{5}}{2}$. We have

$$\theta^{-1} = \frac{2}{3 - \sqrt{5}} = \frac{6 + 2\sqrt{5}}{4} = \frac{3 + \sqrt{5}}{2},$$

hence $a_1 = 2$, and

$$\theta_2 = 3 - \frac{3 + \sqrt{5}}{2} = \frac{3 - \sqrt{5}}{2} = \theta.$$

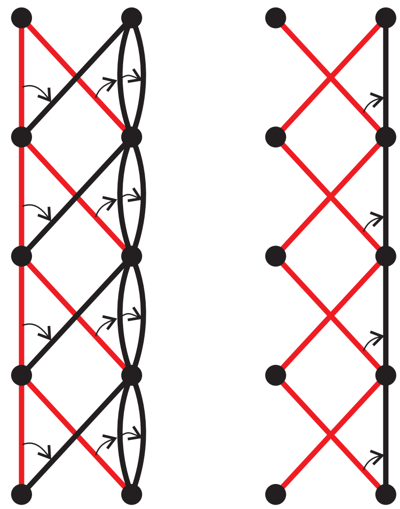

Figure 1.25. Two Vershik-Bratteli diagrams of the golden mean rotation

It follows that

$$\theta = 1 - \varphi = \cfrac{1}{2 + 1 - \cfrac{1}{2 + 1 - \cfrac{1}{2 + 1 - \cfrac{1}{\cdots}}}}$$

It is also well known that

$$\varphi = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}$$

Consequently, the rotation by φ can be described by the Vershik-Bratteli diagrams shown on Figure 1.25. The left-hand side part is obtained by changing to the opposite the ordering of the edges of the diagram associated with the sequence $(\mathcal{R}_n)$ for the rotation by $\theta = 1 - \varphi$, while the right-hands side part shows the diagram associated with the sequence $(\mathcal{R}'_n)$ for φ. We have highlighted the minimal edges. Note that there are two minimal paths in the right-hand side diagram.

1.3.10. Linearly repetitive systems. By Proposition 1.2.4, if $(\mathcal{F}, \mathsf{s})$ is a minimal subshift, then for every finite word $v \in W_{\mathcal{F}}$ there exists N_v such that for every $w \in \mathcal{F}$ there exists $0 \leqslant k \leqslant N_v - 1$ such that the subword

$w'(1)w'(2)\dots w'(|v|)$ of $w' = \mathsf{s}^k(w)$ is equal to v. Denote by $R_{\mathcal{F}}(n)$ the maximum of N_v for all words v of length at most n.

Definition 1.3.42. We say that a subshift $\mathcal{F}$ is linearly repetitive if there exists $C > 1$ such that $R_{\mathcal{F}}(n) \leqslant Cn$ for all $n \geqslant 1$.

The following is straightforward.

Proposition 1.3.43. *Suppose that $\mathcal{F}' \subset (\mathsf{X}^k)^{\mathbb{Z}}$ is the image of a minimal subshift $\mathcal{F} \subset \mathsf{X}^{\mathbb{Z}}$ under a sliding block map. Then*

$$R_{\mathcal{F}'}(n) = R_{\mathcal{F}}(n+k)$$

for all $n \geqslant 1$. In particular, $\mathcal{F}$ is linearly repetitive if and only if so is $\mathcal{F}'$.

Theorem 1.3.44. *Minimal substitutional subshifts are linearly repetitive.*

Proof. It is shown in the proof of Proposition 1.2.27 that the image of every minimal substitutional subshift under some sliding block map is generated by a primitive substitution. It follows then from Proposition 1.3.43 that it is enough to consider only the case of a subshift $\mathcal{F}$ generated by a primitive substitution $\sigma : \mathsf{X} \longrightarrow \mathsf{X}^*$.

It was shown in the proof of Theorem 1.2.34 that there exists a constant C, depending only on $\mathcal{F}$, such that for every $n > 1$ there exists k such that for every $x \in \mathsf{X}$ we have $n \leqslant |\sigma^k(x)| \leqslant Cn$. Then every word $v \in W_{\mathcal{F}}$ of length n is a subword of a word of the form $\sigma^k(xy)$ for $xy \in W_{\mathcal{F}}$. It follows that $R_{\mathcal{F}}(n) \leqslant R_{\mathcal{F}}(2) \cdot Cn$. $\square$

See another proof in [**DL06**, Theorem 1], where this theorem was proved for the first time.

More generally, we have the following characterization of linearly repetitive subshifts, see [**Dur03**].

Theorem 1.3.45. *A subshift is linearly repetitive if and only if it is topologically conjugate to the subshift generated by a sequence of substitutions*

$$\mathsf{X}_1^* \xleftarrow{\sigma_1} \mathsf{X}_2^* \xleftarrow{\sigma_2} \mathsf{X}_3^* \xleftarrow{\sigma_3} \cdots$$

such that

(1) *the set $\{\sigma_n : n = 1, 2, \dots\}$ is finite;*
(2) *there exists s such that for every $x \in \mathsf{X}_n$ and $y \in \mathsf{X}_{n+s+1}$ the letter x appears in the word $\sigma_n \circ \sigma_{n+1} \circ \cdots \circ \sigma_{n+s}(y)$;*
(3) *there exist $a_n, b_n \in \mathsf{X}_n$ such that for every $x \in \mathsf{X}_{n+1}$ the word $\sigma_n(x)$ begins with a_n and ends with b_n.*

Denote by $p_{\mathcal{F}}(n)$ the complexity of the subshift $\mathcal{F}$, i.e., the number of subwords of length n in elements of $\mathcal{F}$. We have an obvious estimate $p_{\mathcal{F}}(n) \leqslant R_{\mathcal{F}}(n)$, since every word $v \in W_{\mathcal{F}}$ of length n appears as a subword of every the word $u \in W_{\mathcal{F}}$ of length $n+R_{\mathcal{F}}(n)-1$. On the other hand, $R_{\mathcal{F}}(n)$ can grow much faster than $p_{\mathcal{F}}(n)$. For example, $R_{\mathcal{F}}$ can grow arbitrarily fast even for Sturmian sequences. Namely, we have the following theorem, see ...

Theorem 1.3.46. *Let $\mathcal{X}_\theta \subset \{0,1\}^{\mathbb{Z}}$ be the Sturmian shift defined by an irrational number $\theta \in (0,1)$, as in Proposition 1.2.37, and let $a_1, a_2, \ldots$ be the terms of the continued fraction expansion of θ (the* partial quotients*). Then $\mathcal{X}_\theta$ is linearly repetitive if and only if a_n is bounded (i.e., if θ is of* bounded type*).*

Proof. (Sketch) Let ϕ_{a_i} be as in Theorem 1.3.40. Consider the matrix $B_n = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$, where b_{xy} is the number of letters x in the word $\phi_{a_1} \circ \phi_{a_2} \circ \cdots \circ \phi_{a_n}(y)$. We have $B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It follows from the definition of the morphism ϕ_{a_i} that

$$B_{n+1} = B_n \cdot \begin{pmatrix} a_{n+1} - 1 & a_{n+1} \\ 1 & 1 \end{pmatrix}.$$

Consequently,

$$B_n = \begin{pmatrix} a_1 - 1 & a_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_2 - 1 & a_2 \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} a_n - 1 & a_n \\ 1 & 1 \end{pmatrix}.$$

Denote by p_n, q_n the numerator and the denominator of the fraction $\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$, respectively. We have then

$$p_{n+1} = a_{n+1} p_n + p_{n-1}, \qquad q_{n+1} = a_{n+1} q_n + q_{n-1}.$$

It is easy to check by induction that then

$$B_n = \begin{pmatrix} q_n - p_n & q_n - p_n + q_{n-1} - p_{n-1} \\ p_n & p_n + p_{n-1} \end{pmatrix}.$$

In particular, the lengths of $\phi_{a_1} \circ \phi_{a_2} \circ \cdots \circ \phi_{a_n}(0)$ and $\phi_{a_1} \circ \phi_{a_2} \circ \cdots \circ \phi_{a_n}(1)$ are q_n and $q_n + q_{n-1}$, respectively.

Let $r_n = \frac{q_n}{q_{n-1}}$. Then $r_1 = a_1$, and we have

$$r_{n+1} = \frac{a_{n+1}q_n + q_{n-1}}{q_n} = a_{n+1} + \frac{1}{r_n},$$

so that

$$r_n = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{\ddots + \cfrac{1}{a_1}}}},$$

which implies $a_n \leqslant r_n \leqslant a_n + 1$. We have

$$\frac{|\phi_{a_1} \circ \phi_{a_2} \circ \cdots \circ \phi_{a_n}(1)|}{|\phi_{a_1} \circ \phi_{a_2} \circ \cdots \circ \phi_{a_n}(0)|} = \frac{q_n + q_{n-1}}{q_n} = 1 + \frac{1}{r_n}$$

and

$$\frac{||\phi_{a_1} \circ \phi_{a_2} \circ \cdots \circ \phi_{a_n}(0)|}{||\phi_{a_1} \circ \phi_{a_2} \circ \cdots \circ \phi_{a_{n-1}}(0)|} = \frac{q_n}{q_{n-1}} = r_n.$$

It follows that these ratios are uniformly bounded away from zero and infinity if and only if a_n is bounded from above. If this is the case, then the proofs Theorems 1.2.34 and 1.3.44 can be repeated for $\mathcal{X}_\theta$, which will show the "if" direction.

For the "only if" direction it is enough to notice that the word between two neighboring occurrences of $\phi_{a_1} \circ \phi_{a_2} \circ \cdots \circ \phi_{a_n}(1)$ is either $\phi_{a_1} \circ \phi_{a_2} \circ \cdots \circ \phi_{a_n}(0^{a_{n+1}-1})$ or $\phi_{a_1} \circ \phi_{a_2} \circ \cdots \circ \phi_{a_n}(0^{a_{n+1}})$, which shows that

$$R_{\mathcal{X}_\theta}(q_n + q_{n-1}) \geqslant q_n + q_{n-1} + a_{n+1}q_n = q_n + q_{n+1}.$$

We have

$$\frac{q_n + q_{n+1}}{q_n + q_{n-1}} = \frac{1 + r_{n+1}}{1 + r_n^{-1}} = \frac{1 + a_{n+1} + r_n^{-1}}{1 + r_n^{-1}} \geqslant 1 + \frac{a_{n+1}}{2},$$

as $r_n \geqslant 1$ and the expression is increasing with r_n. It follows that if the sequence a_n is not bounded from above, then so is $\frac{R_{\mathcal{X}_\theta}(n)}{n}$. □

1.4. Hyperbolic dynamics

General discussion.. Expanding maps are very well understood and have rich analytic and algebraic structure. See, for instance [Haisinki-Pilgrim...] and... One of the main subjects of Chapter 5 is the algebraic theory of expanding maps, where we will establish a functorial bijection between expanding covering maps and a class of groups (more precisely groups bisets...)... The case of hyperbolic homeomorphisms (Ruelle-Smale spaces) is much less understood, and many questions that are relatively easy in the expanding case are wide open in the case of homeomorphisms....

1.4.1. Expanding maps: definitions and examples.

Definition 1.4.1. Let $\mathcal{X}$ be a compact space. A map $f \circlearrowleft \mathcal{X}$ is *expanding* if it generates an expansive action of the semigroup $\mathbb{N}$, i.e., if there exists a neighborhood $U \subset \mathcal{X} \times \mathcal{X}$ of the diagonal such that $(f^k(x), f^k(y)) \in U$ for all $k \geqslant 0$ implies $x = y$.

Definition 1.4.2. We say that a map $f \circlearrowleft \mathcal{X}$ on a compact metric space is *metrically expanding* if there exist $\epsilon > 0$, $L > 1$ such that if $x, y \in \mathcal{X}$ are such that $d(x, y) < \epsilon$ then $d(f(x), f(y)) \geqslant Ld(x, y)$.

It is obvious that every metrically expanding map is expanding.

Example 1.4.3. Consider the circle $\mathbb{R}/\mathbb{Z}$ and the self-covering $f \circlearrowleft \mathbb{R}/\mathbb{Z}$ given by $f(x) = kx \pmod 1$, for an integer k, $|k| > 1$. It is metrically expanding, for example with $L = |k|$ and $\epsilon = 1/|k|$.

It is easier in some cases to show that an iteration of a map is metrically expanding. A simple change of the metric shows that then the map itself expanding.

Lemma 1.4.4. *Suppose that $f \circlearrowleft \mathcal{X}$ is a map on a metric space $(\mathcal{X}, d)$ such that $f^n \circlearrowleft \mathcal{X}$ is metrically expanding for some $n \geqslant 1$. Then there exists a metric d' on $\mathcal{X}$ such that $f \circlearrowleft \mathcal{X}$ is metrically expanding with respect to d'.*

Proof. Suppose that f^n is expanding with respect to a metric d. Let ϵ and L be as in Definition 1.4.2 for f^n.

Consider the metric

$$d'(x, y) = \sum_{k=0}^{n-1} L^{-k/n} d(f^k(x), f^k(y)).$$

Then

$$d'(f(x), f(y)) = \sum_{k=1}^{n} L^{-(k-1)/n} d(f^k(x), f^k(y)) =$$

$$L^{-(n-1)/n} d(f^n(x), f^n(y)) + L^{1/n} \sum_{k=1}^{n-1} L^{-k/n} d(f^k(x), f^k(y)) \geqslant$$

$$L^{-1+1/n} L d(x, y) + L^{1/n} \sum_{k=1}^{n-1} L^{-k/n} d(f^k(x), f^k(y)) = L^{1/n} d'(x, y).$$

It follows that f is expanding with respect to d'. □

Example 1.4.5. An endomorphism $f \circlearrowleft M$ of a Riemannian manifold is called *expanding* if there exist $C > 0$ and $L > 1$ such that $\|Df^n \vec{v}\| \geqslant CL^n \|\vec{v}\|$

for every tangent vector $\vec{v}$. It follows from Lemma 1.4.4 that every expanding endomorphism of a compact Riemannian manifold is an expanding self-covering.

Expanding endomorphisms of Riemannian manifolds were studied by M. Shub in [**Shu69, Shu70, Hir70**]. One of applications of M. Gromov's theorem on groups of polynomial growth is showing that all expanding endomorphisms of compact Riemannian manifolds are generalizations of Examle 1.4.3, see [**Gro81**]. Namely, they are all topologically conjugate to *endomorphisms of infra-nil-manifolds*. Here a manifold M is an infra-nil-manifold if there exists a nilpotent connected Lie group L and a subgroup $G < L \rtimes \mathrm{Aut}(L)$ such that G acts freely and properly on L, and M is diffeomorphic to L/G. If $F \circlearrowleft L$ is an expanding automorphism such that $FGF^{-1} < G$, then F induces an expanding endomorphism of $L/G = M$. We will revisit this result in ...

Example 1.4.6. We get many more examples of expanding self-coverings $f \circlearrowleft \mathcal{X}$, if we do not require $\mathcal{X}$ to be a manifold. A big class of examples is provided by holomorphic dynamics. Namely, every *hyperbolic* complex rational function is expanding on its Julia set, see 1.5.3.

Example 1.4.7. Let $\mathsf{s} \circlearrowleft \mathsf{X}^{\mathbb{N}}$ be the one-sided shift. Consider the metric $d(w_1, w_2) = 2^{-n}$, where n is the largest non-negative integer such that the beginnings of length n of w_1 and w_2 coincide. Then s is expanding for $\epsilon = 1/2$ and $L = 2$. In particular, every one-sided subshift $\mathcal{F} \subset \mathsf{X}^{\mathbb{N}}$ is expanding. This shows that the class of all expanding maps is very big (for example, there exists uncountably many conjugacy classes of one-sided shifts). On the other hand, we will see later that the class of *expanding self-coverings* is much more rigid. In particular, it contains only countably many topological conjugacy classes, see...

1.4.2. Natural log-scale. Let $f \circlearrowleft \mathcal{X}$ be a map, where $\mathcal{X}$ is compact. If $U \subset \mathcal{X} \times \mathcal{X}$ is an expansion entourage for $f \circlearrowleft \mathcal{X}$, then $U^- = \{(x, y) : (y, x) \in U\}$ is also an expansion entourage. Then $U^- \cap U$ is a symmetric expansion entourage. It follows that we may assume without loss of generality that expansion entourages are symmetric. We will also assume that they are closed.

Suppose that U is an expansion entourage for a map $f \circlearrowleft \mathcal{X}$. Denote

$$U_n = \bigcap_{k=0}^{n} f^{-k}(U).$$

In other words, U_n is the set pairs of points (x, y) such that $(f^k(x), f^k(y)) \in U$ for all $k = 0, 1, \ldots, n$. In particular, $U_0 = U$. Denote $U_{-1} = \mathcal{X} \times \mathcal{X}$. By the definition of an expansion entourage, $\bigcap_{n \geqslant 0} U_n$ is equal to the diagonal.

The following is a particular case of Lemma 1.2.7.

Lemma 1.4.8. *For every neighborhood V of the diagonal, there exists n such $U_n \subset V$.*

For subsets A, B of $\mathcal{X} \times \mathcal{X}$, denote by $A \circ B$ the set of pairs (x, y) such that there exists z such that $(x, z) \in A$ and $(z, y) \in B$. Note that $A \circ B$ is the image of the closed subset $D = \{((x_1, y_1), (x_2, y_2)) \,:\, y_1 = x_2\}$ of $A \times B$ under the map $((x_1, y_1), (x_2, y_2)) \mapsto (x_1, y_2)$. If A and B are compact, then D is a closed subset of a compact space $A \times B$, hence D is compact, which implies that $A \circ B$ is compact.

Lemma 1.4.9. *There exists $\Delta \in \mathbb{N}$ such that $U_{n+\Delta} \circ U_{n+\Delta} \subset U_n$ for all $n \geqslant 1$.*

Proof. Suppose that there is no Δ such that $U_\Delta \circ U_\Delta \subset Int(U)$. Denote $B_k = (\mathcal{X}^2 \setminus Int(U)) \cap (U_k \circ U_k)$, for $k \geqslant 0$. Then the sets B_k are closed non-empty, and $B_{k+1} \subset B_k$. It follows from compactness of $\mathcal{X}^2$ that the intersection $\bigcap_{k \geqslant 1} B_k$ is non-empty. Let (x, y) be such that $(x, y) \in B_k$ for all k. Let $Z_k \subset \mathcal{X}$ be the set of points z such that $(x, z) \in U_k$ and $(z, y) \in U_k$. Since U_k is closed, the set Z_k is closed. It is non-empty, by the choice of (x, y). We also have $Z_{k+1} \subset Z_k$. It follows that the intersection of all Z_k is non-empty. Let $z_0 \in \bigcap_{k \geqslant 1} Z_k$. Then $(x, z_0) \in U_k$ for all k, hence $x = z_0$, and $(z_0, y) \in U_k$ for all k, hence $z_0 = y$, which implies $x = y$, which is a contradiction.

We have shown that there exists Δ such that $U_\Delta \circ U_\Delta \subset U$. If $(x, y) \in U_{\Delta+n} \circ U_{\Delta+n}$, then there exists z such that $(x, z) \in U_{\Delta+n}$ and $(z, y) \in U_{\Delta+n}$. Then $(f^i(x), f^i(z)) \in U_{\Delta+n-i} \subseteq U_\Delta$ and $(f^i(z), f^i(y)) \in U_{\Delta+n-i} \subseteq U_\Delta$ for all $i = 0, 1, \ldots, n$. It follows that $(f^i(x), f^i(y)) \in U_\Delta \circ U_\Delta \subset U$, hence $(x, y) \in U_n$. We have shown that $U_{n+\Delta} \circ U_{n+\Delta} \subset U_n$ for all $n \geqslant 0$. ☐

Definition 1.4.10. Denote, for $(x, y) \in \mathcal{X}^2$, by $\ell(x, y)$ the maximal value of n such that $(x, y) \in U_n$, and ∞ if $x = y$.

Lemma 1.4.9 is reformulated then as follows.

Proposition 1.4.11. *There exists $\Delta > 0$ such that*

$$\ell(x, y) \geqslant \min(\ell(x, z), \ell(z, y)) - \Delta$$

for all $x, y, z \in \mathcal{X}$.

The function $\ell(x, y)$ measures "closeness" of points of $\mathcal{X}$. The closer points x and y are, the bigger is the value of $\ell(x, y)$. We will transform this function into a metric in the next subsection.

1.4.3. Log-scales and associated metrics. Generalizing Propositin 1.4.11, we adopt the following definition.

Definition 1.4.12. A *log-scale* on a set X is a map $\ell : X \times X \to \mathbb{R} \cup \{\infty\}$ satisfying the following conditions.

(1) $\ell(x, y) = \ell(y, x)$ for all $x, y \in X$;
(2) $\ell(x, y) = \infty$ if and only if $x = y$;
(3) there exists $\Delta > 0$ such that $\ell(x, y) \geqslant \min(\ell(x, z), \ell(z, y)) - \Delta$ for all $x, y, z \in X$.

We say that a metric d on X is *associated* with the log-scale ℓ if there exist constants $\alpha > 0$, $c > 1$, such that

$$c^{-1}e^{-\alpha\ell(x,y)} \leqslant d(x, y) \leqslant ce^{-\alpha\ell(x,y)}.$$

The number α is called the *exponent* of the metric.

It is easy to check that if d is any metric on X, then $\ell(x, y) = -\log d(x, y)$ is a log-scale such that d is associated with ℓ.

With this connection between log-scales and metrics in mind, we give the following definition.

Definition 1.4.13. We say that two log-scales ℓ_1, ℓ_2 on $\mathcal{X}$ are *bi-Lipschitz equivalent* if

$$\sup_{x,y\in X, x\neq y} |\ell_1(x, y) - \ell_2(x, y)| < \infty.$$

Theorem 1.4.14. *Let ℓ be a log-scale on a set X. Then there exists $\alpha_c \in (0, \infty]$ such that for every $\alpha \in (0, \alpha_c)$ there exists a metric d on X of exponent α associated with ℓ, and for every $\alpha > \alpha_c$ such a metric does not exist.*

Proof. Consider, for every $n \in \mathbb{N}$ the graph Γ_n with the set of vertices $\mathcal{X}$ in which two points x, y are connected by an edge if $\ell(x, y) \geqslant n$. Let d_n be the combinatorial distance between the vertices of Γ_n.

Lemma 1.4.15. *There exists $\alpha > 0$ and $C > 0$ such that*

$$d_n(x, y) \geqslant Ce^{\alpha(n-\ell(x,y))}$$

for all $x, y \in \mathcal{X}$ and $n \in \mathbb{N}$.

Proof. Let Δ be as in Proposition 1.4.11, and let us prove the lemma for $\alpha = \frac{\ln 2}{\Delta}$. If x_0, x_1, x_2 is a path in Γ_n, then $\ell(x_0, x_2) \geqslant n - \Delta$, hence x_0, x_2 is a path in $\Gamma_{n-\Delta}$. It follows that $d_{n-\Delta}(x, y) \leqslant \frac{1}{2}(d_n(x, y) + 1)$, or

$$d_{n+\Delta}(x, y) \geqslant 2d_n(x, y) - 1.$$

If $\ell(x, y) = m$, then $d_{m+1}(x, y) \geqslant 2$, and hence

$$d_{m+1+t\Delta}(x, y) \geqslant 2^t + 1.$$

It follows that for every n and $t = \left\lfloor \frac{n-\ell(x,y)-1}{\Delta} \right\rfloor > \frac{n-\ell(x,y)-1}{\Delta} - 1$ we have

$$d_n(x,y) > 2^t > 2^{(n-\ell(x,y)-1-\Delta)/\Delta} = Ce^{\alpha(n-\ell(x,y))},$$

where $C = 2^{(-1-\Delta)/\Delta}$ and $\alpha = \frac{\ln 2}{\Delta}$. □

We say that $\alpha > 0$ is a *lower exponent* if there exists $C > 0$ such that α and C satisfy the conditions of Lemma 1.4.15. If α is a lower exponent, then all numbers in the interval $(0, \alpha)$ are lower exponents. Hence, the set of lower exponents is either an interval $(0, \alpha_c)$ (including the case $\alpha_c = +\infty$) or an interval $(0, \alpha_c]$. The number α_c is called the *critical lower exponent*, and we are going to prove the theorem for this value.

It is easy to see that if α is such that there exists a metric of exponent α, then α is a lower exponent.

Let α be a lower exponent, and let $\beta \in (0, \alpha)$. Let us show that there exists a metric d of exponent β. Denote by $d_\beta(x,y)$ the infimum of the sum $\sum_{i=1}^m e^{-\beta\ell(x_i,x_{i+1})}$ over all sequences $x_0 = x, x_1, x_2, \ldots, x_m = y$. The function d_β obviously satisfies the triangle inequality, $d_\beta(x,y) = d_\beta(y,x)$, and $d_\beta(x,y) \leqslant e^{-\beta\ell(x,y)}$ for all $x, y \in \mathcal{X}$. It remains to show that there exists $C > 0$ such that $Ce^{-\beta\ell(x,y)} \leqslant d_\beta(x,y)$. In other words, there exists C is such that

$$Ce^{-\beta\ell(x,y)} \leqslant \sum_{i=1}^m e^{-\beta\ell(x_i,x_{i+1})} \tag{1.7}$$

for all sequences $x_0, x_1, \ldots, x_m$ such that $x = x_0$ and $y = x_m$.

Let $C_0 \in (0,1)$ be such that $d_n(x,y) \geqslant C_0 e^{-\alpha(n-\ell(x,y))}$ for all $x, y \in \mathcal{X}$ and $n \in \mathbb{N}$. Let us prove inequality (1.7) for $C = \exp\left(\frac{\beta(\ln C_0 - 2\alpha\Delta)}{\alpha-\beta}\right)$.

Lemma 1.4.16. *Let $x_0, x_1, \ldots, x_m$ be a sequence such that $\ell(x_i, x_{i+1}) \geqslant n$ for all $i = 0, 1, \ldots, m-1$. Let $n_0 \leqslant n$. Then there exists a sub-sequence $y_0 = x_0, y_1, \ldots, y_{t-1}, y_t = x_m$ of the sequence $(x_i)_{i=0}^m$ such that*

$$n_0 - 2\Delta \leqslant \ell(y_i, y_{i+1}) < n_0$$

for all $i = 0, 1, \ldots, t-1$.

Proof. Let us construct the subsequence y_i by the following algorithm. Define $y_0 = x_0$. Suppose we have defined $y_i = x_r$ for $r < m$. Let s be the largest index such that $s > r$ and $\ell(x_r, x_s) \geqslant n_0$. Note that since $\ell(x_r, x_{r+1}) \geqslant n \geqslant n_0$, such s exists.

If $s < m$, then $\ell(x_r, x_{s+1}) < n_0$, and

$$\ell(x_r, x_{s+1}) \geqslant \min\{\ell(x_r, x_s), \ell(x_s, x_{s+1})\} - \Delta \geqslant \min\{n_0, \ell(x_s, x_{s+1})\} - \Delta = n_0 - \Delta.$$

Define then $y_{i+1} = x_{s+1}$. We have

$$n_0 - k \leqslant \ell(y_i, y_{i+1}) < n_0.$$

If $s+1 = m$, then we stop and get our sequence $y_0, \dots, y_t$, for $t = i+1$.

If $s = m$, then $\ell(x_r, x_m) = \ell(y_i, x_m) \geqslant n_0$, and

$$\ell(y_{i-1}, x_m) \geqslant \min\{\ell(y_{i-1}, y_i), \ell(y_i, x_m)\} - \Delta \geqslant \min\{n_0 - \Delta, n_0\} - k = n_0 - 2\Delta$$

and

$$\ell(y_{i-1}, x_m) < n_0,$$

since y_i was defined and was not equal to x_m. Then we redefine $y_i = x_m$ and stop the algorithm.

In all the other cases we repeat the procedure. It is easy to see that at the end we get a sequence y_i satisfying the conditions of the lemma. □

Let $x_0 = x, x_1, \dots, x_m = y$ be an arbitrary sequence of points of X. Let n_0 be the minimal value of $\ell(x_i, x_{i+1})$. Let $y_0 = x, y_1, \dots, y_t = y$ be a sub-sequence of the sequence x_i satisfying conditions of Lemma 1.4.16.

Suppose at first that

$$n_0 < \ell(x, y) + \frac{2\alpha\Delta - \ln C_0}{\alpha - \beta}.$$

Remember that $n_0 = \ell(x_i, x_{i+1})$ for some i, hence

$$\sum_{i=1}^{m} e^{-\beta\ell(x_{i-1}, x_i)} \geqslant e^{-\beta n_0} > \exp\left(-\beta\ell(x, y) - \frac{\beta(2\alpha\Delta - \ln C_0)}{\alpha - \beta}\right) = Ce^{-\beta\ell(x,y)},$$

and the statement is proved.

Suppose now that $n_0 \geqslant \ell(x, y) + \frac{2\alpha k - \ln C_0}{\alpha - \beta}$, which is equivalent to

$$(\alpha - \beta)n_0 - (\alpha - \beta)\ell(x, y) - 2\alpha\Delta + \ln C_0 \geqslant 0. \tag{1.8}$$

If $t = 1$, then $n_0 - 2\Delta \leqslant \ell(x, y) < n_0$, hence

$$n_0 \leqslant \ell(x, y) + 2\Delta = \ell(x, y) + \frac{2\alpha\Delta - 2\beta\Delta}{\alpha - \beta} < \ell(x, y) + \frac{2\alpha\Delta - \ln C_0}{\alpha - \beta},$$

since $\ln C_0 < 0 < 2\beta\Delta$. But this contradicts our assumption.

Therefore, we have $t > 1$, so that the inductive assumption implies

$$\sum_{i=1}^{m} e^{-\beta\ell(x_{i-1}, x_i)} \geqslant \sum_{i=0}^{t-1} Ce^{-\beta\ell(y_i, y_{i+1})} > tCe^{-\beta n_0}.$$

We have $t \geqslant d_{n_0-2\Delta}(x,y) \geqslant C_0 e^{\alpha(n_0-2\Delta-\ell(x,y))}$, hence

$$\sum_{i=1}^{m} e^{-\beta\ell(x_{i-1},x_i)} \geqslant C_0 C e^{-\beta n_0+\alpha n_0-2\alpha\Delta-\alpha\ell(x,y)} =$$

$$C\exp\left(\ln C_0 - \beta n_0 + \alpha n_0 - 2\alpha\Delta - \alpha\ell(x,y)\right) =$$

$$C\exp\left(-\beta\ell(x,y) + (\alpha-\beta)n_0 - (\alpha-\beta)\ell(x,y) - 2\alpha\Delta + \ln C_0\right) \geqslant Ce^{-\beta\ell(x,y)},$$

by (1.8). Which finishes the proof. □

1.4.4. Metrics associated with expanding maps.

Theorem 1.4.17. *Let $\mathcal{X}$ be a compact space. A continuous map $f \circlearrowleft \mathcal{X}$ is expanding (in the sense of Definition 1.4.1) if and only if f is metrically expanding for some metric on $\mathcal{X}$.*

Proof. Suppose that there exists a metric d and numbers $\epsilon > 0$ and $L > 1$ such that $d(f(x), f(y)) > Ld(x,y)$ for all $(x,y) \in \mathcal{X}^2$ such that $d(x,y) \leqslant \epsilon$. Then the set $\{(x,y) : d(x,y) \leqslant \epsilon\}$ is an expansion entourage and the action of $\mathbb{N}$ is expansive.

Suppose now that the action of $\mathbb{N}$ is expansive. Then there exists a symmetric expansion entourage U. Suppose that d is a metric associated with the log-scale defined by U, see Definition 1.4.10. Let α be the exponent of the metric d, and let $C > 1$ be such that

$$C^{-1}e^{-\alpha\ell(x,y)} \leqslant d(x,y) \leqslant Ce^{-\alpha\ell(x,y)}$$

for all $x, y \in \mathcal{X}$. Let k be a positive integer, and suppose that $\ell(x,y) \geqslant k$. Then $\ell(f^k(x), f^k(y)) = \ell(x,y) - k$, and

$$\frac{d(f^k(x), f^k(y))}{d(x,y)} \leqslant C^2 e^{-\alpha k}.$$

It follows that for any integer k greater than $\frac{\ln C^2}{\alpha}$ we have $d(f^k(x), f^k(y)) \leqslant Ld(x,y)$, where $L = C^2 e^{-\alpha k} < 1$, for all $(x,y) \in U_k$. If $\epsilon < C^{-1}e^{-\alpha k}$, then $\ell(x,y) \geqslant k$ for all $x, y \in \mathcal{X}$ such that $d(x,y) \leqslant \epsilon$. It follows that f^k is metrically expanding. Lemma 1.4.4 shows that f is also expanding. □

Let us investigate how canonical is the metric constructed in the proof of Theorem 1.4.17.

Proposition 1.4.18. *Let U and V be expansion entourages for a map $f \circlearrowleft \mathcal{X}$. Then the sets of lower exponents for U and V coincide. If d_U and d_V are metrics associated with U and V of exponent α, then there exists $C > 1$ such that $C^{-1}d_U(x,y) \leqslant d_V(x,y) \leqslant Cd_U(x,y)$ (i.e., the metrics are* bi-Lipschitz equivalent*).*

Proof. Let ℓ_U and ℓ_V be defined by U and V, respectively. By Lemma 1.4.8, there exists k such that $U_k \subset V$, hence $U_{n+k} \subset V_n$ for all $n \in \mathbb{N}$. It follows that $\ell_V(x, y) \geqslant \ell_U(x, y) + k$. The same arguments show that $\ell_U(x, y) \geqslant \ell_V(x, y) + k$ for some k, i.e., that $|\ell_U(x, y) - \ell_V(x, y)|$ is uniformly bounded. The statements of the proposition easily follow from this fact. □

Proposition 1.4.18 implies that for any expanding map $f \circlearrowright \mathcal{X}$ the critical lower exponent α_c is well defined (i.e., does not depend on the choice of the expansion entourage), and for every $\beta \in (0, \alpha)$ the corresponding metric of exponent β is uniquely defined, up to a bi-Lipschitz equivalence. Note that if d is a metric of exponent β, then any metric bi-Lipschitz equivalent to d is also a metric of exponent β. The map f is metrically expanding with respect to some metric of exponent β for every $\beta \in (0, \alpha_c)$. This class of metrics is studied in detail in the works of P. Haïsinsky and K. Pilgrim (see [**HP09**] and references therein). The critical exponent α_c is one of several numerical invariants of expanding dynamical systems...

1.4.5. Expansive actions of $\mathbb{Z}$. Recall that a homeomorphism $f \circlearrowright \mathcal{X}$ of a metric space is said to be *expansive* if there exists a closed neighborhood $U \subset \mathcal{X} \times \mathcal{X}$ of the diagonal such that $\bigcap_{n\in\mathbb{Z}} f^n(U)$ is equal to the diagonal.

Let $f \circlearrowright \mathcal{X}$ be an expansive homeomorphism of a compact metric space, and let U be the corresponding expansion entourage as above.

Define $\ell(x, y)$ as the maximal $n \geqslant 0$ such that $(f^k(x), f^k(y)) \in U$ for all $-n < k < n$. If such n does not exist, then we set $\ell(x, y) = 0$. It is obvious that $\ell(x, y) = \ell(y, x)$ and that $\ell(x, y) = \infty$ if and only if $x = y$. Define, as in the expanding case, $U_n = \{(x, y) \in \mathcal{X} \times \mathcal{X} \ : \ \ell(x, y) \geqslant n\}$, i.e., $U_n = \bigcap_{|k|<n} f^k(U)$.

Lemma 1.4.19. *The map ℓ is a log-scale on $\mathcal{X}$ compatible with the topology (i.e., such that every metric associated with ℓ is compatible with the topology on $\mathcal{X}$. The log-scale ℓ does not depend, up to bi-Lipschitz equivalence, on the choice of U.*

Proof. It is proved in the same way as for expanding maps that for every neighborhood V of the diagonal there exists n such that $U_n \subset V$ (see Lemma 1.4.8). The statement of the Lemma 1.4.9 is also true in our case with the same proof (the only thing to change is to replace "$i = 0, 1, \ldots, n$" by "$-n \leqslant i \leqslant n$". These two lemmas imply that the metric associated with the log-scale ℓ is compatible with the topology. Independence on the choice of U is also proven in the same way as for the expanding case, see the proof of Proposition 1.4.18. □

We see that, in the same way as for expanding maps, we have a canonical class of metrics associated with an expansive homeomorphism $f \circlearrowleft \mathcal{X}$. These metrics were defined by D. Fried in [**Fri83**].

Definition 1.4.20. We say that $x, y \in \mathcal{X}$ are *stably equivalent* if $d(f^n(x), f^n(x)) \to 0$ as $n \to \infty$. We say that they are *unstably equivalent* if $d(f^{-n}(x), f^{-n}(y)) \to 0$ as $n \to \infty$. Two points are *homoclinic* if they are simultaneously stably and unstably equivalent.

It is easy to see that the defined relations are equivalences.

Lemma 1.4.21. *Let U be the expansivity entourage for an expansive homeomorphism $f \circlearrowleft \mathcal{X}$. Two points $x, y \in \mathcal{X}$ are stably (resp. unstably) equivalent if and only if there exists $n_0 \in \mathbb{Z}$ such that $(f^n(x), f^n(y)) \in U$ for all $n \geqslant n_0$ (resp. all $n \leqslant n_0$).*

Proof. The "only if" direction is obvious. Suppose that $(f^n(x), f^n(y)) \in U$ for all $n \geqslant n_0$. Then $\ell(f^n(x), f^n(y)) \geqslant n - n_0$ for all $n \geqslant n_0$. It follows that for every metric associated with ℓ we have $d(f^n(x), f^n(y)) \leqslant Ce^{-\alpha(n-n_0)} \to 0$ as $n \to \infty$. $\square$

Define, for $x \in \mathcal{X}$, and an expansivity entourage U

$$W_{+,U}(x) = \{y \in \mathcal{X} \ : \ (f^n(x), f^n(y)) \in U \ \forall n \geqslant -\Delta - 1\}$$

and

$$W_{-,U}(x) = \{y \in \mathcal{X} \ : \ (f^n(x), f^n(y)) \in U \ \forall n \leqslant \Delta + 1\},$$

where Δ satisfies the condition of Definition 1.4.12 for the log-scale ℓ associated with U. Equivalently, $W_{+,U}(x)$ is the set of points y such that $\ell(f^n(x), f^n(y)) \geqslant \Delta + 1$ for all $n \geqslant 0$, and $W_{-,U}(x)$ is the set of points y such that $\ell(f^n(x), f^n(y)) \geqslant \Delta + 1$ for all $n \leqslant 0$.

Lemma 1.4.22. *For every pair $x, y \in \mathcal{X}$ the intersection $W_{+,U}(x) \cap W_{-,U}(y)$ consists of at most one point.*

Proof. Suppose that $z_1, z_2 \in W_{+,U}(x) \cap W_{-,U}(y)$. Then for every $n \in \mathbb{Z}$ either

$$\min(\ell(f^n(z_1), f^n(x)), \ell(f^n(z_2), f^n(x))) \geqslant \Delta + 1$$

or

$$\min(\ell(f^n(z_1), f^n(y)), \ell(f^n(z_2), f^n(y))) \geqslant \Delta + 1,$$

depending on the sign of n. But this implies that $\ell(f^n(z_1), f^n(z_2)) \geqslant 1$ for all $n \in \mathbb{Z}$, i.e., that $(f^n(z_1), f^n(z_2)) \in U$ for all $n \in \mathbb{Z}$. It follows that $z_1 = z_2$. $\square$

Definition 1.4.23. We will denote by $[x, y]_U$ or just $[x, y]$ the unique intersection point of $W_{+,U}(x)$ and $W_{-,U}(y)$, if it exists.

1.4.6. Local product structures.

1.4.6.1. *Rectangles.* A *rectangle* is a topological space R together with a decomposition into a direct product $R = A \times B$ of two topological spaces. In order to make this structure more intrinsic (so that we do not introduce new spaces A and B), we can define the direct product structure as a binary operation $[(a_1, b_1), (a_2, b_2)] = (a_1, b_2)$ on $A \times B = R$. Then the structure of the direct product decomposition can be axiomatized in the following way.

Definition 1.4.24. A *rectangle* is a topological space R together with a continuous map $[\cdot, \cdot] : R \times R \longrightarrow R$ such that

(1) $[x, x] = x$ for all $x \in R$;
(2) $[x, [y, z]] = [x, z]$ and $[[x, y], z] = [x, z]$ for all $x, y, z \in R$.

Example 1.4.25. Let $f \circlearrowright \mathcal{X}$ be an expansive homeomorphism of a compact space. We say that $R \subset \mathcal{X}$ is a *rectangle* for f if the map $[\cdot, \cdot]$, given in Definition 1.4.23 is defined on whole $R \times R$. Then, by Proposition **??**, $(R, [\cdot, \cdot])$ is a rectangle in the sense of Definition 1.4.24.

Suppose that $(R, [\cdot, \cdot])$ is a rectangle. Then *plaques* of $x \in R$ are defined as

$$\mathsf{P}_1(R, x) = \{y \in R \ : \ [x, y] = x\}, \qquad \mathsf{P}_2(R, x) = \{y \in R \ : \ [x, y] = y\}.$$

Note that we have the implication

$$[x, y] = x \Longrightarrow [y, x] = [y, [x, y]] = [y, y] = y.$$

It follows that $[x, y] = x$ is equivalent to $[y, x] = y$. It is shown in the same way that $[x, y] = y$ is equivalent to $[y, x] = x$. In other words, $y \in \mathsf{P}_i(R, x)$ is equivalent to $x \in \mathsf{P}_i(R, y)$ for every $i = 1, 2$.

Lemma 1.4.26. *For every $x \in R$ the map $[\cdot, \cdot] : \mathsf{P}_1(R, x) \times \mathsf{P}_2(R, x) \longrightarrow R$ is a homeomorphism.*

Proof. For every $y \in R$ we have $[x, [y, x]] = x$, hence $[y, x] \in \mathsf{P}_1(x)$. Similarly, $[[x, y], x] = x$, hence $[x, y] \in \mathsf{P}_2(x)$. Then $[[y, x], [x, y]] = [[y, x], y] = y$. It follows that $y \mapsto ([y, x], [x, y])$ is a continuous map $R \mapsto \mathsf{P}_1(R, x) \times \mathsf{P}_2(R, x)$ inverse to the map $[\cdot, \cdot]$. ☐

Note that if $(a_1, b_1), (a_2, b_2) \in \mathsf{P}_1(R, x) \times \mathsf{P}_2(R, x)$, then $[[a_1, b_1], [a_2, b_2]] = [a_1, b_2]$, i.e., the map $[\cdot, \cdot]$, after the identification of R with $\mathsf{P}_1(R, x) \times \mathsf{P}_2(R, x)$, becomes $[(a_1, b_1), (a_2, b_2)] = (a_1, b_2)$. See Figure 1.26, where the structure of a rectangle is shown.

Note that two different plaques $\mathsf{P}_i(R, x)$ and $\mathsf{P}_i(R, y)$ are naturally identified by a canonical homeomorphism, so that the decomposition of R into

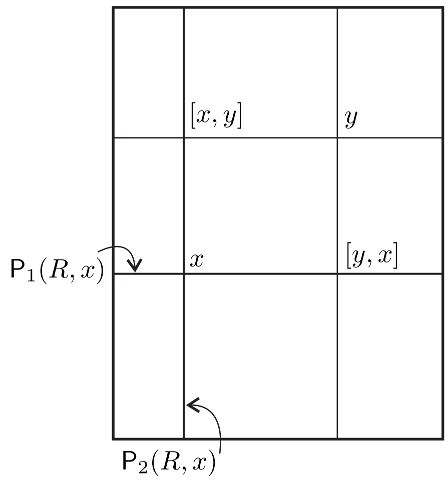

Figure 1.26. A rectangle

the direct product of plaques does not depend on the reference point (x or y). Namely, the respective homeomorphisms are

$$H_{1,x,y} : z \mapsto [z, y] : \mathsf{P}_1(R, x) \longrightarrow \mathsf{P}_1(R, y)$$

and

$$H_{2,x,y} : z \mapsto [y, z] : \mathsf{P}_2(R, x) \longrightarrow \mathsf{P}_2(R, y).$$

It is checked directly that $H_{1,x,y} \circ H_{1,y,x}$ and $H_{2,x,y} \circ H_{2,y,x}$ are identity homeomorphisms, and that the decomposition of R into the direct product $\mathsf{P}_1(R, x) \times \mathsf{P}_2(R, x)$ is transformed by the homeomorphisms $H_{i,x,y}$ to the decomposition of R into the direct product of plaques of y.

We will therefore denote sometimes by $\mathsf{P}_1(R)$ and $\mathsf{P}_2(R)$ the plaques of R as abstract topological spaces, without any reference to points of R.

1.4.6.2. *Local product structures.*

Definition 1.4.27. Let $\mathcal{X}$ be a topological space. An *atlas of a local product structure* on $\mathcal{X}$ is a cover of $\mathcal{X}$ by open subsets R_i, $i \in I$, together with structures of rectangles $(R_i, [\cdot, \cdot]_i)$ on each of them, such that for every pair $i, j \in I$ and every $x \in \mathcal{X}$ there exists a neighborhood U of x such that $[y, z]_i = [y, z]_j$ for all $y, z \in R_i \cap R_j \cap U$.

Two atlases are *compatible* if their union is also an atlas. A *local product structure* on $\mathcal{X}$ is a compatibility class of atlases of local product structures on $\mathcal{X}$.

Note that the condition of Definition 1.4.27 is void in the case when x does not belong to the intersection of the closures of R_i and R_j.

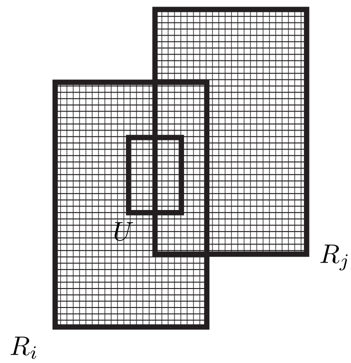

Figure 1.27. Local product structure

An open subset $R \subset \mathcal{X}$ together with a rectangle structure $[\cdot,\cdot]$ on R is a *rectangle of* $\mathcal{X}$ if the union of an atlas of $\mathcal{X}$ with $\{(R,[\cdot,\cdot])\}$ is an atlas of $\mathcal{X}$, i.e., if the structure of a rectangle on R is compatible with the local product structure on $\mathcal{X}$.

Example 1.4.28. Let $F : B \longrightarrow X$ be a locally trivial bundle with fiber P. It means that for every point $x \in \mathcal{X}$ there is a neighborhood U of x and a homeomorphism $\phi_U : P \times U \longrightarrow F^{-1}(U)$ such that $F(\phi_U(p, y)) = y$ for all $y \in U$ and $p \in P$. Moreover, the maps ϕ_U naturally agree with each other, i.e., if U_1 and U_2 intersect, then $\phi_{U_1}(p, y) = \phi_{U_2}(p, y)$ for all $p \in P$ and $y \in U_1 \cap U_2$.

It is easy to see that the set of the rectangles $\phi_U(P \times U)$ defines a local product structure on B.

Definition 1.4.29. Let $\mathcal{X}_1, \mathcal{X}_2$ be two spaces with local product structures on them. We say that a continuous map $f : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ *preserves the local product structures* if for every point $x \in \mathcal{X}_1$ there exist rectangular neighborhoods $(R_1, [\cdot,\cdot]_1)$ and $(R_2, [\cdot,\cdot]_2)$ of x and $f(x)$, respectively, such that $f([y, z]_1) = [f(y), f(z)]_2$ for all $y, z \in R_1$.

Example 1.4.30. Consider a rectangle $\mathcal{X} = A \times B$, and let G be a group acting properly and freely on $\mathcal{X}$ by homeomorphisms preserving the local product structure on $\mathcal{X}$. Then the quotient $G\backslash\mathcal{X}$ by the action is naturally a space with a local product structure. We call such local product structures *splittable*. For example, for any decomposition of $\mathbb{R}^n$ into a direct sum of subspaces, we get the corresponding local product structure on the torus $\mathbb{R}^n/\mathbb{Z}^n$. See, for example, the decomposition into the direct sum of the eigenspaces for the "Arnold's Cat" map 1.1.5.

1.4.7. Ruelle-Smale systems. The following class of dynamical systems was introduced by D. Ruelle in [**Rue78**] as a generalization of basic sets of hyperbolic diffeomorphisms.

Definition 1.4.31. A *Ruelle-Smale system* (also called *Smale space*) is a homeomorphism $f \circlearrowleft \mathcal{X}$ of a compact metric space with a local product structure satisfying the following conditions.

(1) The map f preserves the local product structure.
(2) There exists $\lambda \in (0,1)$ and a cover of $\mathcal{X}$ by a finite number of rectangles $(R_i, [\cdot,\cdot])$ such that for any two points x, y belonging to one plaque $\mathsf{P}_1(R_i, x) = \mathsf{P}_2(R_i, y)$ we have $d(f(x), f(y)) \leqslant \lambda d(x,y)$; and for any two points x, y belonging to one plaque $\mathsf{P}_2(R_i, x) = \mathsf{P}_2(R_i, y)$ we have $d(f^{-1}(x), f^{-1}(y)) \leqslant \lambda d(x,y)$.

In other words, a homeomorphism is a Ruelle-Smale system if it is contracting in one direction and expanding in the other direction of a local product structure preserved by it.

We will denote $\mathsf{P}_1 = \mathsf{W}_+$ and $\mathsf{P}_2 = \mathsf{W}_-$. The plaques $\mathsf{W}_+(R,x)$ and $\mathsf{W}_-(R,x)$ are called the *stable* and the *unstable* plaques, respectively.

Note that by the Lebesgue's covering lemma, if $\{(R_i, [\cdot,\cdot]_i)\}_{i\in I}$ is a finite atlas of the local product structure of a compact space $\mathcal{X}$, then there exists $\epsilon > 0$ such that $[x,y]_i = [x,y]_j$ for all $i, j \in I$ and all x, y such that $d(x,y) < \epsilon$ and the corresponding expressions are defined. It follows that we may assume that we have one map $[\cdot,\cdot]$ defined on a neighborhood of the diagonal of $\mathcal{X} \times \mathcal{X}$.

It follows from the definition that if x and y belong to the same stable plaque, then the distance $d(f^n(x), f^n(y))$ exponentially converges to zero. Similarly, if x and y belong to the same unstable plaque, then $d(f^{-n}(x), f^{-n}(y))$ exponentially converges to zero.

Proposition 1.4.32. *A homeomorphism $f \circlearrowleft \mathcal{X}$ of a compact space is a Ruelle-Smale system if and only if f is expansive and for every expansivity entourage U the map $[\cdot,\cdot]_U$ from Definition 1.4.23 is defined on a neighborhood of the diagonal.*

In particular, the local product structure satisfying the conditions of Definition 1.4.31 is unique and depends only on the topological conjugacy class of $f \circlearrowleft \mathcal{X}$.

Proof. Let us prove at first that every Ruelle-Smale systems is expansive. Let ϵ be a Lebesgue number of a finite cover $\{R_i\}_{i\in I}$ of $\mathcal{X}$ by rectangles. Suppose that $x, y \in \mathcal{X}$ are such that $d(f^n(x), f^n(y)) < \epsilon$ for all $n \in \mathbb{Z}$ and $x \neq y$. Then for every n there exists $i_n \in I$ such that $f^n(x), f^n(y)$ both belong to R_{i_n}. Consider then the corresponding point $z_n = [f^n(x), f^n(y)] \in R_{i_n}$. We have then $f(z_n) = z_{n+1}$ for all $n \in \mathbb{Z}$, i.e., $z_n = f^n(z_0)$. The distance $d(f^n(x), f^n(z_0))$ is bounded from above by the maximum of diameters of the rectangles R_i. But we must have $d(f^n(x), f^n(z_0)) \geqslant \lambda^{-1} d(f^{n-1}(x), f^{n-1}(z_0))$ for all $n \in \mathbb{Z}$, which is a contradiction. It follows that ϵ is an expansivity constant for $f \circlearrowleft \mathcal{X}$.

The fact that the local product structure on the Ruelle-Smale system coincides with the local product structure given in Definition 1.4.23 for expansive systems is now straightforward.

In the other direction, suppose that $f \circlearrowleft \mathcal{X}$ is an expansive homeomorphism of a compact space such that $[\cdot,\cdot]_U$ is defined on a neighborhood of the diagonal for every expansivity entourage U. Note that if $U_1 \subset U_2$, then $[\cdot,\cdot]_{U_1}$ is a restriction of $[\cdot,\cdot]_{U_2}$. We have $(x, [x,y]) \in U$ and $(y, [x,y]) \in U$, which implies that $[\cdot,\cdot]_U$ is continuous. The axioms of a local product structure are checked directly using Lemma 1.4.22.

Let d be a metric on $\mathcal{X}$ associated with U (see beginning of 1.4.5). It is checked directly that some iterate of $f \circlearrowleft \mathcal{X}$ is Ruelle-Smale system with respect to d. Using then the same trick as in Lemma 1.4.4, we can modify the metric so that $f \circlearrowleft \mathcal{X}$ is a Ruelle-Smale system. $\square$

1.4.8. Examples of Ruelle-Smale systems.

1.4.8.1. *Shifts of finite type.*

Proposition 1.4.33. *A subshift $\mathcal{S} \subset A^{\mathbb{Z}}$ is a Ruelle-Smale system if and only if it is of finite type.*

Proof. Two points $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$ of $\mathsf{X}^{\mathbb{Z}}$ are stably equivalent if and only if there exists $n_0 \in \mathbb{Z}$ such that $a_n = b_n$ for all $n \geqslant n_0$. Similarly, they are unstably equivalent if and only if there exists n_0 such that $a_n = b_n$ for all $n \leqslant n_0$.

Suppose that $\mathcal{F} \subset \mathsf{X}^{\mathbb{Z}}$ is a subshift. Denote by U_N the entourage consisting of all pairs $(w_1, w_2) \in \mathcal{F} \times \mathcal{F}$ such that $w_1(n) = w_2(n)$ for all $-N \leqslant n \leqslant N$.

Suppose that $\mathcal{F}$ is a Ruelle-Smale system. Then there exists N such that for any pair of sequences $(w_1, w_2) \in U_N$ the intersection $W_{+,U_N}(w_1) \cap W_{-,U_N}(w_2)$ is non-empty. In other words, there exists N such that if sequences w_1 and w_2 from $\mathcal{F}$ coincide on the interval $[-N, N]$, then the sequence w equal to w_1 on $[-N, +\infty)$ and to w_2 on $(-\infty, N]$ also belongs to $\mathcal{F}$. The converse statement is also true: if $\mathcal{F}$ satisfies the last condition, then it is a Ruelle-Smale system. We leave it to the reader to use this condition to prove that every shift of finite type is a Ruelle-Smale system.

Let $L \subset \mathsf{X}^{2N+2}$ be the set of all subwords of length $2N + 2$ of elements of $\mathcal{F}$. Let us prove that if $w \in \mathsf{X}^{\mathbb{Z}}$ is a sequence such that every subword of length $2N + 2$ of w belongs to L, then $w \in \mathcal{F}$. This will prove that $\mathcal{F}$ is a shift of finite type. It is enough to show that if v is finite word such that every subword of v of length $2N + 2$ belongs to L, then v is a subword of an element of $\mathcal{F}$. Let us prove this statement by induction on the length of v. The statement is trivially true for $|v| = 2N + 2$. Suppose that we have proved it of $|v| = k$, let us prove it for vx, where $x \in \mathsf{X}$. Write $v = yu$ for $y \in \mathsf{X}$. Then $|v| = |ux| = k$, so there exist sequences $w_1, w_2 \in \mathcal{F}$ such that the restriction of w_1 and w_2 onto an interval $[a, b] \subset \mathbb{Z}$ containing $[-N, N]$ is equal to u, $w_1(a-1) = y$ and $w_2(b+1) = x$. Then there exists a sequence $w \in \mathcal{F}$ such that $w|_{(-\infty,N]} = w_1|_{(-\infty,N]}$ and $w|_{[-N,\infty)} = w_2|_{[-N,\infty)}$. Then $yux = vx$ is equal to $w|_{[a-1,b+1]}$. $\square$

1.4.8.2. *Anosov diffeomorphisms.*

Definition 1.4.34. An *Anosov diffeomorphism* is a diffeomorphism $f \circlearrowleft \mathcal{M}$ of a compact Riemannian manifold such that there exists a decomposition $T\mathcal{M} = T_+ \oplus T_-$ of the tangent bundle into a direct sum of f-invariant sub-bundles, and constants $C > 0$ and $\lambda \in (0, 1)$ such that

(1) $\|Df^n(\vec{v})\| \leqslant C\lambda^n \|\vec{v}\|$ for all $n \geqslant 0$ and $\vec{v} \in T_+$,
(2) $\|Df^{-n}(\vec{v})\| \leqslant C\lambda^n \|\vec{v}\|$ for all $n \geqslant 0$ and $\vec{v} \in T_-$.

By classical theory (see, for example, [**Sma67**, Theorem 7.4], [**BS02**, Theorem 5.6.4, Theorem 5.7.2]) every Anosov diffeomorphism is a Ruelle-Smale system.

An example of an Anosov diffeomorphism is the Arnold's Cat Map from 1.1.5. It can be generalized in the following way. Let $\mathfrak{L}$ be a simply connected nilpotent Lie group, and let $\phi : \mathfrak{L} \longrightarrow \mathfrak{L}$ be its automorphism such that the differential $D\phi$ at the identity $1 \in \mathfrak{L}$ is hyperbolic (i.e., its spectrum is disjoint with the unit circle). Let G be a subgroup of the affine group $\mathfrak{L} \ltimes \mathrm{Aut}\,\mathfrak{L}$ acting naturally on $\mathfrak{L}$, and suppose that the action $G \curvearrowright \mathfrak{L}$ is free and co-compact, so that $G\backslash\mathfrak{L}$ is a compact manifold. Such manifolds are called *infra nil-manifolds.* Assume also that the G-action is ϕ-invariant, so that ϕ induces a diffeomorphism of $G\backslash\mathfrak{L}$. Then this diffeomorphism is Anosov, see [**Sma67**, pp. 760–764]. We call such Anosov diffeomorphisms *algebraic.*

All currently known Ansov diffeomorphisms are topologically conjugate to algebraic diffeomorphisms. It is an open question if this is a complete description. Moreover, the only known examples of Ruelle-Smale systems $f \circlearrowleft \mathcal{X}$ such that $\mathcal{X}$ is a connected and locally connected space are algebraic Anosov diffeomorphisms.

1.4.8.3. *Hyperbolic sets.* More generally, let $\mathcal{M}$ be a Riemanian manifold, $U \subset \mathcal{M}$ a non-empty open subset, and let $f : U \longrightarrow f(U)$ be a diffeomorphism. A closed totally f-invariant subset $\mathcal{X} \subset U$ (i.e., such that $f(U) = U = f^{-1}(U)$) is said to be *hyperbolic* if there exist decompositions $T_x\mathcal{M} = T_+(x) \oplus T_-(x)$ of the tangent spaces at $x \in \mathcal{X}$ and constants $C > 0, \lambda \in (0, 1)$ such that

(1) $DfT_+(x) = T_+(f(x))$, $DfT_-(x) = T_-(f(x))$;
(2) $\|Df^n\vec{v}\| \leqslant C\lambda^n\|\vec{v}\|$ for all $\vec{v} \in T_+(x)$, $x \in \mathcal{X}$, and $n \geqslant 0$;
(3) $\|Df^{-n}\vec{v}\| \leqslant C\lambda^n\|\vec{v}\|$ for all $\vec{v} \in T_-(x)$, $x \in \mathcal{X}$, and $n \geqslant 0$.

A hyperbolic set $\mathcal{X}$ is *locally maximal* if there exists an open neighborhood $V \supset \mathcal{X}$ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V)$.

If $\mathcal{X}$ is a locally maximal compact hyperbolic set, then $f \circlearrowright \mathcal{X}$ is a Ruelle-Smale system, see [**BS02**, Proposition 5.9.1, 5.9.3]. Examples of locally maximal hyperbolic sets are the Smale horseshoe attractor W from 1.1.3, and the solenoid from 1.1.4 (in its concrete version in $\mathbb{R}^3$).

1.4.8.4. *DA attractors.* We have seen in 1.1.5 that there exists a semiconjugacy $\phi : \mathcal{F} \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$ from a shift of finite type $\mathcal{F}$ to the Anosov diffeomorphism $A \circlearrowright \mathbb{R}^2/\mathbb{Z}^2$ defined by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. We will see later that this is true for any Ruelle-Smale system.

The shift of finite type $\mathcal{F}$ can be defined as a result of cutting the torus along the boundaries of the elements of the Markov partition, and propagating the cuts by the $\mathbb{Z}$-action of the dynamical system, exactly in the same way as we did it with the circle rotation in 1.3.1.2.

We could also cut, i.e., make slits in the torus, only along the stable boundaries of the Markov partition. This will produce another Ruelle-Smale system $f \circlearrowright \mathcal{X}$ with a semiconjugacy $\phi : \mathcal{X} \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$. It is called a *DA-attractor* ("Derived from Anosov"). We will give an alternative description of this dynamical system later in....

The DA-attractor can be realized as a hyperbolic set of a diffeomorphism by modifying the Anosov diffeomorphism in a small neighborhood of a point (essentially by imitating the slits described above), see [**Sma67**, p. 788].

1.4.9. Natural extension of an expanding covering map.

Proposition 1.4.35. *If $f \circlearrowright \mathcal{X}$ is an expanding map on a compact space, then its natural extension $\hat{f} \circlearrowright \hat{\mathcal{X}}$ is expansive.*

For the notion of a natural extension, see Definition 1.1.8.

Proof. Let $\epsilon > 0$ be an expansivity constant for $f \circlearrowright \mathcal{X}$ with respect to a metric d on $\mathcal{X}$. Consider the entourage $U \subset \hat{\mathcal{X}} \times \hat{\mathcal{X}}$ consisting of pairs of sequences $((x_1, x_2, \ldots), (y_1, y_2, \ldots))$ such that $d(x_1, y_1) < \epsilon$. Suppose that $\xi = (x_1, x_2, \ldots)$ and $\zeta = (y_1, y_2, \ldots)$ are such that $(\hat{f}^n(\xi), \hat{f}^n(\zeta)) \in U$ for all $n \in \mathbb{Z}$. Then for every $i \geqslant 1$ we have $d(f^n(x_i), f^n(y_i)) < \epsilon$ for all $n \geqslant 0$, since $f^{n+i-1}(\xi) = (f^n(x_i), f^n(x_{i+1}), \ldots)$ and $f^{n+i-1}(\zeta) = (f^n(y_i), f^n(y_{i+1}), \ldots)$. This implies, by expansivity of $f \circlearrowleft \mathcal{X}$, that $x_i = y_i$ for all i. □

Theorem 1.4.36. *Let $f \circlearrowleft \mathcal{X}$ be an expanding covering map of a compact space, and let $\hat{f} \circlearrowleft \hat{\mathcal{X}}$ be its natural extension. Then $\hat{\mathcal{X}}$ is a fiber bundle with respect to the natural projection $P : \hat{\mathcal{X}} \longrightarrow \mathcal{X}$: for every $x \in \mathcal{X}$ there exists a neighborhood U of x such that $P^{-1}(U)$ is naturally homeomorphic to the direct product $C \times U$, where C is the inverse limit of the sets $f^{-n}(x)$.*

The natural extension $\hat{f} \circlearrowleft \hat{\mathcal{X}}$ is a Ruelle-Smale system, where the local product structure coincides with the local product structure coming from the described fiber bundle.

Proof. Let $\epsilon > 0$, $L > 1$ be such that $d(f(x), f(y)) \geqslant Ld(x, y)$ for all $x, y \in \mathcal{X}$ such that $d(x, y) \leqslant \epsilon$.

For every $x \in \mathcal{X}$ there exists an open neighborhood U of x that is *evenly covered*, i.e., such that $f^{-1}(U)$ can be decomposed into a disjoint union $f^{-1}(U) = U_1 \cup U_2 \cup \cdots \cup U_m$ such that $f : U_i \longrightarrow U$ is a homeomorphism for every i. The decomposition is finite, since $\mathcal{X}$ is compact (hence $f^{-1}(x)$ is compact for every $x \in \mathcal{X}$).

Note that in general (if $\mathcal{X}$ is not locally connected) the decomposition is not unique. But we can use the fact that f is expanding to choose canonical decompositions for sets U of small diameter as follows.

Since $\mathcal{X}$ is compact, there exists a finite cover $\mathcal{U}$ of $\mathcal{X}$ by open evenly covered sets. Then, by Lebesgue's lemma, there exists $\delta_0 > 0$ such that for every set B of diameter less than δ_0 there exists $U \in \mathcal{U}$ such that $B \subset U$. It follows that every set of diameter less than δ_0 is evenly covered.

Consider decompositions of $f^{-1}(U)$, for $U \in \mathcal{U}$, into disjoint unions $U = U_1 \cup \cdots \cup U_m$ such that $f : U_i \longrightarrow U$ are homeomorphisms, and consider the corresponding inverse maps $f^{-1} : U \longrightarrow U_i$. By the continuity of the maps $f^{-1} : U \longrightarrow U_i$, there exists $\delta < \delta_0$ such that for every set A of diameter less than δ the set $f^{-1}(A)$ can be decomposed into a disjoint union of sets $A_1 \cup \cdots \cup A_m$ of sets of diameter less than ϵ. Then the diameters of A_i will be less than $L^{-1}\delta$. Note that then the distance between any two different points of $f^{-1}(x)$ for $x \in \mathcal{X}$ is not less than ϵ. Consequently, for any $x_1 \in A_i$ and $x_2 \in A_j$ for $i \neq j$ we have $d(x_1, x_2) > \epsilon - 2L^{-1}\delta$. If δ is small enough, then $\epsilon - 2L^{-1}\delta > \delta$, and we get the following.

Lemma 1.4.37. *If $\delta > 0$ is small enough, then for every set $A \subset \mathcal{X}$ of diameter less than δ the set $f^{-1}(A)$ is decomposed in a unique way into a* *disjoint union $f^{-1}(A) = A_1 \cup \cdots \cup A_m$ such that $f : A_i \longrightarrow A$ are homeomorphisms, the sets A_i have diameters less than δ, and distance between any two points belonging to different sets A_i is greater than δ.*

Definition 1.4.38. We say that δ is a *strong injectivity constant* of the expanding covering $f \circlearrowright \mathcal{X}$ if it satisfies the conditions of Lemma 1.4.37.

We will call the sets A_i the *components* of $f^{-1}(A)$. For $n > 1$, the components of $f^{-n}(A)$ are defined inductively as components of $f^{-1}(A_i)$, where A_i is a component of $f^{-(n-1)}(A)$. Note that since components of $f^{-1}(A)$ are of diameter less than $L^{-1}\delta < \delta$, we have a unique decomposition of $f^{-n}(A)$ into components. If A is connected, then components of $f^{-n}(A)$ are its connected components.

Fix some strong injectivity constant $\delta > 0$. Let $U \subset \mathcal{X}$ be a set of diameter less than δ. Consider the rooted tree T_U with the set of vertices equal to the disjoint union of the sets of components of $f^{-n}(U)$ for $n \geqslant 0$, where a component A of $f^{-n}(U)$ is connected to the component $f(A)$ of $f^{-(n-1)}(U)$. The root is $f^{-0}(U) = \{U\}$.

Similarly, for every $x \in \mathcal{X}$, denote by T_x the tree $T_{\{x\}}$ with the set of vertices equal to the disjoint union of the sets $f^{-n}(x)$ for $n \geqslant 0$. For every $x \in U$ the trees T_x and T_U are naturally isomorphic: the isomorphism maps a vertex $t \in f^{-n}(x)$ of T_x to the unique component of $f^{-n}(U)$ containing t.

The *boundary* ∂T_U of the tree T_U is the inverse limit of the sets of components of $f^{-n}(U)$ with respect to the maps induced by f. Similarly, ∂T_x is the inverse limit of the sets $f^{-n}(x)$ with respect to $f : f^{-n}(x) \longrightarrow f^{-(n-1)}(x)$.

For every set $U \subset \mathcal{X}$ of diameter less than δ we get a natural homeomorphism $\phi_U : P^{-1}(U) \subset \hat{\mathcal{X}} \longrightarrow U \times \partial T_U$ defined in the following way. Let $\xi = (t_0, t_1, \ldots) \in \hat{\mathcal{X}}$ be a point of the inverse limit $\hat{\mathcal{X}}$, where $t_0 \in U = U_0$. Let U_n be the component of $f^{-n}(U)$ such that $t_n \in U_n$. Then $\phi_U(\xi) = (t_0, (U_0, U_1, \ldots))$. Recall that $t_0 = P(\xi)$. It is easy to see that this is a homeomorphism.

It also follows from the fact that T_U is naturally isomorphic to T_x for every $x \in U$, that the homeomorphisms ϕ_U agree on the intersections of the sets U, i.e., that we have a fiber bundle.

If $\xi_1 = (t'_0, t'_1, \ldots), \xi_2 = (t''_0, t''_1, \ldots) \in U$ are such that the second coordinates of $\phi_U(\xi_1)$ and $\phi_U(\xi_2)$ are equal, then t'_i and t''_i belong for every i to the same component of $f^{-n}(U)$. This implies that $d(t'_i, t''_i) \to 0$ as $i \to \infty$, hence ξ_1 and ξ_2 are unstably equivalent.

Suppose that $\xi_1 = (t'_0, t'_1, \ldots)$ and $\xi_2 = (t''_0, t''_1, \ldots)$ are such that $P(\xi_1) = P(\xi_2) \in U$. Consider the points $\hat{f}^n(\xi_1)$ and $\hat{f}^n(\xi_2)$. They are equal to

$$(f^n(t_0), f^{n-1}(t_0), \ldots, f(t_0), t_0, t'_1, t'_2, \ldots)$$

and

$$(f^n(t_0), f^{n-1}(t_0), \ldots, f(t_0), t_0, t'_1, t'_2, \ldots),$$

respectively, where $t_0 = t'_0 = t''_0$. We see that the distance between $\hat{f}^n(\xi_1)$ and $\hat{f}^n(\xi_2)$ in the inverse limit $\hat{\mathcal{X}}$ goes to zero, i.e., that ξ_1 and ξ_2 are stably equivalent.

We have shown that the local product structure defined by the homeomorphisms ϕ_U agrees with the stable and unstable equivalence classes, which finishes the proof of the theorem. □

Example 1.4.39. Let $f \circlearrowleft \mathcal{X}$ be an expanding endomorphism of a Riemannian manifold. Then its natural extension $\hat{f} \circlearrowleft \hat{\mathcal{X}}$ can be realized as an attractor of a diffeomorphism, see [**Sma67**, p. 788]. We have seen an example of such a realization in the case of the angle doubling map and the solenoid in 1.1.4. The general case is very similar to the solenoid example.

1.4.10. Williams solenoids. The natural extension of a map $f \circlearrowleft \mathcal{X}$ is a Ruelle-Smale system not only in the case of expanding coverings. As a starting point of a more general setting, let us consider the following class of examples.

Let $\sigma : \mathsf{X} \longrightarrow \mathsf{X}^*$ be a substitution such that the length of $\sigma^n(x)$ goes to infinity for all $x \in \mathsf{X}$. Let B_σ be the corresponding stationary Vershik-Bratteli diagram, see 1.3.7. Suppose that B_σ is properly ordered, see Proposition 1.3.31. An example of such a substitution is

$$\sigma(0) = \mathsf{01}, \qquad \sigma(1) = \mathsf{011}, \tag{1.9}$$

The associated Vershik-Bratteli diagram is shown on Figure 1.17.

Let $\mathcal{X}$ be a bouquet of $|\mathsf{X}|$ oriented loops labeled by the letters of X. Consider the map $f_\sigma \circlearrowleft \mathcal{X}$ realizing σ: it maps every loop labeled by x to the path in $\mathcal{X}$ on which the word $\sigma(x)$ is read. We parametrize each loop by $[0, 1]$, so that the common point of the loops is parametrized by 0 and 1, and assume that the word $\sigma(x)$ is read in the positive (increasing) direction.

See, for instance Figure 1.28 where the corresponding map f_σ for the substitution (1.9) is shown.

By choosing appropriate lengths of the loops in $\mathcal{X}$, we can make f_σ expanding on each loop. The map f_σ will be expanding the length of paths, but it is not expanding on any neighborhood of the common point of the loops, since it is not injective there.

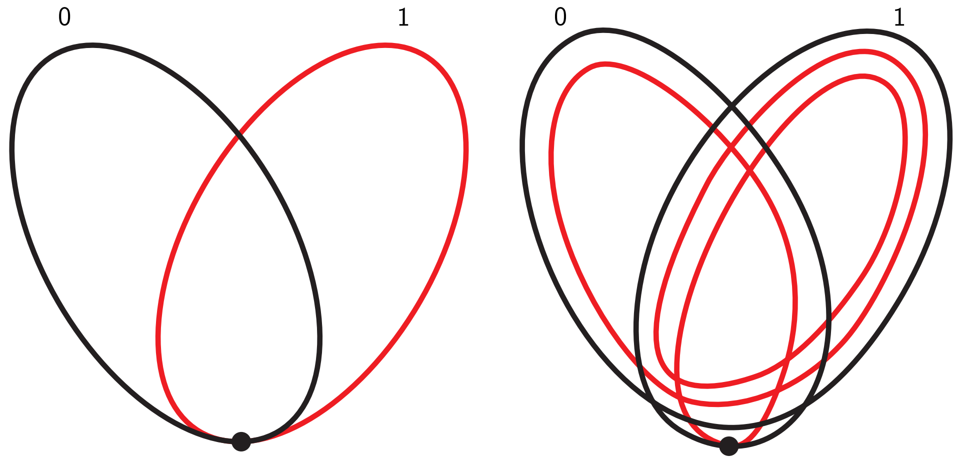

Figure 1.28. Geometric realization of a substitution

Note that for every positive n the map $f_\sigma^n \circlearrowleft \mathcal{X}$ is the realization of σ^n: the loop labeled by x is mapped to the path on which $\sigma^n(x)$ is read.

Proposition 1.4.40. *The natural extension $\hat{f}_\sigma \circlearrowleft \hat{\mathcal{X}}$ is a Ruelle-Smale system. The space $\hat{\mathcal{X}}$ is homeomorphic to the mapping torus of the adic transformation defined by the Vershik-Bratteli diagram B_σ.*

Proof. Let us denote by γ_x the loop of $\mathcal{X}$ labeled by $x \in \mathsf{X}$. Consider a point $(t_1, t_2, \ldots) \in \hat{\mathcal{X}}$. Suppose that t_1 belongs to the interior of γ_x (i.e., is not equal to the common point of the loops) for some $x \in \mathsf{X}$. Let $a_n \in \mathsf{X}$ be such that t_n belongs to γ_{a_n}. In particular, $x = a_1$. Then f_σ maps γ_{a_n} to a path containing x_{n-1}. We get an occurence of the letter a_{n-1} in the word $\sigma(a_n)$ corresponding to the segment of γ_{a_n} mapped by f_σ to $\gamma_{a_{n-1}}$ and containing t_{n-1}. Let e_{n-1} be the edge of the Vershik-Bratteli diagram B_σ corresponding to this occurence. It is an edge connecting a_n to a_{n-1} in B_σ. We get a path $(e_1, e_2, \ldots)$ in B_σ. It is clear that the point $(t_1, t_2, \ldots) \in \hat{\mathcal{X}}$ is uniquely determined by t_1 and $(e_1, e_2, \ldots)$. We will denote the point $(t_1, t_2, \ldots)$ by $(t; e_1, e_2, \ldots)$, where $t \in (0, 1)$ is the parameter corresponding to the point t_1. We see that the subset of $\hat{\mathcal{X}}$ consisting of points $(t_1, t_2, \ldots)$ such that t_1 different from the common point of the loops γ_x is naturally identified with the product of the open unit interval $(0, 1)$ with the space of paths in B_σ. It is easy to show that this identification is a homeomorphism.

Denote by $(0; e_1, e_2, \ldots)$ and $(1; e_1, e_2, \ldots)$ the limits of $(t; e_1, e_2, \ldots)$ as $t \to 0$ and $t \to 1$, respectively. It follows from the construction that $(1; e_1, e_2, \ldots)$ is equal to $(0; f_1, f_2, \ldots)$, where $(f_1, f_2, \ldots)$ is the image of $(e_1, e_2, \ldots)$ under the adic transformation, provided $(e_1, e_2, \ldots)$ is not maximal. If $(e_1, e_2, \ldots)$ is maximal, then the point $(1; e_1, e_2, \ldots)$ is equal to

$(p, p, \ldots) \in \hat{\mathcal{X}}$ where p is the common point of the loops of $\mathcal{X}$. The same is true for $(0; e_1, e_2, \ldots)$ if $(e_1, e_2, \ldots)$ is minimal.

We see that the space $\hat{\mathcal{X}}$ can be obtained from the product of $[0, 1]$ with the space of paths $\mathcal{P}(\mathsf{B}_\sigma)$ by identifying $(1; e_1, e_2, \ldots)$ with $(0; f_1, f_2, \ldots)$, where $(f_1, f_2, \ldots)$ is the image of $(e_1, e_2, \ldots)$ by the adic transformation. It is not hard to check that $\hat{f}_\sigma$ expands the distances in the direction of the unit interval and contracts them in the direction of $\mathcal{P}(\mathsf{B}_\sigma)$. $\square$

The construction from Proposition 1.4.40 and its generalizations were studied by R. F. Williams in [**Wil67**] and [**Wil74**].

The main feature of this approaches is that even if the map $f \circlearrowright \mathcal{X}$ is not an expanding covering in the sense of Definition 1.4.1, it is "eventually" an expanding covering map.

R. F. Williams, for example, uses *branched manifolds*, i.e., spaces equal to unions of closed pieces of $\mathbb{R}^n$ pasted together in such a way that every point still has one well defined tangent space. We will not give a precise definition, but note that the above example of a rose of loops in Proposition 1.4.40 is a branced manifold (you should imagine it as a union of circles tangent to each other at the common point). Then the expansion condition can be defined in the same way as for manifolds (see Example 1.4.5). The covering condition is replaced by a "flattening" condition: for every $x \in \mathcal{X}$ there exists $k \geqslant 1$ and a neighborhood N of x such that $f^k(N)$ is contained in a subset of $\mathcal{X}$ diffeomorphic to an open ball of $\mathbb{R}^n$. Note that this condition is satisfied for the example shown on Figure 1.28 with $k = 1$: the image of a neighborhood of the singular point (the common point of the two circles) under the map is a smooth interval equal to the union of one black and one red half-intervals; it is trivially true for the interior points of the loops.

The following more combinatorial version of Williams' conditions (in the one-dimensional case) are given in [**Yi01**].

Definition 1.4.41. Let Γ be a graph seen as a topological space (a one-dimensional CW-complex) with a metric d compatible with the topology. Consider the following conditions.

(1) **Expansion:** there exist constants $C > 0$ and $L > 1$ such that if x, y are points on an edge of Γ, and f^n, for $n \geqslant 1$ maps the interval $[x, y] \subset \Gamma$ to an edge of Γ, then $d(f^n(x), f^n(y)) \geqslant CL^n d(x, y)$.

(2) **Markov:** The set V of vertices of Γ is forward-invariant: $f(V) \subset V$.

(3) **Nonfolding:** The map $f^n : \Gamma \backslash V \longrightarrow \Gamma \backslash V$ is locally one-to-one for every $n \geqslant 1$.

(4) **Flattening:** There exists $k \geqslant 1$ such that for every $x \in \Gamma$ there exists an open neighborhood N of x such that $f^k(N)$ is homeomorphic to an open interval.

It is proved in [**Yi01**] that if $f \circlearrowleft \Gamma$ satisfies the conditions of Definition 1.4.41, then the natural extension $\hat{f} \circlearrowleft \hat{\Gamma}$ is a Ruelle-Smale system. (The paper includes additional irreducibility and non-wandering conditions.)

More general conditions are given in S. Wieler's paper [**Wie14**]. We present here a modified version.

For an entourage U and $x \in \mathcal{X}$, we denote by $B(U, x)$ the "ball of radius U" with center in x:

$$B(U, x) = \{y \in \mathcal{X} \ : \ (x, y) \in U\}.$$

Theorem 1.4.42. *Let $f \circlearrowleft \mathcal{X}$ be a map, where $\mathcal{X}$ is compact and Hausdorff. Suppose that the following conditions hold:*

(1) ***Eventual expansion:*** *There exists an entourage U and a number $k \geqslant 1$ such that if $(f^n(x), f^n(y)) \in U$ for all $n \geqslant 0$, then $f^k(x) = f^k(y)$.*
(2) ***Eventually open map:*** *For every entourage U there exists an entourage $V \subset U$ and $k \geqslant 1$ such that for every $x \in \mathcal{X}$ we have $f^k(B(f^k(x), V)) \subset f^{2k}(B(x, V))$.*

Then the natural extension $\hat{f} \circlearrowleft \hat{\mathcal{X}}$ is a Ruelle-Smale system.

Proof. Let us prove at first that the first condition implies that the natural extension is expansive. The proof essentially repeats the proof of Proposition 1.4.35. Consider the entourage $\hat{U} \subset \hat{\mathcal{X}} \times \hat{\mathcal{X}}$ consisting of all pairs $((x_1, x_2, \ldots), (y_1, y_2, \ldots))$ such that $(x_1, y_1) \in U$. Suppose that $\xi = (x_1, x_2, \ldots)$ and $\zeta = (y_1, y_2, \ldots)$ are such that $(\hat{f}^n(\xi), \hat{f}^n(\zeta)) \in \hat{U}$ for all $n \in \mathbb{Z}$. Then we have $(f^n(x_i), f^n(y_i)) \in U$ for all $n \geqslant 0$ and $i \geqslant 1$. It follows from eventual expansion of f that $f^k(x_i) = f^k(y_i)$, hence $x_{i-k} = y_{i-k}$ for all $i \geqslant k$. The latter implies that $\xi = \zeta$, i.e., that $\hat{f}$ is expansive.

Let us prove now that the map $[\cdot, \cdot]$ from Definition 1.4.23 is defined on a neighborhood of the diagonal of $\hat{\mathcal{X}} \times \hat{\mathcal{X}}$, if the map $f \circlearrowleft \mathcal{X}$ satisfies both conditions of the theorem. Let U be an entourage satisfying the first condition of the theorem, and let V be an entourage satisfying the second condition and such that $V \subset U$ and $f(V) \subset U$. We can replace f by f^k, so it is enough to prove the statement for the case $k = 1$.

Let $\xi = (x_1, x_2, \ldots)$ and $\zeta = (y_1, y_2, \ldots)$ be points of $\hat{\mathcal{X}}$ such that $(x_i, y_i) \in V$ for $i = 1$ and $i = 2$.

Let us construct by induction a sequence y'_i such that $(x_i, y'_i) \in V$ and $f^2(y'_{i+1}) = f(y'_i)$ for all $i \geqslant 1$. Set $y'_1 = y_1$ and $y'_2 = y_2$. Then since $y'_i \in B(x_i, V) = B(f(x_{i+1}), V)$, we have $f(y'_i) \in f(B(f(x_{i+1}), V)) \subset f^2(B(x_{i+1}, V))$. It follows that we can choose $y'_{i+1} \in B(x_{i+1}, V)$ such that $f^2(y'_{i+1}) = f(y'_i)$. Denote $y''_i = f(y'_{i+1})$. Then we have $f(y''_{i+1}) = f^2(y'_{i+2}) = f(y'_{i+1}) = y''_i$, $y''_1 = f(y'_2) = f(y_2) = y_1$, and $(x_i, y''_i) = (f(x_{i+1}, f(y'_{i+1})) \in f(V) \subset U$ for all $i \geqslant 1$.

We have found a point $\zeta'' = (y''_1, y''_2, \ldots) \in \hat{\mathcal{X}}$ such that $y''_1 = y_1$ and $(x_i, y''_i) \in U$ for all i. We have $[\xi, \zeta] = \zeta''$ (check the order....), hence $[\cdot, \cdot]$ is defined on a neighborhood of the diagonal, i.e., $\hat{f} \circlearrowleft \hat{\mathcal{X}}$ is a Ruelle-Smale system. □

S. Wieler proved in [**Wie14**] that for every Ruelle-Smale system $g \circlearrowleft \mathcal{S}$ with totally disconnected stable direction there exists a map $f \circlearrowleft \mathcal{X}$ on a compact metric space $\mathcal{X}$ satisfying the conditions of Theorem 1.4.42 such that $\hat{f} \circlearrowleft \hat{\mathcal{X}}$ is conjugate to $g \circlearrowleft \mathcal{S}$.

Example 1.4.43. Consider a map $f \circlearrowleft \mathcal{X}$ defined on a domain in $\mathbb{R}^2$ as it is schematically shown on the left-hand side of Figure 1.29. We can choose f in such a way that it is contracting along an f-invariant foliation consisting of vertical lines in the orange rectangle and radial lines in the semi-annular shapes. We also may assume that iterations of f are expanding in a transversal foliation, so that the intersection of the ranges of f^n is a hyperbolic set in the sense of 1.4.8.3. This set is locally maximal, and is called the *Plykin attractor* [**Ply74**].

If we collapse the domain of f along the leaves of the stable foliation, i.e., collapse the orange rectangle to a horizontal segment, and the semianular regions to loops, we will get a graph shown on the top part of the right-hand side of Figure 1.29. Since the foliation is f-invariant, f induces a well-defined self-map of the graph, as it is shown schematically on the lower part of the right-hand side of the figure. This map satisfies the conditions of Definition 1.4.41, and its natural extension is topologically conjugate to the Plykin attractor.

Example 1.4.44. DA attractor...

1.4.11. Symbolic encoding and shadowing. Let $H \curvearrowright \mathcal{X}_1$ and $H \curvearrowright \mathcal{X}_2$ be topological dynamical systems, where H is a semigroup. Recall that a *semiconjugacy* from the first system to the second one is a continuous map $\phi : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ such that

$$\phi(h(x)) = h(\phi(x))$$

for all $x \in \mathcal{X}_1$ and $h \in H$. If there exists a surjective semiconjugacy $\phi : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$, then $H \curvearrowright \mathcal{X}_2$ is called a *factor* of $H \curvearrowright \mathcal{X}_1$.

The *kernel* of the semiconjugacy is the set

$$\mathcal{E}_\phi = \{(x, y) \in \mathcal{X}_1^2 \ : \ \phi(x) = \phi(y)\}.$$

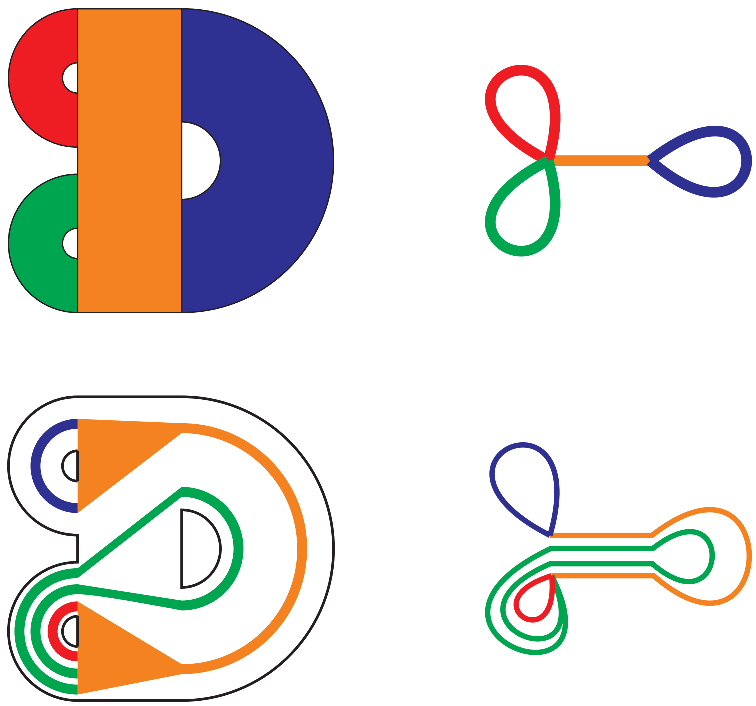

Figure 1.29. Plykin attractor and its one-dimensional model

It is a subset of $\mathcal{X}_1^2$ invariant under the diagonal action of H: if $(x, y) \in \mathcal{E}_\phi$, then $(h(x), h(y)) \in \mathcal{E}_\phi$. We consider the kernel as the topological dynamical system $H \curvearrowright \mathcal{E}_\phi$.

Definition 1.4.45. We say that a system $H \curvearrowright \mathcal{X}$ is *finitely presented* if there exists a subshift of finite type $H \curvearrowright \mathcal{S}$, $\mathcal{S} \subset \mathsf{X}^H$, and a surjective semiconjugacy $\phi : \mathcal{S} \longrightarrow \mathcal{X}$ such that the kernel $H \curvearrowright \mathcal{E}_\phi$ is also a shift of finite type. Here we naturally identify $\mathsf{X}^H \times \mathsf{X}^H$ with $(\mathsf{X} \times \mathsf{X})^H$, so that $\mathcal{E}_\phi \subset (\mathsf{X} \times \mathsf{X})^H$.

The terminology is attributed to M. Gromov... references...

Proposition 1.4.46. *Let $H \curvearrowright \mathcal{S} \subset \mathsf{X}^H$ be a subshift, and let $\phi : \mathcal{S} \longrightarrow \mathcal{X}$ be a surjective semiconjugacy to a dynamical system $H \curvearrowright \mathcal{X}$. If $H \curvearrowright \mathcal{X}$ is expansive, then the kernel $H \curvearrowright \mathcal{E}_\phi$ is a subshift of relative finite type in $H \curvearrowright \mathcal{S} \times \mathcal{S}$.*

For the notion of relative finite type, see Definition 1.2.13.

Proof. Let $U \subset \mathcal{X} \times \mathcal{X}$ be an expansion entourage. Consider its full preimage $\phi^{-1}(U) \subset \mathcal{S} \times \mathcal{S}$. We have $\mathcal{E}_\phi \subset \phi^{-1}(U)$. Since $\mathcal{E}_\phi$ is compact, and $\mathcal{S} \times \mathcal{S}$ is zero-dimensional, there exists a clopen set $U' \subset \mathcal{S} \times \mathcal{S}$ such that

$\mathcal{E}_\phi \subset U' \subset \phi^{-1}(U)$. Let us prove that $\mathcal{E}_\phi = \bigcap_{h\in H} h^{-1}(U')$. We obviously have $\mathcal{E}_\phi \subset \bigcap_{h\in H} h^{-1}(U')$, since $\mathcal{E}_\phi \subset U'$ and $\mathcal{E}_\phi$ is H-invariant. Suppose that $(w_1, w_2) \in h^{-1}(U')$ for all h. Then $(h(w_1), h(w_2)) \in U' \subset \phi^{-1}(U)$ for all $h \in H$, hence $(h(\phi(w_1)), h(\phi(w_2))) \in U$ for all $h \in H$, which implies $\phi(w_1) = \phi(w_2)$, i.e., $(w_1, w_2) \in \mathcal{E}_\phi$. This proves that $\mathcal{E}_\phi$ is of relative finite type in $\mathcal{S} \times \mathcal{S}$. □

Example 1.4.47. A subshift $\mathcal{F} \subset \mathsf{X}^{\mathbb{Z}}$ is called *sofic* if there exists a shift of finite type $\tilde{\mathcal{F}} \subset \mathsf{Y}^{\mathbb{Z}}$ and a surjective semi-conjugacy $\tilde{\mathcal{F}} \longrightarrow \mathcal{F}$. It follows from Proposition 1.4.46 that every sofic subshift is finitely presented. Consider, for example, the *even subshift* $\mathcal{F} \subset \{0, 1\}^{\mathbb{Z}}$ consisting of all sequences $(x_n)_{n\in\mathbb{Z}}$ such that there is an even number of 1s between any two consecutive 0s. It is nor hard to show that it is not a shift of finite type (exercise ...). On the other hand, if $\tilde{\mathcal{F}}$ is the shift of finite type consisting of all sequences $(x_n)_{n\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ with no subword 11, then the sliding block map defined by $(00) \mapsto 0$, $(01) \mapsto 1$, $(10) \mapsto 1$ maps $\tilde{\mathcal{F}}$ surjectively to $\mathcal{F}$. Exercise... describe $\mathcal{E}_\phi$ in this case...

Example 1.4.48. Let $f \circlearrowleft \mathbb{R}/\mathbb{Z}$ be the angle doubling map $x \mapsto 2x \pmod 1$. We have seen in 1.1.2 that it admits a surjective semiconjugacy $\{0, 1\}^* \longrightarrow \mathbb{R}/\mathbb{Z}$ from the one-sided full shift. The system $f \circlearrowleft \mathbb{R}/\mathbb{Z}$ is expansive, therefore it follows from Proposition 1.4.46 that it is finitely presented. Check that the corresponding kernel is of finite type...

It follows from Proposition 1.4.46 that a system $H \curvearrowright \mathcal{X}$ is finitely presented if and only if it is expansive and is a factor of a shift of finite type. The following approach to factors of shifts of finite type is due to R. Bowen, see...

Definition 1.4.49. Let H be a semigroup generated by a finite set $S \subset H$ (our main examples will be $H = \mathbb{N}$ and $H = \mathbb{Z}$ with $S = \{1\}$), and let $H \curvearrowright \mathcal{X}$ be an action on a metric space. We say that a sequence $h \mapsto x_h : H \longrightarrow \mathcal{X}$ is an *ϵ-pseudo-orbit* if

$$d(s(x_h), x_{sh}) < \epsilon$$

for all $h \in H$ and $s \in S$.

We say that a system $H \curvearrowright \mathcal{X}$ satisfies the *shadowing property* if for every $\delta > 0$ there exists a positive number $\epsilon > 0$ such that for every ϵ-pseudo-orbit x_h there exists a point $y \in \mathcal{X}$ such that $d(x_h, h(y)) < \delta$ for all $h \in H$.

Picture of a pseudo-orbit...

Proposition 1.4.50. *Every Ruelle-Smale system satisfies the orbit shadowing property.*

Proof. Let $x_n, n \in \mathbb{Z}$ be an ϵ-pseudo-orbit for a Ruelle-Smale system $f \circlearrowleft \mathcal{X}$, where ϵ will be selected later. We assume that the metric d on $\mathcal{X}$ belongs to the canonical class of metrics ... Let $\lambda \in (0,1)$ be as in the definition ... Let δ be such that the δ-neighborhood every point of $\mathcal{X}$ is contained in a rectangle. We assume that $\epsilon < \delta$. Choose for every n a rectangle R_n containing the δ-neighborhood of x_n.

Let us choose a metric d_n on R_n equal to the direct product metric of the expanding and the contracting directions:

$$d_n(x, y) = \max\{d([x_n, x], [x_n, y]), d([x, x_n], [y, x_n])\}.$$

This metric is compatible with the topology on R_n (in fact, it is bi-Lipschitz equivalent to d).

Define $x^+_{n,0} = x_n$ and then inductively, for $k \geqslant 0$,

$$x^+_{n,k+1} = [x^+_{n,k}, f(x^+_{n-1,k})].$$

Then all $x^+_{n,k}$ belong to the stable plaque of x_n, and we have

$$d(x^+_{n,k}, x^+_{n,k+1}) \leqslant \lambda d(x^+_{n-1,k-1}, x^+_{n-1,k}),$$

see Figure... It follows that there exists $C > 0$ such that $d(x^+_{n,k}, x^+_{n,k+1}) \leqslant C\delta\lambda^k$ for all k for some C depending only on the metric d. It follows that the sequence $x^+_{n,k}$, $k = 1, 2, \ldots$ converges, provided all $x^+_{n,k}$ are defined. Each point $x^+_{n,k}$ and the limit are on the distance not more than $\frac{C\epsilon}{1-\lambda}$ from x_n. It follows that if ϵ is small enough, the points $x^+_{n,k}$ are defined and converge in R_n. Let x^+_n be its limit. Note that it follows from the definitions that $[x^+_{n+1}, f(x^+_n)] = x^+_{n+1}$.

Changing the direction, we will find a sequence $x^-_{n,k}$ satisfying $x^-_{n,k+1} = [f^{-1}(x^-_{n+1,k}), y^-_{n,k}] = [f^{-1}(x^-_{n+1,k}), x_n]$ and converging to a point x^-_n on distance not more than $\frac{C\epsilon}{1-\lambda}$ from x_n. We will also have $[f^{-1}(x^-_n), x^-_{n-1}] = x^-_{n-1}$.

Then $y_n = [x^+_n, x^-_n]$ is an orbit such that $d(x_n, y_n) \leqslant \frac{2C\epsilon}{1-\lambda}$ for all n. $\square$

Proposition 1.4.51. *Suppose that $H \curvearrowright \mathcal{X}$ is an expansive dynamical system on a compact metric space satisfying the shadowing property. Then $H \curvearrowright \mathcal{X}$ is a factor of a shift of finite type, and hence is finitely presented.*

Corollary 1.4.52. *Every Ruelle-Smale system is finitely presented.*

Proof. Let $\delta > 0$ be a number less than half of the expansivity constant ... Let $\epsilon > 0$ be the corresponding constant from Definition 1.4.49. Let $N \subset \mathcal{X}$ be a finite ϵ-net. Consider the set $\mathcal{S} \subset N^H$ consisting of all ϵ-pseudo-orbits, i.e., all sequences $w : H \longrightarrow N$ such that $d(s(w(h)), w(sh)) < \epsilon$ for every $h \in H$, $s \in S$. It is a topological Markov shift, see... In particular, it is a shift of finite type. For every $w \in \mathcal{S}$, by Definition 1.4.49, there exists $y \in \mathcal{X}$

such that $d(w(h), h(y)) < \delta$. Define $\phi(w) = y$. Note that if y' is another point satisfying this condition, then $d(h(y), h(y')) < 2\delta$ for all $h \in H$, hence $y = y'$, by expansivity.

We get a well defined map $\phi : \mathcal{S} \longrightarrow \mathcal{X}$ such that $\phi(h(w)) = h(\phi(w))$ for all $w \in \mathcal{S}$ and $h \in H$. The map ϕ is surjective, since for every $y \in \mathcal{X}$ and $h \in H$ we can choose $w(h)$ such that $d(w(h), h(y)) < \epsilon$. It is continuous, since if $w_1, w_2 \in \mathcal{S}$ are such that $w_1(h) = w_2(h)$ for all $h \in A$ for a set $A \subset H$, then $d(h(\phi(w_1)), h(\phi(w_2))) < 2\epsilon$ for all $h \in A$. By increasing A, we can get an arbitrarily small upper estimate of $d(\phi(w_1), \phi(w_2))$, by Lemma 1.2.7. It follows that $\phi : \mathcal{S} \longrightarrow \mathcal{X}$ is a surjective semiconjugacy. □

Example 1.4.53. We have seen that the binary solenoid from 1.1.4 has a finite presentation given by the semiconjugacy from the two-sided shift $\{0, 1\}^{\mathbb{Z}}$. The corresponding kernel consists of the diagonal, the pair $(\ldots 111\ldots, \ldots 000\ldots)$, and all the pairs of the form $(\ldots x_{n-1}x_n 0111\ldots, \ldots x_{n-1}x_n 1000\ldots)$. It is easy to check that this set is a subshift of finite type.

Suppose that $\phi : \mathcal{F} \longrightarrow \mathcal{X}$ is a surjective semi-conjugacy from a subshift of finite type $\mathsf{s} \circlearrowright \mathcal{F} \subset \mathsf{X}^{\mathbb{Z}}$ to a Ruelle-Smale system $f \circlearrowright \mathcal{X}$. Consider the cylindrical subsets $C_x = \{(x_n)_{n\in\mathbb{Z}} \in \mathcal{F} \ : \ x_0 = x\}$ for $x \in \mathsf{X}$. If the corresponding subsets $\phi(C_x)$ have disjoint interiors, then we say that $\{\phi(C_x)\}$ is a *Markov partition* of $f \circlearrowright \mathcal{X}$. Note that the subshift $\mathcal{F}$ is uniquely determined by the Markov parition. Namely, for every generic $t \in \mathcal{X}$ the point $f^n(t)$ belongs to the interior of a unique element C_{x_n} of the Markov partition for every $n \in \mathbb{Z}$ (by Bair's Category Theorem). The sequence $(x_n)_{n\in\mathbb{Z}}$ is the *itinerary* of t, and we obviously have $\phi((x_n)_{n\in\mathbb{Z}}) = t$. Then $\mathcal{F}$ is equal to the closure of the set of such itineraries. The subshift $\mathcal{F}$ can be seen as a result of "cutting" $\mathcal{X}$ along the boundaries of the elements of the Markov partition, and then propagating the cuts by the dynamics, similarly to what we did with an irrational rotation in 1.3.1.2 (compare it with the definition of $\mathcal{X}_\theta$ in Proposition 1.2.37).

It was proven by R. Bowen that every Ruelle-Smale system has a Markov partition, see [**Bow70**] (he proved it for hyperbolic sets of diffeomorphisms, but the proof of the general statement is the same).

The class of finitely presented actions of $\mathbb{Z}$ is wider than the class of Ruelle-Smale systems. The following theorem of D. Fried [**Fri87**] clarifies the relation between these two classes.

Theorem 1.4.54. *An expansive system $f \circlearrowright \mathcal{X}$ is finitely presented if and only if $\mathcal{X}$ can be covered by a finite number of closed rectangles.*

Here a *rectangle* is a subset $R \subset \mathcal{X}$ such that the operation $[\cdot, \cdot]$ given in Definition 1.4.23 is defined and continuous on $R \times R$ and takes values in R.

Example 1.4.55. Consider the Arnold's Cat map $f \circlearrowleft \mathbb{R}^2/\mathbb{Z}^2$ defined by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Note that $f(-x) = -x$, so f induces a well defined homeomorphism of the space obtained from the torus by identifying every element $x \in \mathbb{R}^2/\mathbb{Z}^2$ with $-x$. This space is homeomorphic to the sphere, and can be visualized as a result of folding the rectangle $[0, 1/2] \times [-1/2, 1/2]$ into a square pillow, see Figure..., where the foliation by the stable and unstable equivalence classes (manifolds) are shown... Show the partition into a finite number of rectangles... This is an example of a *pseudo-Anosov diffeomorphism*, see...

Example 1.4.56. Consider the even subshift $\mathcal{F}$ from Example 1.4.47. It is the set of all sequences $(x_n)_{n\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ such that there is an even number of 1s between any two consecutive 0s. Let R_0 and R_1 be the sets of sequences $(x_n)_{n\in\mathbb{Z}}$ such that the number of leading 1s of $x_0x_1x_2\ldots$ is even and odd, respectively (where infinity is considered to be even and odd). If $w_1 = (x_n)_{n\in\mathbb{Z}}, w_2 = (y_n)_{n\in\mathbb{Z}} \in R_i$, then the sequence $[w_1, w_2] = \ldots y_{-2}y_{-1} \,.\, x_0x_1x_2\ldots$ belongs to $R_i \subset \mathcal{F}$. It follows that $\mathcal{F}$ is a union of two closed rectangles R_0 and R_1. Their intersection is the set of all sequences $(x_n)_{n\in\mathbb{Z}} \in \mathcal{F}$ such that $x_0x_1x_2\ldots = 111\ldots$.

1.4.12. Structural stability. Corollary 1.4.52 implies that Ruelle-Smale systems can be described, up to topological conjugacy, by a finite amount of information: by a shift of finite type $\mathcal{F}$ and a shift of finite type $\mathcal{E} \subset \mathcal{F} \times \mathcal{F}$. A shift of finite type is, by definition, described by a finite number of prohibited subwords. In particular, there only countably many Ruelle-Smale systems, up to topological conjugacy. The same is true for expanding self-coverings (write more above)...

In other words, hyperbolic dynamical systems are essentially combinatorial objects. One of aspects of the combinatorial nature of hyperbolic dynamical systems is their *rigidity*, or *structural stability*. It can be formulated, for example in the following way.

Theorem 1.4.57. *Let $f \circlearrowleft \mathcal{X}$ be a Ruelle-Smale system, and let d be a metric on $\mathcal{X}$. Then there exists $\epsilon > 0$ such that if $f' \circlearrowleft \mathcal{X}$ is another Ruelle-Smale system such that $d(f(x), f'(x)) < \epsilon$ for all $x \in \mathcal{X}$, then $f \circlearrowleft \mathcal{X}$ and $f' \circlearrowleft \mathcal{X}$ are topologically conjugate.*

Proof. (Sketch.) If $d(f(x), f'(x)) < \epsilon$ for all $x \in \mathcal{X}$, then the f-orbit $(f^n(x))_{n\in\mathbb{Z}}$ is an ϵ-pseudo-orbit for f', since we have $d(f'(f^n(x)), f^{n+1}(x)) = d(f'(f^n(x)), f(f^n(x))) < \epsilon$. By Proposition 1.4.50, for every $\delta > 0$ we can find $\epsilon > 0$ such that if f and f' satisfy the condition of the theorem for ϵ, then for every $x \in \mathcal{X}$ there exists $x' \in \mathcal{X}$ such that $d((f')^n(x'), f^n(x)) < \delta$ for all $n \in \mathbb{Z}$. If δ is smaller than the expansivity constant for f', the point x' is unique. Consider the map $\phi : x \mapsto x'$. It follows from the definition that ϕ is a semiconjugacy from $f \circlearrowleft \mathcal{X}$ to $f' \circlearrowleft \mathcal{X}$. On the other hand, x and x' play the same role in the definition, so it follows from the uniqueness that ϕ is a bijection conjugating the systems. It remains to show that ϕ is continuous. This can be deduced from the proof of Proposition 1.4.50, where the orbit shadowing a pseudo-orbit was constructed as $y_n = [x_n^+, x_n^-]$, where x_n^+ and x_n^- were constructed as limits of sequences defined using the map and the operation $[\cdot, \cdot]$. Using uniform continuity of f, f', and $[\cdot, \cdot]$, one can show that $\phi(x)$ depends continuously on x. □

Structural stability suggests that one should be able to describe and study hyperbolic dynamical systems using some algebraic or combinatorial techniques. Finite presentations and symbolic dynamics is in some sense such technique, though it is not easy to work with it, and to deduce topological properties of a dynamical system from its symbolic presentation. We will see later in Chapter 4 that there exists an algebraic approach to expanding covering maps, which have computationally efficient encoding by a *self-similar group* (or a *biset*). A similar encoding of Ruelle-Smale systems is still missing...

1.5. Holomorphic dynamics

Here we present a very short collection of classical introductory results in holomorphic dynamics, which will be used later. For a more detailed exposition, see the books [**Mil06, Bea91**]...

1.5.1. Preliminaries from complex analysis.

Theorem 1.5.1 (Uniformization Theorem)**.** *Any simply connected Riemann surface (i.e., a one dimensional smooth complex manifold) is conformally isomorphic to exactly one of the following surfaces.*

(1) *The Riemann sphere $\widehat{\mathbb{C}}$.*
(2) *The (Euclidean) plane $\mathbb{C}$.*
(3) *the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$, or, equivalently, the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \ : \ \Im z > 0\}$.*

Theorem 1.5.2 (Schwarz Lemma)**.** *If $f \circlearrowleft \mathbb{D}$ is holomorphic and $f(0) = 0$, then $|f'(0)| \leqslant 1$. If $|f'(0)| = 1$, then f is a rotation $z \mapsto cz$ about 0 (for $|c| = 1$). If $|f'(z)| < 1$, then $|f(z)| < |z|$ for all $z \neq 0$.*

As a corollary we get

Theorem 1.5.3 (Liouville Theorem)**.** *If $f \circlearrowleft \mathbb{C}$ is holomorphic and bounded, then it is constant.*

The automorphism groups (i.e., groups of bi-holomorphic automorphisms) of the simply connected Riemannian surfaces are as follows:

(1) $Aut(\widehat{\mathbb{C}})$ is the group of all *Möbius transformations* $z \mapsto \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ such that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. It is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$.

(2) $Aut(\mathbb{C})$ is the group of all affine transformations $z \mapsto az + b$ for $a, b \in \mathbb{C}$, $a \neq 0$.

(3) $Aut(\mathbb{H})$ is the group of all transformations $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ are such that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$. It is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. Every automorphism of $\mathbb{D}$ is of the form $z \mapsto e^{i\theta} \frac{z-a}{1-\bar{a}z}$, where $\theta \in \mathbb{R}$, $|a| < 1$.

If S is a connected Riemann surface, then its universal covering $\widetilde{S}$ is one of the simply connected surfaces $\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{D}$, and the fundamental group $\pi_1(S)$ acts on $\widetilde{S}$ by conformal automorphisms. We say that S is *Euclidean* or *hyperbolic*, if $\widetilde{S}$ is isomorphic to $\mathbb{C}$ or $\mathbb{D}$, respectively.

Note that the action of $\pi_1(S)$ on $\widetilde{S}$ is fixed point free. Since every non-identical Möbius transformation has a fixed point, the only surface with universal covering $\widehat{\mathbb{C}}$ is the sphere $\widehat{\mathbb{C}}$ itself.

Any transformation $z \mapsto az + b$ for $a \neq 1$ has a fixed point, hence in the Euclidean case the fundamental group acts on the universal covering $\mathbb{C}$ by translations. It is easy to see that this implies that a Euclidean surface is isomorphic either to the cylinder $\mathbb{C}/\mathbb{Z}$, or to a torus $\mathbb{C}/\Lambda$, where Λ is the subgroup of the additive group of $\mathbb{C}$ generated by two non-zero complex numbers a, b such that $a/b \notin \mathbb{R}$. All the other Riemann surfaces are hyperbolic.

It is a direct corollary of the Liouville theorem that every holomorphic map from $\mathbb{C}$ to a hyperbolic surface is constant (since we can lift it to the universal covering). In particular, every holomorphic map $f \circlearrowleft \mathbb{C}$ such that $\mathbb{C} \setminus f(\mathbb{C})$ has more than one point is constant (*Picard's Theorem*).

Theorem 1.5.4 (Poincaré metric)**.** *There exists a unique Riemannian metric (up to multiplication by a constant) on* $\mathbb{D}$ *invariant under every conformal automorphism of* $\mathbb{D}$*. It is given by* $ds = \frac{2|dz|}{1-|z|^2}$*. Every orientation preserving isometry of* $\mathbb{D}$ *is a conformal automorphism.*

Every hyperbolic surface S has then a unique Poincaré metric coming from the Poincaré metric on the universal covering $\widetilde{S} \cong \mathbb{D}$ of S (since the fundamental group $\pi_1(S)$ acts on $\widetilde{S}$ by conformal automorphisms).

The following is a corollary of Schwarz Lemma.

Theorem 1.5.5 (Pick Theorem)**.** *Let $f : S \longrightarrow S'$ be a holomorphic map between hyperbolic surfaces. Then exactly one of the following cases is taking place.*

(1) *f is a conformal isomorphism and an isometry with respect to the Poincaré metrics.*
(2) *f is a covering map and is a local isometry.*
(3) *f is strictly contracting, i.e., for every compact set $K \subset S$ there is a constant $c_K < 1$ such that $d(f(x), f(y)) \leqslant c_k d(x, y)$ for all $x, y \in K$.*

1.5.2. Fatou and Julia sets.

Definition 1.5.6 (Compact-open topology)**.** Let $\mathcal{X}$ be a locally compact space, and let $\mathcal{Y}$ an arbitrary topological space. *Compact open topology* on the space $Map(\mathcal{X}, \mathcal{Y})$ of continuous maps $\mathcal{X} \longrightarrow \mathcal{Y}$ is given by the basis of neighborhoods of a map $f : \mathcal{X} \longrightarrow \mathcal{Y}$ consisting of sets

$$N_{K,\epsilon}(f) = \{g \in Map(\mathcal{X}, \mathcal{Y}) \ : \ d(f(x), g(x)) < \epsilon \text{ for all } x \in K\}$$

where $K \subset \mathcal{X}$ is compact and $\epsilon > 0$.

In fact, the compact-open topology does not depend on the metric on $\mathcal{Y}$. Convergence in the compact-open topology is called the *uniform convergence on compact subsets.*

Definition 1.5.7. A set $\mathcal{F}$ of holomorphic functions from a Riemann surface S to a compact Rieman surface T is called a *normal family* if its closure is compact in $Map(S, T)$. In the case when T is not compact, we replace T by its one-point compactification.

Thus, a family $\mathcal{F} \subset Hol(S, T)$ is normal if every sequence f_n of elements of $\mathcal{F}$ has either a subsequence f_{n_k} convergent uniformly on compact subsets, or a subsequence f_{n_k} converging to infinity uniformly on compact subsets (i.e., such that for all compact $K_1 \subset S$ and $K_2 \subset S$ the intersection $f_{n_k}(K_1) \cap K_2$ is empty for all k big enough).

Definition 1.5.8. Let $f \circlearrowleft \widehat{\mathbb{C}}$ be a rational function. The *Fatou set* of f is the set of points $z \in \widehat{\mathbb{C}}$ such that there exists a neighborhood U of z such that $f^{\circ n} : U \longrightarrow \widehat{\mathbb{C}}$, for $n \geqslant 0$, is normal. The complement of the *Fatou set* is called the *Julia set*.

Example 1.5.9. Consider the function $f(z) = z^n$ for $n \geqslant 2$. If $|z_0| < 1$, then $f^n(z)$ uniformly converges on a neighborhood of z_0 to the constant 0 function, hence z_0 belongs to the Fatou set. If $|z_0| > 1$, then $f^n(z)$ uniformly converges to ∞ on a neighborhood of z_0, hence z_0 also belongs to the Fatou set in this case. On the other hand, if $|z_0| = 1$, then for any neighborhood U

of z_0 and any sequence $n_k \to \infty$ the sequence $f^{n_k} : U \longrightarrow \widehat{\mathbb{C}}$ does not have a continuous limit. Consequently, the Julia set of z^n is the unit circle.

It easily follows from the definitions that the Julia set J is totally invariant, i.e., $f(J) = J = f^{-1}(J)$. The Julia sets of f and f^n coincide. It is always non-empty (unless f is a Möbius transformation) and compact. On the other hand, the Fatou set can be empty.

Another possible definition of the Julia set is given by the following theorem (see, for instance [**Mil06**, Theorem 14.1].

Theorem 1.5.10. *Let $f(z)$ be a rational function of degree > 1. Then the Julia set of f is equal to the closure of the union of its repelling cycles.*

Here a cycle $z_0 = f(z_{n-1}), z_1 = f(z_0), z_2 = f(z_1), \ldots, z_{n-1} = f(z_{n-2})$ is called *repelling* if the *multiplier* $\left.\frac{df^n(z)}{dz}\right|_{z=z_i} = f'(z_0)f'(z_1)\cdots f'(z_{n-1})$ is greater than one in absolute value. It is called *attracting* if the multiplier is less than one in absolute value.

A *Fatou component* of a rational function $f(z)$ is a connected component of the Fatou set of f. If U is a Fatou component, then $f(U)$ is also a Fatou component. By D. Sullivan's *Nonwandering Theorem...* the forward orbit of every Fatou component is finite, i.e., eventually belongs to a cycle. It follows that every Fatou component is a branched covering of a Fatou component fixed under some iteration of f. The fixed Fatou components are classified in the following way.

Theorem 1.5.11. *Let U be a Fatou of f such that $f(U) = U$. Then one of the following four cases takes place.*

(1) *U is the immediate basin of attraction of an attracting fixed point.*
(2) *U is one petal of a parabolic fixed point of multiplier* 1.
(3) *U is a Siegel disc.*
(4) *U is a Herman ring.*

Here the immediate basin of attraction of a fixed point z_0 is the connected component containing z_0 of the set of points $z \in \widehat{C}$ such that $\lim_{n\to\infty} f^n(z) = z_0$. Similarly, if z_0 is a fixed point such that $f'(z_0) = 1$, then there is a Fatou component, called a *petal* of points whose forward orbits converge to z_0. A *Siegel disc* is an open domain such that the action of f on it is biholomorphically conjugate to an irrational rotation of a disc. Similarly, a *Herman* ring is domain the action of f on which is conjugate to an irrational rotation of an annulus. For more detail, see [**Mil06**], in particular Section 16.

1.5.3. Hyperbolic rational functions.

Definition 1.5.12. A rational function f is *hyperbolic* if it is expanding on a neighborhood of its Julia set.

Hyperbolic rational functions are important examples of expanding dynamical systems in the sense of Definitions 1.4.1 and 1.4.2.

A *post-critical set* of f is the set of all points of the form $f^n(c)$, where c is a critical point of f, and $n \geqslant 1$.

Theorem 1.5.13. *Let f be a rational function of degree $\geqslant 2$. Then the following conditions are equivalent.*

(1) *f is hyperbolic.*
(2) *The closure of the post-critical set of f is disjoint from its Julia set.*
(3) *The orbit of every critical point converges to an attracting cycle.*

Sketch of the proof. It is easy to see that (3) implies (2), since basins of attraction belong to the Fatou set. Let us show only that (2) implies (1) in the case when $\overline{P}$ has more than two points. Then $\mathcal{X} = \widehat{\mathbb{C}} \setminus \overline{P}$ is a hyperbolic surface containing the Julia set. Note that $f(\overline{P}) \subset \overline{P}$, hence $f^{-1}(\mathcal{X}) \subset \mathcal{X}$. The map $f : f^{-1}(\mathcal{X}) \longrightarrow \mathcal{X}$ is a covering. Consider the Poincaré metrics on $\mathcal{X}$ and $f^{-1}(\mathcal{X})$. The map f is a local isometry with respect to these metrics. The inclusion map $Id : f^{-1}(\mathcal{X}) \longrightarrow \mathcal{X}$ is not a covering map, hence it is strictly contracting, see Theorem 1.5.5. It follows that if we consider the restriction of the Poincaré metric of $\mathcal{X}$ onto the subset $f^{-1}(\mathcal{X})$, then the map $f : f^{-1}(\mathcal{X}) \longrightarrow \mathcal{X}$ is expanding. Since the Julia set is compact and contained in $f^{-1}(\mathcal{X})$, the map f will be uniformly expanding on the Julia set.

If closure of the post-critical set has less than three points, then they belong to attracting cycles, and we can take $\mathcal{X}$ equal to $\widehat{\mathbb{C}}$ minus a small neighborhood of $\overline{P}$, and repeat the proof.

Let us show that (1) implies (2). Let W be a neighborhood of the Julia set J such that f is expanding on W. Taking an ϵ-neighborhood of J in W, we get an open neighborhood U of J such that f is expanding on U, and $f^{-1}(U) \subset U$. Then $f^{-n}(U) \subset U$ for all $n \geqslant 1$. The set U does not contain critical points of f, since otherwise f is not one-to-one, hence not expanding on any neighborhood of a critical point. If c is critical, and $f^n(c) \in U$, then $c \in f^{-n}(U) \subset U$, which is a contradiction. Consequently, U does not contain any post-critical points. This implies that intersection of U with the closure of the post-critical set is empty.

The fact that (2) implies (3) follows from classification of components of the Fatou set, see Theorem 1.5.11. □

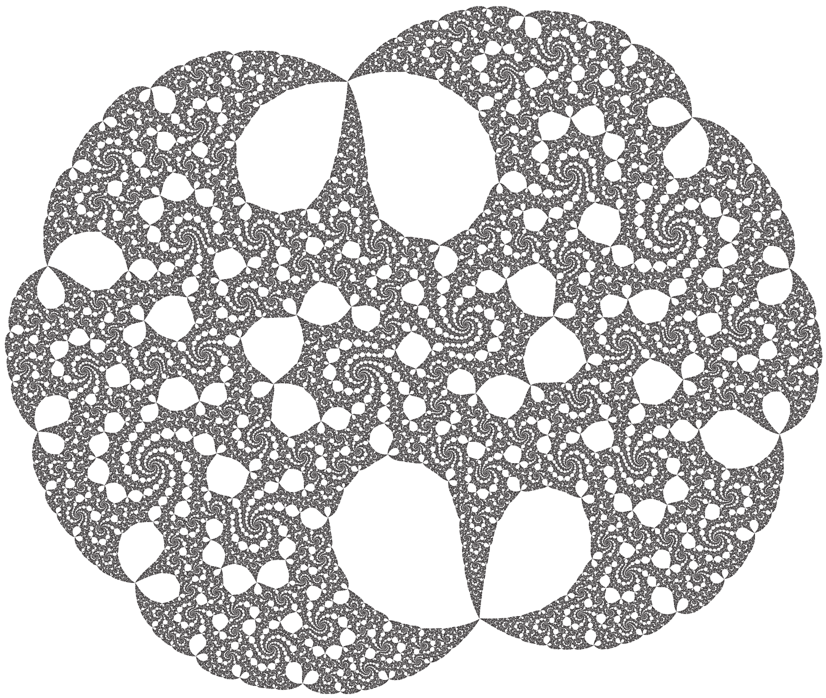

Figure 1.30. Julia set of a hyperbolic rational function

If f is a hypebolic rational function, then all its Fatou components belong to the basins of attraction to attracting cycle. In other words, only the first type in the classification of Fatou components in Theorem 1.5.11 is possible in this case.

See examples of Julia sets of hyperbolic rational functions on Figure 1.30 and Figure 1.31.

1.5.4. Subhyperbolic rational functions. Post-critically finite rational functions in general, orbifolds and orbifold metrics... (define using the universal cover and lengths of paths)...

1.5.5. Quadratic polynomials and the Mandelbrot set. Let $f(z)$ be a complex polynomial. Its critical points are zeros of the derivative and the infinity. The infinity is totally invariant, i.e., $f(\infty) = f^{-1}(\infty) = \infty$ and superattracting. It follows that the basin of attraction of infinity is

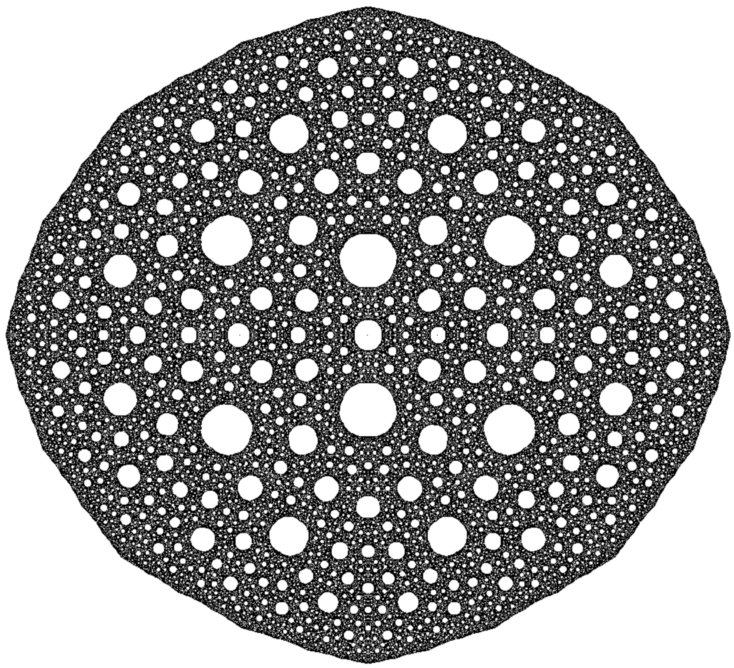

Figure 1.31. A Sierpinski carpet Julia set

connected. Points belonging to it *escape to infinity.* The *filled Julia set* of the polynomial is the complement of the basin of attraction of infinity. In other words, it is the set of points whose forward orbit is bounded. The filled Julia set is compact. The Julia set is its boundary.....

Every quadratic polynomial is conjugate (by an affine transformation) to a polynomial of the form $z^2 + c$. It has only one finite critical point 0.

Theorem 1.5.14. *The Julia set of $z^2 + c$ is connected if and only if the orbit of the critical point* 0 *is bounded. Otherwise it is homeomorphic to the Cantor set.*

Proof. If the orbit of 0 is not bounded, then c belongs to the basin of infinity. Consider a curve γ connecting c to infinity inside the basin of infinity. Then $f^{-1}(\gamma)$ disconnects the complex plane in two connected components, both of which are homeomorphically mapped onto the complement of γ. Since the Julia set of f is totally invariant, and does not intersect γ, it follows that $f^{-1}(\gamma)$ disconnects the Julia set J into two closed subsets J_0 and J_1 such that $f : J_0 \longrightarrow J$ and $f : J_1 \longrightarrow J$ are homeomorphisms....

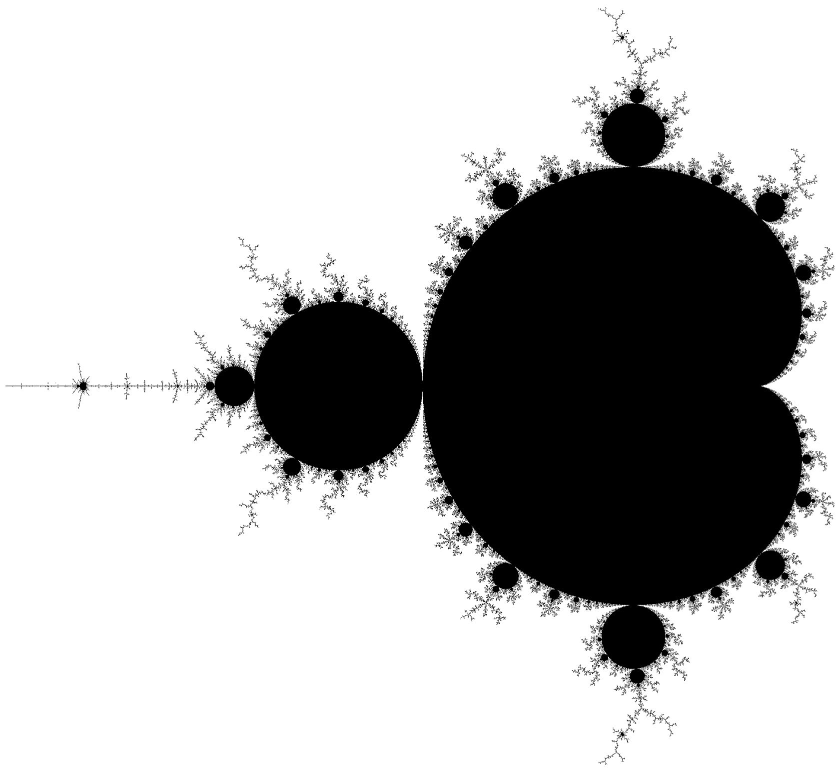

Figure 1.32. The Mandelbrot set

The main ideas of the proof... □

Definition 1.5.15. The *Mandelbrot set* $\mathcal{M}$, see Figure 1.32, is the set of points c such that the Julia set of $z^2 + c$ is connected. In other words, it is the set of points c such that the orbit of 0 under the iterations of $z^2 + c$ is bounded.

It is known, see... that the Mandelbrot set is connected, and that its complement in the complex plane is homeomorphic to the complement of the closed unit disc. External angles... rational angles, hyperbolic components, their parametrization by the multiplier of the attracting cycle, Misiurewicz points, external angles in the dynamical plane...

1.5.6. Lyubich-Minski lamination. The natural extension, leaves, conformal type of the leaves...

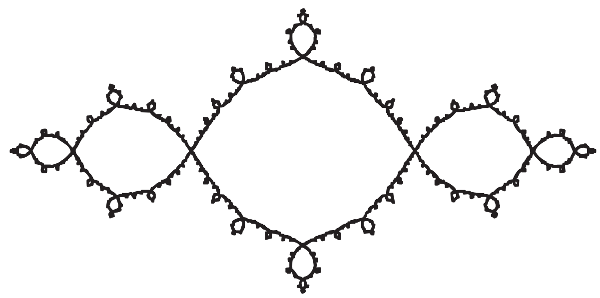

Figure 1.33. Julia set of $z^2 - 1$

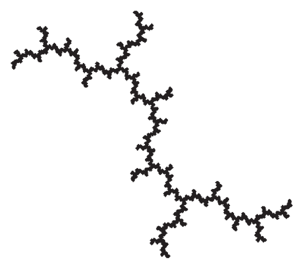

Figure 1.34. Julia set of $z^2 + i$

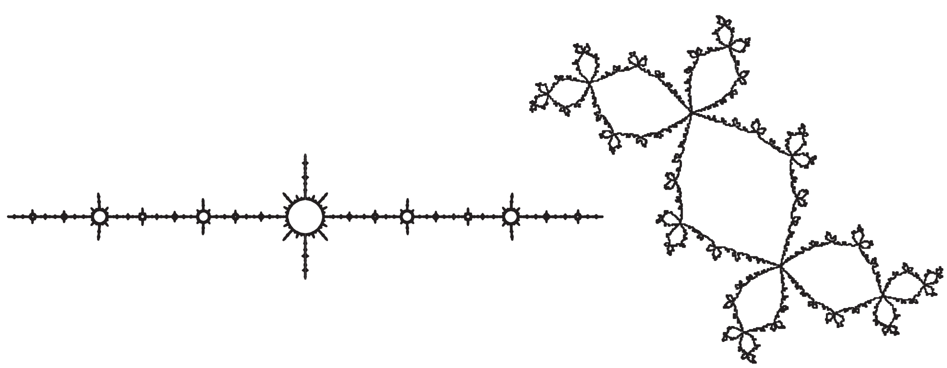

Figure 1.35. "Airplane" and "Rabbit"

Exercises

1.1. Prove that an action of a group G on a space $\mathcal{X}$ is minimal if and only if the space of orbits $G\backslash\mathcal{X}$ has trivial (i.e., antidiscrete) topology.

1.2. Let $(a_n)_{n\in\mathbb{Z}}$ be the sequence defined in 1.1.1. Let d_n be the number of letters d in the sequence $(a_1, a_2, \ldots, a_n)$. Find $\lim_{n\to\infty} \frac{d_n}{n}$.

1.3. Find a number $x \in \mathbb{R}/\mathbb{Z}$ such that the closure of its orbit under the angle doubling map is homeomorphic to the Cantor set.

1.4. Show that for the one-sided shift $\mathsf{s} \circlearrowleft X^\omega$ there is no nonempty proper closed subset $F \subset X^\omega$ such that $\mathsf{s}^{-1}(F) \subset F$.

1.5. Tent map and a semiconjugacy with the one-sided shift... Describe the identifications...

1.6. Prove that the natural extension of the one-sided shift $\mathsf{s} \circlearrowleft X^\omega$ is topologically conjugate to the two-sided shift $\circlearrowleft X^{\mathbb{Z}}$.

1.7. Prove Proposition 1.1.11.

1.8. Prove that a point of the torus is represented by at most three sequences in the encoding of the map $A \circlearrowleft \mathbb{R}^2/\mathbb{Z}^2$ constructed in Subsection 1.1.5 for the Markov partition given in Figure 1.11.

1.9. Prove Proposition 1.2.3.

1.10. Prove that $\{\mathsf{0}, \mathsf{1}\}^{\mathbb{Z}}$ has uncountably many subshifts.

1.11. Prove that for every subshift of finite type $\mathcal{F} \subset \mathsf{X}^{\mathbb{Z}}$ the union of finite orbits is dense.

1.12. Let $\mathsf{s} \circlearrowleft \mathcal{F}$ be a subshift of finite type. Let p_n be the number of points $w \in \mathcal{F}$ such that $\mathsf{s}^n(w) = w$. Prove that the formal power series $\sum_{n=0}^{\infty} p_n x^n$ is a rational function.

1.13. Prove that the intersection of two shift of finite type is a shift of finite type.

1.14. Let $x_0 x_1 x_2 \ldots = \mathsf{01101}\ldots$ be the Thue-Morse word from Example 1.2.21. Prove that x_n is equal to the sum modulo 2 of the digits of the binary expansion of n.

1.15. Let $x_0 x_1 x_2 \ldots$ be, as in the previous problem, the Thue-Morse sequence. Let A and B be the sets of numbers $i = 0, 1, 2, \ldots$ such that $x_i = \mathsf{0}$ and $x_i = \mathsf{1}$, respectively. Prove that for every $k = 1, 2, \ldots,$ the sets $A_k = A \cap \{0, 1, 2, \ldots, 2^{k+1} - 1\}$ and $B_k = B \cap \{0, 1, 2, \ldots 2^{k+1} - 1\}$ satisfy $\sum_{x \in A_k} x^d = \sum_{x \in B_k} x^d$ for all $d = 0, 1, 2, \ldots, k$. (solve...)

1.16. Prove that the Thue-Morse word is *cube-free*, i.e., that it has no subwords of the form vvv for a non-empty $v \in \{0, 1\}^*$.

1.17. Show that the Fibonacci substitutional shift is palindromic...

1.18. Prove that every Sturmian subshift is palindromic...

1.19. Let A be a dense countable subset of $[0, 1]$ disjoint from $\{0, 1\}$. Replace every point $a \in A$ by two copies $a - 0$ and $a + 0$ with the natural order on the obtained set $\mathcal{X}$. Prove that $\mathcal{X}$ is homeomorphic to the Cantor set with respect to the order topology.

1.20. Consider the substitution

$$\sigma : a \mapsto aca, \quad b \mapsto d, \quad c \mapsto b, \quad d \mapsto c$$

from [**Lys85**]. Show that it generates a minimal subshift. Find a primitive substitution generating a conjugate subshift.

1.21. Prove that the subshift from the previous problem is Toeplitz.

1.22. Paper folding sequence... Prove that it is Toeplitz.

1.23. Let $\sigma : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ be a substitution such that for every $x \in \mathsf{X}$ the word $\sigma(x)$ is non-empty (such substitutions are called *non-erasing*). Let β_S be a block code (see ...), and let $\mathcal{F}_\sigma$ be the subshift defined by the substitution σ. Prove that $\beta_S(\mathcal{F}_\sigma)$ is a substitutional subshift.

1.24. Let W be a non-empty set of words. Show that W is the set of all finite subwords of elements of a subshift $\mathcal{F} \subset \mathsf{X}^{\mathbb{Z}}$ if and only if the following conditions are satisfied. (1) Every subword of an element of W belongs to W. (2) For every $v \in W$ there exists a word $v_1 v v_2 \in W$, where $v_1, v_2 \in \mathsf{X}^*$ are non-empty.

Formulate a similar criterion for one-sided subshifts.

1.25. We say that a dynamical system $\mathbb{Z} \curvearrowright \mathcal{X}$ is *essentially minimal* if there exists a unique closed $\mathbb{Z}$-invariant set $\mathcal{Y} \subset \mathcal{X}$ such that $\mathbb{Z} \curvearrowright \mathcal{Y}$ is minimal. Prove that a dynamical system $\mathbb{Z} \curvearrowright \mathcal{X}$, where $\mathcal{X}$ is totally disconnected compact and metrizable, is essentially minimal if and only if there exists a properly ordered Vershik-Bratteli diagram B such that $\mathbb{Z} \curvearrowright \mathcal{X}$ is topologically conjugate to the system generated by the adic transformation on $\mathcal{P}(\mathsf{B})$. In other words, prove Theorem 1.3.20 for essentially minimal systems without the condition that B is simple.

1.26. Let $G \curvearrowright \mathbb{R}/\mathbb{Z}$ be an action of a group on the circle by homeomorphisms. Prove that either $G \curvearrowright \mathbb{R}/\mathbb{Z}$ has a finite orbit, or $G \curvearrowright \mathbb{R}/\mathbb{Z}$ is essentially minimal. Give an example of an action $\mathbb{Z} \curvearrowright \mathbb{R}/\mathbb{Z}$ that has no finite orbits but is not minimal.

1.27. Prove the statement of Example 1.3.9, i.e., construct and explicit isomorphism of the direct limit with the group of continuous maps $\mathcal{P}(\mathsf{B}) \longrightarrow G$.

1.28. Find a properly ordered Vershik-Bratteli diagram realizing the Lysenok subshift from Propblem 20.

1.29. Suppose that B is a properly ordered Vershik-Bratteli diagram such that the sizes of the sets of vertices V_i and the sets of edges E_i are uniformly bounded, and the adic transformation generates an expansive $\mathbb{Z}$-action. Show then that the complexity $p_{\mathcal{F}}(n)$ of the adic transformation (see Definition 1.2.28) is bounded from above by a linear function. (Hint: generalize the proof of Theorem 1.2.34.)

1.30. Prove the converse S. Ferenczi, Rank and symbolic complexity, Ergodic Theory Dyn. Systems 16 (1996) 663–682....

1.31. Complexity of Toeplitz subshifts... (realizability of a particular class of functions)...

1.32. Consider the sets of words $W_0 = \{0^n 1 0^m : n, m \geqslant 0\}$, $W_1 = \{1^n 0 1^m : n, m \geqslant 0\}$, $W_2 = \{0^n 1^m : n, m \geqslant 0\}$ and $W_3 = \{1^n 0^m : n, m \geqslant 0\}$. Prove that if $\mathcal{X}$ is a *countable* subshift of complexity $p_{\mathcal{X}}(n) = n + 1$, then there exists a finite sequence $k_1, k_2, \ldots, k_n$ and $i \in \{0, 1, 2, 3\}$ such that the $W_{\mathcal{X}} = \psi_{k_1+1}\psi_{k_2+1}\cdots\psi_{k_n+1}(W_i \cup \{\varnothing\})$, where ψ_{k_i+1} are as in Theorem 1.3.36. (Such subshifts are called *skew-Sturmian*, see...).

1.33. Show that Toeplitz subshifts can have arbitrarily large repetitivity functions...

1.34. Prove that the critical exponent α_c (see Theorem 1.4.17) is infinite for every one-sided subshift.

1.35. Find the critical exponent of the circle doubling map $x \mapsto 2x \pmod 1$.

1.36. Prove that every map satisfying the conditions of Definition 1.4.41 satisfies the conditions of Theorem 1.4.42.

1.37. Consider the following endomorphism σ of the free group

$$a \mapsto b, \qquad b \mapsto b^{-1}cb, \qquad c \mapsto cac^{-1}.$$

Consider a rose $\mathcal{X}$ of three circles labeled by a, b, c, and a map $f \circlearrowright \mathcal{X}$ realizing the substitution σ similarly to Proposition 1.4.40, i.e., mapping the circle labeled by x to the path $\sigma(x)$ in $\mathcal{X}$. Choose a realization such that iterations of f expand the lengths of paths in $\mathcal{X}$. Show that the natural extension $\hat{f} \circlearrowright \hat{\mathcal{X}}$ is topologically conjugate to the Plykin attractor (see Exampe 1.4.43).

1.38. Find the set of values of $c \in \mathbb{C}$ such that $z^2 + c$ has an attracting fixed point (i.e., a cycle of length 1).

1.39. Find the set of values of $c \in \mathbb{C}$ such that $z^2 + c$ has an attracting cycle of length 2.

1.40. Show that every repelling cycle belongs to the Julia set.

1.41. Consider the Tchebyshev polynomials $T_d(x) = \cos(d \arccos x)$. Describe the Julia sets of T_d for $d \geqslant 1$.

1.42. Let $\mathbb{C}/\mathbb{Z}[i]$ be the torus, and let $A \circlearrowright \mathbb{C}/\mathbb{Z}[i]$ be the map given by $A(z) = (1 + i)z$. Find the Julia set of A. Using the fact that any holomorphic map $f \circlearrowright \mathbb{C}/\Lambda$ on a torus is induced by a linear map on $\mathbb{C}$, describe all possible Julia sets of holomorphic maps on the torus.

1.43. Consider the group G of all maps of the form $z \mapsto (-1)^k z + a + ib$, where $k \in \{0, 1\}$, and $a, b \in \mathbb{Z}$. Show that $\mathbb{C}/G$ is homeomorphic to a sphere. Consider the map $A(z) = (1+i)z$. Show that it induces a well defined map on the sphere $\mathbb{C}/G$. Since the group G and the map A act by holomorphic maps, there is a well defined structure of a complex manifold on $\mathbb{C}/G$, and A induces a holomorphic map on $\mathbb{C}/G$, hence is can be realized by a rational function. What is the Julia set of this rational function?

1.44. Prove that $f_c(z) = 1 + \frac{c}{z^2}$ has a unique cycle of length 2 consisting of the roots of the polynomial $x^2 - cx + c$ (except for the case $c = 4$ when it degenerates to a fixed point). Show that this cycle is attracting if and only if $|c| > 4$.

1.45. Prove that the set of values c such that $z \mapsto 1+\frac{c}{z^2}$ has an attracting fixed point is equal to the image of the open unit disc under the transformation $u \mapsto \frac{4u}{(2-u)^3}$. It is the largest "cardioid" on Figure 6.7.

1.46. Let $c \in (-1, -4/27)$, denote $f_c(x) = 1 + \frac{c}{x^2}$.

a) Prove that for every positive real number x we have $f_c(x) < x$ and that there exists n such that $f_c^{\circ n}(x) \leqslant 0$.

b) Prove that for every $n \geqslant 1$ there exists c_n such that the sequence $a_k = f^{\circ k}(1)$ satisfies $1 > a_1 > a_2 > \ldots > a_n = 0$.

c) Find the limit $\lim_{n\to\infty} c_n$.

1.47. Prove that every quadratic polynomial can be conjugated by an affine map to $z^2 + c$ for some c.

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