Groups and topological dynamics

Volodymyr Nekrashevych

E-mail address: nekrash@math.tamu.edu

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Chapter 2

Group actions

2.1. Structure of orbits

2.1.1. Orbital graphs and defining groups by graphs. Let G be a group acting by permutations on a set X. Suppose that S is a finite generating set of G.

Definition 2.1.1. The *graph of the action* is the graph $\Gamma_{X,S}$ with the set of vertices X, the set of edges $S \times X$, the source and range maps $\mathbf{s}(g, x) = x$, $\mathbf{r}(g, x) = g(x)$, and the labelling $(g, x) \mapsto g$.

The *orbital graph* $\Gamma_{x,S}$ (or Γ_x) for $x \in X$ is the subgraph of $\Gamma_{X,S}$ spanned by the G-orbit of x. In other words, it is the graph with the set of vertices equal to the orbit $G(x)$ in which for every $y \in G(x)$ and $g \in S$ there is an arrow from y to $g(y)$ labeled by g.

For example, if H is a subgroup of G, then we can consider the natural action of G on the set $G/H = \{gH \ : \ g \in G\}$ of left cosets of G modulo H. The corresponding graph of the action is called the Schreier graph of G modulo H.

Conversely, every orbital graph Γ_x is naturally isomorphic to the Schreier graph of G modulo the stabilizer G_x of the point x. The isomorphism maps a coset hG_x to the vertex $h(x)$ of Γ_x.

Describing the action of the generators of a group on a set is identical to describing the corresponding graph of the action. (In fact, the *graph* of a function $g \circlearrowleft X$ is, by definition, the subset $\{(x, g(x)) \ : \ x \in X\}$ of $X \times X$, and it is customary to identify functions with their graphs.)

Let A be a finite set. An edge-labeling of a graph Γ by A is called *perfect* if for every vertex v of Γ and for every $s \in S$ there exists exactly one arrow

e_1 labeled by s starting at v and exactly one arrow e_2 labeled by s ending in v.

Suppose that a graph Γ is perfectly labeled by a finite set S. Let $s \in S$ be a label. For every vertex v of Γ there exists a unique arrow starting at v and labeled by s. Denote the end of this arrow by $s(v)$. Then the map $v \mapsto s(v)$ is a permutation of the set of vertices of Γ, by the definition of a perfect labeling. The set of all permutations $s : v \mapsto s(v)$ generates a group (a subgroup of the symmetric group on the set of vertices of Γ). We call it the *group defined by the labeled graph* Γ.

If $s_1 s_2 \ldots s_n$ is a group word over S, i.e., an element of the free group F_S generated by S, and v is a vertex of Γ, then $s_1 s_2 \ldots s_n(v)$ is obtained by traveling along the arrows of Γ. Namely, first find the arrow starting in v and labeled by s_n or the arrow ending in v and labeled by s_n^{-1}, depending whether $s_n \in S$ or $s_n^{-1} \in S$. The other end of the arrow will be $s_n(v)$. After that find the arrow starting in $s_n(v)$ labeled by s_{n-1} or the arrow ending in $s_n(v)$ labeled by s_{n-1}^{-1}, and so on. At the end you will find a path in Γ *corresponding* to the word $s_1 s_2 \ldots s_n$ starting in v and ending in $s_1 s_2 \ldots s_n(v)$.

The graph Γ is also the graph of an action of the free group F_S generated by the set S. If Γ is connected, then for every vertex v of Γ the graph Γ is naturally the Schreier graph of the free group F_S by the fundamental group $\pi_1(\Gamma, v)$ of the graph, since the fundamental group is precisely the set of elements $g \in F_S$ defining loops at v, i.e., the stabilizer of v for the action defined by Γ. So, describing the graph Γ is equivalent to describing a subgroup of the free group. Note that the group defined by Γ is the quotient of the free group F_S by the intersection of all conjugates of the stabilizer $\pi_1(\Gamma, v)$.

Defining groups by labeled graphs is a surprisingly effective way of constructing groups with special properties. We give here several examples, whose properties will be studied later in more detail.

2.1.1.1. *Linear graphs.* Let X be a finite set. Consider a bi-infinite sequence $w = \ldots A_{-1} A_0 A_1 A_2 \ldots$ of sets $A_n \subset \mathsf{X}$ such that $A_n \cap A_{n+1} = \varnothing$ for every $n \in \mathbb{Z}$. Let Γ_w be the graph with the set of vertices $\mathbb{Z}$ in which for every pair of the form $(n, n+1)$ we have $|A_n|$ edges from n to $n+1$ labeled by every element of A_n. In order to have a perfectly labeled graph, we also add loops: at every vertex n we have loops labeled by all elements of $\mathsf{X} \backslash (A_n \cup A_{n-1})$. Here and later, a non-oriented edge labeled by a symbol a connecting two different vertices represents two oriented edges (one in each direction) both labeled by a. The graph Γ_w defines a group G_w. Note that all generators are of order two.

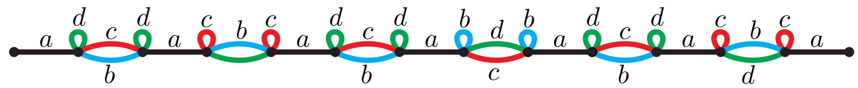

Figure 2.1. Grigorchuk group

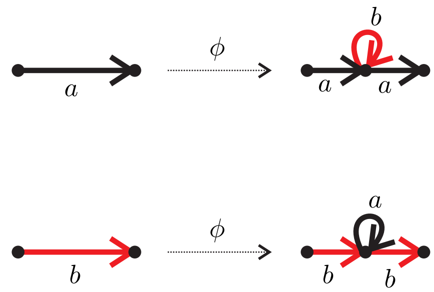

Figure 2.2. Graph substitution

2.1.1.2. *Substitutional subshifts.* The groups from the previous example are defined by a sequence (A_n) of subsets of a finite set X. A natural approach to define such a sequence is by using *substitutions*, see Subsection 1.2.4. For example, consider $\mathsf{X} = \{a, b, c, d\}$, the set $\{A = \{a\}, B = \{c, d\}, C = \{b, d\}, D = \{b, c\}\}$ of subsets of X and the substitution σ given by

$$A \mapsto ADA, \quad B \mapsto D, \quad C \mapsto B, D \mapsto C.$$

Consider the subshift generated by σ (see Exercise 1.20). Recall, that it means considering iterations of σ, and taking all bi-infinite sequences w such that every subword of w is a subword of the word $\sigma^n(A)$ for some $n \geqslant 1$. For example,

$$\sigma^4(A) = DACADABADACADADADACADABADACADA,$$

and the graph shown on Figure 2.1 is a part (corresponding to the word $ADACADABADACA$) of an infinite graph defining a group G_w, as in the previous example.

The group will not depend on the choice of w and is the *first Grigorchuk group* from [**Gri80**]. Figure ... Some other examples of groups that can be defined this way are studied in 5.3.3 and 6.4.

2.1.1.3. *Substitutional graphs.* The previous class of examples can be naturally generalized by using substitutions that not only produce the labeling, but also to produce the graphs. For example, consider the following transformation. If Γ is a graph whose oriented edges are labeled by symbols a and b, denote by $\phi(\Gamma)$ the graph obtained from Γ by subdividing every edge e of Γ into two edges labeled by the same letter as e and adding a loop at the new middle vertex labeled by the other label, see Figure 2.2.

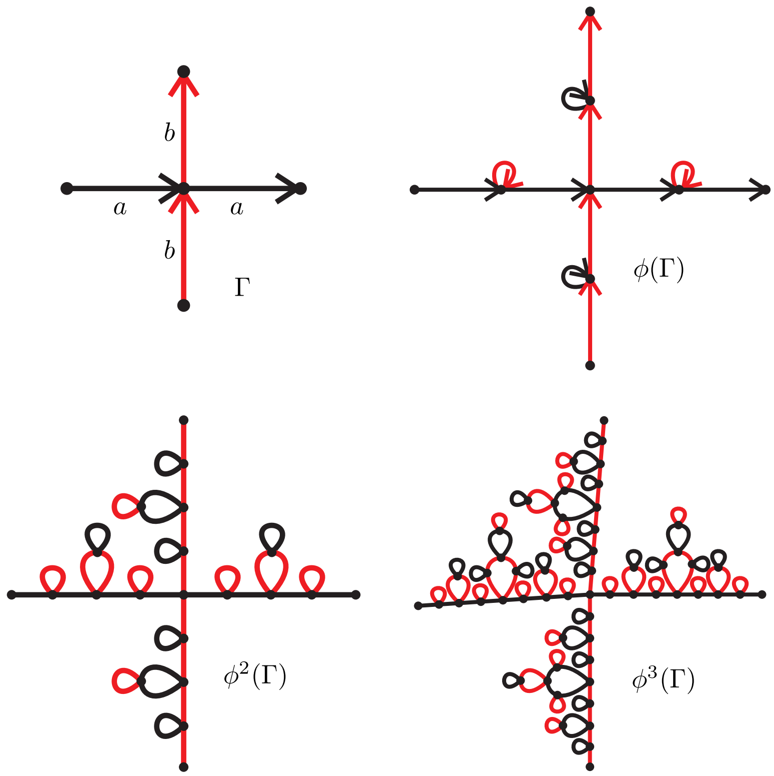

Figure 2.3. IMG $\left((-z^3+3z)/2\right)$

Start with the graph Γ shown on in the upper left corner of Figure **??**. Note that there is an isomorphic embedding of Γ to (the central part of) $\phi(\Gamma)$. It follows that $\phi^n(\Gamma)$ is naturally embedded into $\phi^{n+1}(\Gamma)$, and we can pass to the inductive limit of the graphs $\phi^n(\Gamma)$, in the same way as we did it when generated sequences by substitutions. The inductive limit will be a perfectly labeled graph, and it will define a group generated by two permutations a and b. We will see later that this group is related to the post-critically finite polynomial $\frac{-z^3+3z}{2}$, see 4.1.3.3. In fact, the substitution ϕ and the defined group are closely related to the dynamics of this polynomial.

2.1.1.4. *Houghton's groups.* The following groups were defined in [**Hou79**]. Consider the set $\{0,1,2,\dots,n-1\}\times\mathbb{N}$ and permutations g_i for $i=1,2,\dots,n-1$ acting by $g_i(0,n)=(0,n-1)$, $g(0,1)=(i,1)$, $g_i(i,n)=(i,n+1)$, and $g(j,n)=(j,n)$ for $j\neq i$. Let G_n be the group generated by $g_1,g_2,\dots,g_n$. In other words, consider the graph shown on Figure 2.4 (for $n=4$) and the group defined by it.

Properties...

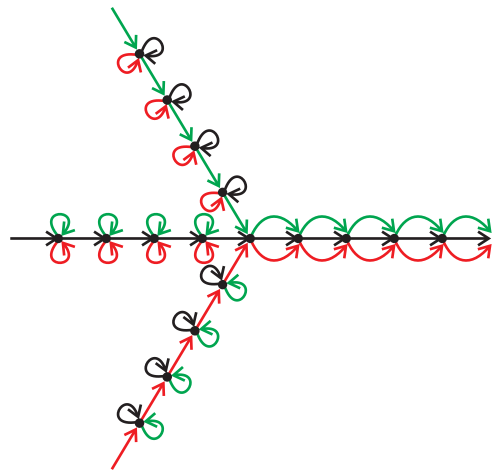

Figure 2.4. Houghton's groups

2.1.1.5. *Long range graphs.* Let $w = x_0x_1\dots$, $x_i \in \{0,1\}$, be an infinite sequence. Denote $w_k = x_0 + 2x_1 + 2^2x_2 + \cdots + 2^kx_k \in \mathbb{Z}$, for $k = 0, 1, \dots$. Let us construct a graph Λ_w with the set of vertices $\mathbb{Z}$ perfectly labeled by the set $S = \{a, b\}$. The arrows labeled by the letter a will start in n and end in $n+1$, so that the corresponding permutation a acts by the shift $n \mapsto n+1$.

The arrows labeled by b will start in $w_k + 2^k(2n+1)$ and end in $w_k + 2^k(2n+3)$ for $k = 0, 1, 2, \dots$, and $n \in \mathbb{Z}$. If the sequence x_i is eventually constant, then, by following the above rule, we will get one vertex not connected to any other vertex by an arrow labeled by b. In this case we add a loop labeled by b to this vertex.

It is not hard to see that the graph Λ_w can be also described in the following way. Construct the edges labeled by a as before, connecting n to $n+1$ in $\mathbb{Z}$. Then connect every other vertex by b-labeled arrows, then among the remaining vertices connect every other vertex, and so on, see Figure 2.5. The choice of vertices that are connected on each stage is done in such way that on the stage number k (starting with $k = 0$) we *do not* connect the vertex w_k. After ω steps we either connect all vertices by b-labeled arrows, or there will remain one vertex. In the latter case we attach to it a loop labeled by b.

The corresponding permutations of the set of vertices $\mathbb{Z}$ of the graph Λ_w are:

$$a : n \mapsto n+1, \qquad b : w_k + 2^k(2n+1) \mapsto w_k + 2^k(2n+3).$$

Let G be the group generated by the permutations a and b. We will see later that it is not defined by a finite set of relations (it is not *finitely presented*),

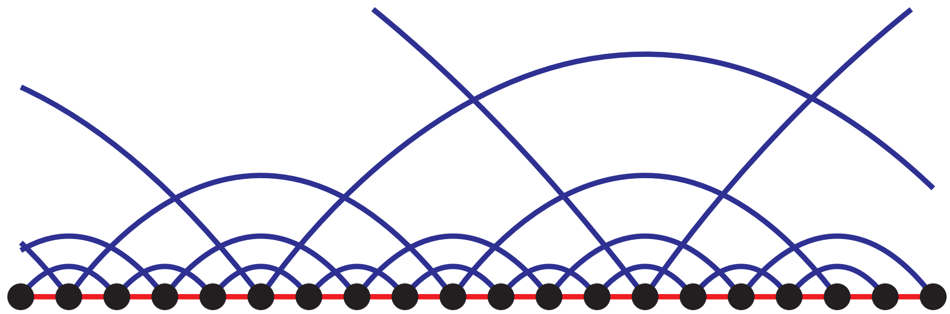

Figure 2.5. Graph Λ_w

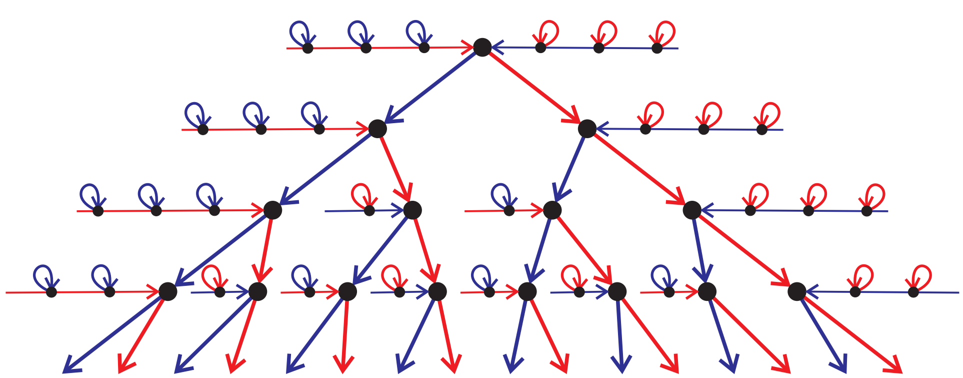

Figure 2.6. Graph defining Thompson group

that it has no non-abelian free subgroups, and that moreover, it is amenable. Not very much more is known about this group.

For some first properties of the graphs Λ_w and the group G, see Exercises... For more, see the papers...

2.1.1.6. *The Thompson group.* Let $S = \{g_0, g_1\}$, and consider the binary rooted tree with the set of vertices equal to the set of finite words over the alphabet S (including the empty word $\varnothing$), where for every $x \in S$ and $v \in S^*$ we connect the vertex v to the vertex vx by an arrow labeled by x. We get a graph in which every vertex has two outgoing arrows (labeled by 0 and 1) and one incoming arrow, unless it is $\varnothing$, when it has no incoming arrows. This is not a perfectly labeled graph. In order to correct this, let us attach to every vertex that is missing an incoming arrow labeled by x an infinite ray of edges labeled by x together with loops at every vertex labeled by the other label $1 - x$, so that we get a perfectly labeled graph. Attach two such rays to the root $\varnothing$, as it is missing both incoming arrows. See Figure 2.6 for the result. We get a perfectly labeled graph defining a group F.

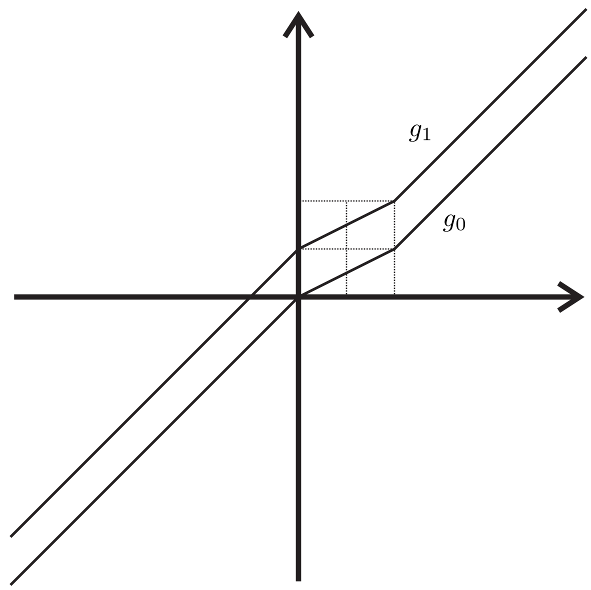

Figure 2.7. Generators of Thompson group

It is isomorphic to the Thompson's group F, which is defined as the group of piecewise affine homeomorphisms $f \circlearrowleft [0, 1]$ of the unit interval such that f is differentiable everywhere except for a finite set of points $B_f \subset \mathbb{Z}\left[\frac{1}{2}\right]$ and such that $f'(x)$ is an integer power of 2 for every $x \in [0, 1]$ where $f'(x)$ exists. It was introduced by R. Thompson in [**Tho80**], see also an expository paper [**CFP96**]. The orbital graph shown on Figure **??** was described by D. Savchuk in [**Sav15**]. It is the orbital graph of the action of F on $\mathbb{Z}\left[\frac{1}{2}\right] \cap (0, 1)$. Note that our graph is given for the generating set $x_1, x_1x_0^{-1}$ (in the left action notation), where x_0, x_1 are the generators for the graph in [**Sav15**]. This choice of the generators producing the graph 2.6 was suggested by K. Juschenko. It is natural to represent them as acting on the real line by the maps

$$g_0(t) = \begin{cases} t & t \in (-\infty, 0], \\ \frac{t}{2} & t \in [0, 2], \\ t-1 & t \in [2, +\infty), \end{cases} \qquad g_1(t) = \begin{cases} t+1 & t \in (-\infty, 0], \\ \frac{t}{2}+1 & t \in [0, 2], \\ t & t \in [2, +\infty], \end{cases}$$

see Exercise 2.8

The graphs of the functions g_0 and g_1 are shown on Figure 2.7.

2.1.2. Local containment and covering. Let Γ be a graph perfectly labeled by a set S. If $s_1s_2 \ldots s_n$ is an element of the free group F_S of length n, then the vertex $s_1s_2 \ldots s_n(v)$ depends only on the ball of radius n around v in the graph Γ, since this ball will contain the path corresponding to

$s_1s_2\dots s_n$ and starting in v. (As usual we measure distances in Γ ignoring the orientation of the edges.)

Definition 2.1.2. Let Γ_1, Γ_2 be labeled graphs. We say that Γ_1 is *locally contained* in Γ_2 (and denote it $\Gamma_1 \sqsubset \Gamma_2$) if every finite subgraph of Γ_1 can be isomorphically embedded into Γ_2 (as a labeled oriented graph). We say that Γ_1 and Γ_2 are *locally isomorphic* if $\Gamma_1 \sqsubset \Gamma_2$ and $\Gamma_2 \sqsubset \Gamma_1$.

Suppose that Γ_1 and Γ_2 are perfectly labeled by S, and let G_1 and G_2 be the groups they define. An element $s_1s_2\dots s_n$ of the free group F_S is non-trivial in the group defined by Γ_i if and only if there exists a vertex v of Γ_i such that $s_1s_2\dots s_n(v) \neq v$, i.e., if there exists a path corresponding to the word $s_1s_2\dots s_n$ which is not a loop. It follows that if $\Gamma_1 \sqsubset \Gamma_2$, then every element of F_S which is non-trivial in G_1 will be also non-trivial in Γ_2. It follows that the identity map $F_S \longrightarrow F_S$ induces an epimorphism $G_2 \longrightarrow G_1$. It also follows that if Γ_1 and Γ_2 are locally isomorphic, then the identity map on F_S induces an isomorphism of G_1 with G_2. In particular, locally isomorphic graphs define isomorphic groups.

Example 2.1.3. Let Γ_{w_1} and Γ_{w_2} be graphs from Example 2.1.1.1 defined by some bi-infinite sequences w_1 and w_2. We have $\Gamma_{w_1} \sqsubset \Gamma_{w_2}$ if and only if every finite subword of w_1 is a subword of w_2, i.e., if and only if the subshift generated by w_1 is contained in the subshift generated by w_2. In particular, if $\mathcal{F}$ is a minimal subshift, and $w_1, w_2 \in \mathcal{F}$, then the groups G_{w_1} and G_{w_2} defined by Γ_{w_1} and Γ_{w_2} are naturally isomorphic. For example, this proves that the Grigorchuk group as defined in 2.1.1.2 does not depend on the sequence w, as the subshift generated by σ is minimal, see Exercise 1.20.

Example 2.1.4. Let T be a tree such that every vertex of T has exactly one incoming arrow labeled by g_0 or g_1 and two outgoing arrows labeled g_0 and g_1. Add to T infinite rays, in the same way as in **??**, so that we get a perfectly labeled graph Γ_T. Then Γ_T is locally contained in the graph Γ from **??** and its local isomorphism class does not depend on T, see Exercise 2.10.

Note that every morphism $f : \Gamma_1 \longrightarrow \Gamma_2$ of perfectly labeled graphs is a covering (i.e., is bijective on the sets of outgoing and incoming edges at every vertex). The image of every path corresponding to a word $s_1s_2\dots s_n \in F_S$ under f is also a path corresponding to the same word, and if the former path was a loop, then so is the latter. It follows that if G_1 and G_2 are groups defined by Γ_1 and Γ_2, then we have a natural epimorphism $G_1 \longrightarrow G_2$ (induced by the identity map on F_S). See Exercise 2.11 for an example of application of this fact.

2.1.3. Orbital graphs on topological spaces. Let G be a group acting by homeomorphisms on a topological space $\mathcal{X}$. Suppose that S is a finite generating set of G. Then the *graph of the action* can be seen as a *topological graph*. Namely, its set of vertices $\mathcal{X}$ and the set of arrows $S \times \mathcal{X}$ are topological spaces, while the source and the range maps $\mathbf{s}(g, x) = x$, $\mathbf{r}(g, x) = g(x)$, and labeling $(g, x) \mapsto g$ are continuous.

We may consider the *topological realization* of the graph of the action, i.e., consider the space obtained by taking the quotient of the space $[0, 1] \times S \times \mathcal{X}$ by the identifications $(0, g, x) \sim (1, h, y)$ for all $g, h \in S$ and $x, y \in \mathcal{X}$ such that $g(x) = y$. We also may consider it as an *abstract graph*, i.e., ignore the topology on $\mathcal{X}$ (and $S \times \mathcal{X}$).

The abstract connected components (i.e., the connected components of the abstract graph of the action) coincide with the path-components of the topological action graph and are the orbital graphs of the action, see Definition 2.1.1.

Denote by $G_{(x)}$ the *neighborhood stabilizer* of the point $x \in \mathcal{X}$, i.e., the set of all elements $g \in G$ such that the interior of the set of fixed points of g contains x. In other words, $g \in G$ belongs to $G_{(x)}$ if there exists a neighborhood N of x such that g fixes every point of N. It is obvious that $G_{(x)}$ is a normal subgroup of G_x.

Definition 2.1.5. The Schreier graph of G modulo $G_{(x)}$ is called the *graph of germs* of x, and is denoted $\widetilde{\Gamma}_x$.

The vertices of $\widetilde{\Gamma}_x$ can be identified with *germs* of the action of G.

Definition 2.1.6. A *germ* is the equivalence class of a pair $(g, x) \in G \times \mathcal{X}$, where two pairs $(g_1, x_1), (g_2, x_2)$ are equivalent (define the same germ) if $x_1 = x_2$ and there exists a neighborhood N of x_1 such that $g_1|_N = g_2|_N$.

For a germ (g, x), we denote by $\mathbf{s}(g, x) = x$ and $\mathbf{r}(g, x) = g(x)$ its *source* and *range*. If $g_1(x_1) = x_2$, then the product

$$(g_2, x_2)(g_1, x_1) = (g_2 g_1, x_1)$$

is well defined. The operation of taking inverse $(g, x) = (g^{-1}, g(x))$ is also well defined, and these two operations define a structure of a *groupoid* on the set of all germs of $G \curvearrowright \mathcal{X}$. *Groupoids of germs* will be main examples of groupoids studied in Chapter 3 and will be an important tool for defining and studying groups in Chapters 5 and 6.

The set of vertices of $\widetilde{\Gamma}_x$ is naturally identified (via the bijection $gG_{(x)} \longrightarrow (g, x)$) with the set of germs of $G \curvearrowright \mathcal{X}$ with the source equal to x. For every generator $s \in S$ and every vertex (g, x) we have an arrow from (g, x) to (sg, x) labeled by s.

Since $G_{(x)}$ is a normal subgroup of G_x, the map $hG_{(x)} \mapsto hG_x$ is a well defined Galois covering of graphs $\widetilde{\Gamma}_x \longrightarrow \Gamma_x$ with the group of deck transformations isomorphic to $G_x/G_{(x)}$.

Definition 2.1.7. A point $x \in \mathcal{X}$ is said to be *G-regular* (or *regular*, for short) if $G_x = G_{(x)}$. Otherwise it is said to be *G-singular*.

If a point $x \in \mathcal{X}$ is G-regular, then the graphs Γ_x and $\widetilde{\Gamma}_x$ coincide, i.e., the natural covering map is an isomorphism.

Proposition 2.1.8. *The set of G-regular points is G-invariant.*

Proof. We obviously have $gG_xg^{-1} = G_{g(x)}$ and $gG_{(x)}g^{-1} = G_{(g(x))}$, hence $G_x = G_{(x)}$ is equivalent to $G_{g(x)} = G_{(g(x))}$. □

Proposition 2.1.9. *Suppose that $\mathcal{X}$ is a Baire space (e.g., locally compact Hausdorff or completely metrizable). If G is at most countable (e.g., is finitely generated), then the set of G-regular points is co-meager.*

Proof. A point $x \in \mathcal{X}$ is regular if and only if for every $g \in G$ either $g(x) \neq x$ or x belongs to the interior of the set of fixed points of g. It follows that the set of singular points is equal to the union of the boundaries of the sets of fixed points of the elements of G. But the boundary of the set of fixed points of an element $g \in G$ is a closed set with empty interior. It follows that the set of singular points is a countable union of closed nowhere dense sets, i.e., is meager. □

2.1.4. Space of rooted labeled graphs. Let $\mathcal{S}_A$ be the set of all isomorphism classes of connected *rooted* perfectly A-labeled graphs. Let $(\Gamma_1, v_1), (\Gamma_2, v_2) \in \mathcal{S}_A$, and let R be the supremum of the radii r such that the balls $(B_{v_1}(r), v_1)$ and $(B_{v_2}(r), v_2)$ of radius r with centers in the roots of the graphs are isomorphic as rooted labeled graphs. Define then the *distance* $d((\Gamma_1, v_1), (\Gamma_2, v_2))$ as 2^{-R}. It is easy to see that this is an ultrametric on $\mathcal{S}_A$.

The metric introduces a natural topology on $\mathcal{S}_A$. Two rooted labeled graphs are close to each other in this topology if big neighborhoods around their roots are isomorphic.

Proposition 2.1.10. *The space $\mathcal{S}_A$ is compact and* 0*-dimensional.*

Proof. Denote by $\mathcal{B}(R)$ the set of all isomorphism classes of balls $(B_v(R), v)$ of elements $(\Gamma, v) \in \mathcal{S}_A$. The sets $\mathcal{B}(R)$ are finite, and we have natural maps $\mathcal{B}(R+1) \longrightarrow \mathcal{B}(R)$ mapping a ball $(B_v(R+1), v)$ to the ball $(B_v(R), v)$ in the same graph (Γ, v). We claim that $\mathcal{S}_A$ is homeomorphic to the inverse limit of the sets $\mathcal{B}(R)$ with respect to these maps. This will prove both statements of the proposition, since inverse limit of finite discrete sets is a compact 0-dimensional space.

For a given graph $(\Gamma, v) \in \mathcal{S}_A$ the sequence of balls $(B_v(1), v), (B_v(2), v), \ldots$ is a point of the inverse limit. This defines a map ι from $\mathcal{S}_A$ to the inverse limit of the sets $\mathcal{B}(R)$. It follows directly from the definitions that this map is continuous and injective.

On the other hand, for every sequence $(B_1, B_2, \ldots) \in \mathcal{B}(1) \times \mathcal{B}(2) \times \ldots$ representing an element of the inverse limit we have a sequence of root-preserving embeddings $B_1 \hookrightarrow B_2 \hookrightarrow \cdots$. The direct limit of these embeddings (i.e., the increasing union of the balls B_R) is an element of $\mathcal{S}_A$. This defines the map inverse to ι. It is also easy to see that this map is continuous. $\square$

Let $G \curvearrowright \mathcal{X}$ be an action of a group on a topological space, and let S be a finite generating set of G. For every $x \in \mathcal{X}$ we have the rooted orbital graph (Γ_x, x), hence we get a map $x \mapsto (\Gamma_x, x)$ from $\mathcal{X}$ to $\mathcal{S}_S$.

The following proposition appears in [**Vor12**].

Proposition 2.1.11. *The map $x \mapsto \Gamma_x : \mathcal{X} \longrightarrow \mathcal{S}_S$ is continuous at x if and only if x is G-regular.*

Proof. Let R be a positive integer. The ball $B_x(R) \subset \Gamma_x$ of radius R is described by a system of equalities and inequalities of the form $g_1(x) = g_2(x)$ or $g_1(x) \neq g_2(x)$ for all pairs $g_1, g_2 \in G$ of products of length at most R of elements of $S \cup S^{-1}$. If x is a G-regular point, then every such an equality or inequality holds on a neighborhood of x. It follows that there exists a neighborhood N of x such that $B_x(R)$ is isomorphic to $B_y(R)$ for all $y \in N$. But this precisely means that the map $x \mapsto (\Gamma_x, x)$ is continuous at x. .... $\square$

Example 2.1.12. Consider the action of the infinite dihedral group D_∞ on $\mathbb{R}$ generated by the transformations

$$a : x \mapsto -x, \qquad b : x \mapsto 1 - x.$$

The group consists of transformations of the form $x \mapsto \pm x + n$, for $n \in \mathbb{Z}$. The transformations of the form $x \mapsto x + n$ are fixed point free (for $n \neq 0$). The transformation $x \mapsto -x + n$ has a unique fixed point $x = n/2$. It follows that the points $\mathbb{R} \setminus \mathbb{Z}/2\mathbb{Z}$ are regular, while the points of $\mathbb{Z}/2\mathbb{Z}$ are singular.

The orbital graph of a regular point $x \in \mathbb{R}$ is a bi-infinite chain of edges alternatively labeled by a and b, see the top part of Figure 2.8. The orbital graphs of singular points are shown on the two lower parts of Figure 2.8. We see that the map $x \mapsto (\Gamma_x, x)$ is constant on the set of regular points, but is discontinuous at singular points.

Note, however, that in the previous example the graphs $\widetilde{\Gamma}_x$ are pairwise isomorphic, so that the map $x \mapsto (\widetilde{\Gamma}_x, x)$ is constant and hence continuous.

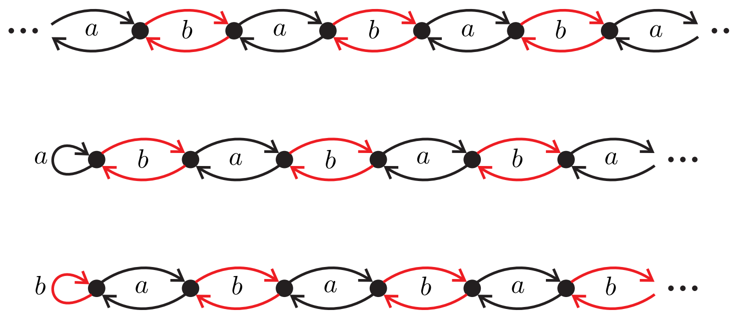

Figure 2.8. Orbital graphs of D_∞

Definition 2.1.13. We say that a point $x \in \mathcal{X}$ has a *Hausdorff group of germs* for the action $G \curvearrowright \mathcal{X}$ if for every $g \in G_x \setminus G_{(x)}$ the interior of the set of fixed points of g does not accumulate on x.

In particular, every regular point x has Hausdorff group of germs, since we have then $G_x \setminus G_{(x)} = \emptyset$.

Proposition 2.1.14. *If a point $x \in \mathcal{X}$ has a Hausdorff group of germs, then the map $x \mapsto (\widetilde{\Gamma}_x, x) : \mathcal{X} \longrightarrow \mathcal{S}_S$ is continuous at x.*

Proof. If the group of germs of x is Hausdorff, then for every $g \in G$ there exists a neighborhood N of x such that either $g|_N$ is identical, or all germs (g, x) for $x \in N$ are non-trivial (i.e., not equal to the germs of the identical map). Then the same argument as in the proof of Proposition 2.1.11 shows that the map $x \mapsto (\widetilde{\Gamma}_x, x)$ is continuous at points with Hausdorff groups of germs. □

Example 2.1.15. Consider the space $\mathcal{Y}$ obtained from $[0, +\infty) \times \{1, 2, 3\}$ by identifying the points $(0, 1), (0, 2)$, and $(0, 3)$, and consider the action of the symmetric group $\mathsf{S}(\{1, 2, 3\})$ acting on the second coordinate of the direct product. The only singular point is the common point $y = (0, 1) = (0, 2) = (0, 3)$ of the three rays. Let us take the generating set $S = \{(1, 2), (2, 3)\}$ of the symmetric group. Then the graphs $\Gamma_x = \widetilde{\Gamma}_x$ for regular points x are all isomorphic to each other, and are chains of three vertices connected by pairs of arrows, see Figure 2.9. The orbital graph Γ_y of the singular point consists of a single vertex, while the graph of germs $\widetilde{\Gamma}_y$ of the singular point is the Cayley graph of the symmetric group.

2.1.5. Topological transitivity and minimality.

Definition 2.1.16. A group action $G \curvearrowright \mathcal{X}$ is said to be *topologically transitive* if for any two non-empty open sets $U, V \subset \mathcal{X}$ there exists $g \in G$ such that $g(U) \cap V = \emptyset$.

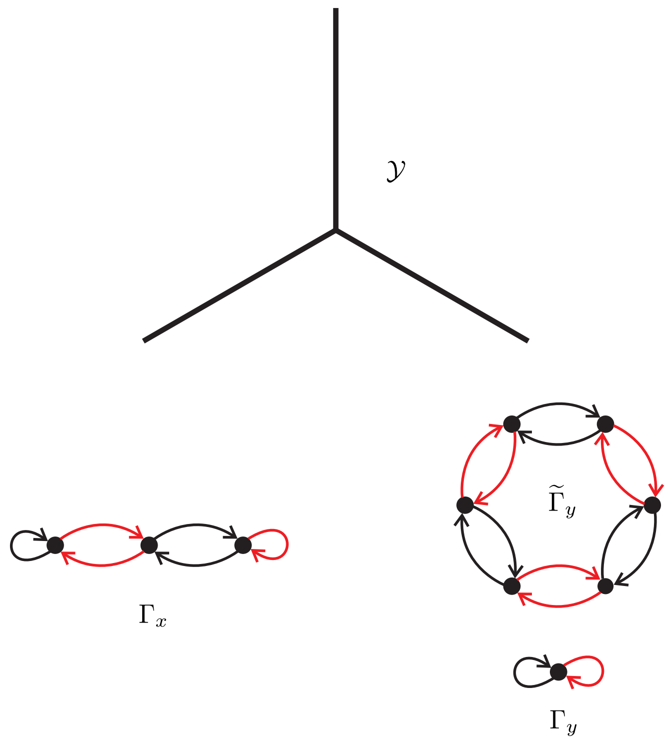

Figure 2.9. Non-Hausdorff singularity

An action $G \curvearrowright \mathcal{X}$ is *minimal* if the only open G-invariant subsets of $\mathcal{X}$ are $\mathcal{X}$ and $\varnothing$.

In other words, $G \curvearrowright \mathcal{X}$ is minimal if and only if the space $G\backslash\mathcal{X}$ of G-orbits has trivial (i.e., antidiscrete) topology: the only open subsets of $G\backslash\mathcal{X}$ are the empty set and the whole space. The action $G \curvearrowright \mathcal{X}$ is topologically transitive if and only if every two non-empty open subsets $U, V \subset G\backslash\mathcal{X}$ have a non-empty intersection.

For every $x \in \mathcal{X}$ the closure of the G-orbit of x is a closed G-invariant set, and its complement is an open G-invariant set. It follows that an action $G \curvearrowright \mathcal{X}$ is minimal if and only if every G-orbit is dense in $\mathcal{X}$.

Topological transitivity can be also formulated in terms of topological properties of G-orbits as follows.

Proposition 2.1.17. *Let $\mathcal{X}$ be a second-countable complete metrizable space (e.g., a second countable locally compact Hausdorff space). An action $G \curvearrowright \mathcal{X}$ is topologically transitive if and only if there exists $x \in \mathcal{X}$ such that the orbit Gx is dense in $\mathcal{X}$. The set of such points x is co-meager.*

Proof. If the action $G \curvearrowright \mathcal{X}$ is topologically transitive, then for every non-empty open subset $U \subset G$ the set $\bigcup_{g \in G} g(U)$ is a dense open set. Let $\mathcal{B}$ be a countable basis of topology of $\mathcal{X}$. Consider the set $B = \bigcap_{U \in \mathcal{B}} \bigcup_{g \in G} g(U)$. It is a countable intersection of open dense sets, hence, by Bair's Category Theorem, the set B is co-meager. For every open set $W \subset \mathcal{X}$ there exists $U \in \mathcal{B}$ such that $U \subset W$. Then for every $x \in B$ there exists $g \in G$ such that $x \in g(U)$, hence $g^{-1}(x) \in W$. We have shown that the G-orbit of every point $x \in B$ is dense in $\mathcal{X}$. $\square$

We saw examples of minimal and topologically transitive actions of the infinite cyclic group $\mathbb{Z}$ in Section 1.1. The action generated by an irrational rotation of the circle is minimal, the action generated by the two-sided shift is topologically transitive, but not minimal.

The following proposition shows that orbital graphs of a regular point of minimal actions on compact spaces are locally contained in every orbital graph, i.e., every finite subgraph of the orbital graph of a regular point is contained (as an isomorphic copy) in every orbital graph. In particular, two orbital graphs of regular points are locally isomorphic.

Proposition 2.1.18. *Let $G \curvearrowright \mathcal{X}$ be a minimal action on a compact space, and let S be a finite generating set of G. Then for every $n > 0$ there exists $R_n > 0$ such that for every regular point $x \in \mathcal{X}$ and every point $y \in \mathcal{X}$ there exists a vertex z of the orbital graph Γ_y on distance not more than R_n from y such that the rooted labeled balls $B_x(n)$ and $B_z(n)$ are isomorphic.*

Proof. Every ball $B_x(n)$ is described by a finite set of equations and inequalities of the form $g_1(x) = g_2(x)$ or $g_1(x) \neq g_2(x)$, where $g_1, g_2 \in G$ are products of length at most n of the elements of $S \cup S^{-1}$. If x is regular, then every such an equation or inequality holds on a neighborhood of x. It follows that there exists a neighborhood N of x such that for every $z \in N$ the balls $B_x(n)$ and $B_z(n)$ are isomorphic. Since the action is minimal, the sets $h(N)$ for $h \in G$ cover the space $\mathcal{X}$. By compactness, there exists a finite set $h_1, h_2, \ldots, h_m \in G$ such that $\mathcal{X} = \bigcup_{i=1}^{m} h_i(N)$. Let R_n be the maximal length of the elements h_i as products of the generators and inverses. Then for every $y \in \mathcal{X}$ there exists h_i such that $z = h_i^{-1}(y) \in N$, and then $B_z(n)$ and $B_x(n)$ are isomorphic and the distance from y to z is not more than R_n. $\square$

2.1.6. Hull of a graph. At the first glance, orbital graphs defining infinite groups as in 2.1.1 seem to be discrete objects without any interesting topological dynamics. But groups defined by orbital graphs have a canonical action on a compact topological space and studying these groups unavoidably leads to the study of the associated topological dynamical systems.

Let Γ be a connected graph perfectly labeled by a set A, and let G be the group it defines. Consider the set of rooted labeled graphs (Γ, v), where v runs through the set of all vertices of Γ. Let $\overline{\Gamma}$ be the closure of this set in the space $\mathcal{S}_A$ rooted perfectly labeled graphs. We call it the *hull* of Γ. The hull is a compact totally disconnected space.

The hull consists of all connected rooted graphs (Γ', v) such that for every $R > 0$ the ball $B_v(R)$ of Γ' is isomorphic as a rooted labeled graph to a ball of Γ. In other words, it is the space of all rooted perfectly labeled graphs that are locally contained in Γ (see Definition 2.1.2). Consequently, the group defined by every graph Γ' belonging to the hull is a quotient of the group G, and G acts on the set of vertices of Γ'. Note that this action is transitive, and the graph Γ' is equal to the graph of the action of G on its set of vertices.

We also get a *natural action* of G on $\overline{\Gamma}$ mapping for every $g \in G$ a rooted graph $(\Gamma', u) \in \overline{\Gamma}$ to the rooted graph $(\Gamma', g(u))$. Note that the orbital graphs of the action $G \curvearrowright \overline{\Gamma}$ may be quotients of the elements of $\overline{\Gamma}$, since two rooted graphs (Γ', u_1) and (Γ', u_2) may be isomorphic even if $u_1 \neq u_2$.

Example 2.1.19. The hull of the graph Γ defined the Houghton's group H_n (see 2.1.1.4) is a compact metrizable space consisting of an infinite countable set of isolated points accumulating on n points. Note that these conditions defined the space uniquely up to a homeomorphism. So, for example, it can be realized as the set $\{0, 1, \ldots, n-1\} \cup_{i=0,1,\ldots,n-1} \{i + 1/n \ : \ n = 2, 3, 4, \ldots\}$. The points of the countable set correspond to different choices of the root in Γ. The limit points correspond to the limits (Γ, v) as v goes to infinity in one of the rays of Γ. The limit will be a bi-infinite version of the ray.

The following is straightforward.

Proposition 2.1.20. *The natural action of G on $\overline{\Gamma}$ is continuous and does not depend, up to topological conjugacy on the choice of the generating set S. The orbit of Γ is dense, hence the action is topologically transitive. The stabilizer of a point $(\Gamma', v) \in \overline{\Gamma}$ is isomorphic to the automorphism group of the labeled graph Γ'. The orbital graph of the action of G on the orbit of $\Gamma' \in \overline{\Gamma}$ is isomorphic to the quotient of Γ' by the automorphism group of Γ'.*

Proposition 2.1.21. *All points of $\overline{\Gamma}$ are regular with respect to the natural G-action.*

Proof. If g is a product of n elements of $S \cup S^{-1}$, and $g(v) = v$ for a vertex v of Γ, then $g(u) = u$ for all vertices u such that $B_u(n)$ is isomorphic to $B_v(n)$. This proves that if g fixes a point $(\Gamma', v) \in \overline{\Gamma}$, then it fixes all points of a neighborhood of (Γ', v). $\square$

Let $G \curvearrowright \mathcal{X}$ be an action of G on a topological space, where G is, as above a group generated by a finite set S. We have the natural map $\Delta : \mathcal{X} \longrightarrow \mathcal{S}_S : x \mapsto (\Gamma_x, x)$. For every $x \in \mathcal{X}$ the closure of the image $\Delta(G(x))$ of the orbit $G(x)$ coincides with the hull $\overline{\Gamma_x}$ of the orbital graph of x. We have the natural action on the closure, and taking union of these actions we get a natural action of G on the closure of $\Delta(\mathcal{X})$.

Definition 2.1.22. A labeled graph $\Gamma \in \mathcal{S}_S$ is said to be *repetitive* if for every ball $B_x(R)$ of Γ there exists $N > 0$ such that for every vertex v of Γ there exists a vertex v' such that $d(v, v') \leqslant N$ and the ball $B_{v'}(R)$ is isomorphic to the ball $B_x(R)$.

Note that since the number of possible isomorphism classes of balls of a given radius is finite, we may assume that $N = N_R$ depends only on R. The smallest N_R satisfying the conditions of Definition 2.1.22 is called the *repetitivity function* of the graph Γ, compare with 1.3.10.

Proposition 2.1.23. *Let $\Gamma \in \mathcal{S}_S$, and let $\overline{\Gamma}$ be its hull. The action of the group G defined by Γ on the space $\overline{\Gamma}$ is minimal if and only if the graph Γ is repetitive.*

Proof. The "only if" direction was proved in Proposition 2.1.18, since every point of the action of G on $\overline{\Gamma}$ is regular, see Proposition 2.1.21. Let us prove the "if" direction.

Suppose that Γ is a repetitive graph, and let $\Gamma_1 \in \Gamma$ be an arbitrary graph in its hull. It is enough to show that Γ belongs to the hull of Γ_1 (since hulls coincide with the closures of the G-orbits). Let v_1 be the root of Γ_1, and let $B_v(R)$ be an arbitrary ball in Γ. Denote by N_R the repetitivity function of Γ. Then the ball of Γ_1 of radius $N_R + R$ with center in v_1 is isomorphic to a ball of Γ (since Γ_1 belongs to the hull of Γ), hence it contains (by repetitivity of Γ) an isomorphic copy of $B_v(R)$. We proved that every ball of Γ is contained in a ball of Γ_1, i.e., that Γ belongs to the hull of Γ_1. $\square$

Example 2.1.24. The long range graphs notice the behavior of the singular point....

Example 2.1.25. Neither the graph Γ from Figure 2.6 nor the graph Γ_0 from Figure 2.26 defining the Thompson group are repetitive, due to the "hairs" attached at every vertex. If v_n is a sequence of vertices going to infinity along one of the "hairs", then (Γ, v_n) converges to a bi-infinite chain of edges labeled by one generator with loops labeled by the other. We get thus two special elements of $\overline{\Gamma}$ that are global fixed points of the Thompson group (the group acts on each of the limit graphs as a translation by one generator and identically by the other.)

If the sequence v_n stays inside the rooted binary subtree of Γ (i.e., does not enter any of the "hairs") and goes to infinity, then the limit of (Γ, v_n) is one of the graphs described in Exercise 2.10. Each such a limit is uniquely described by the sequence of labels of the unique path starting in the root and going against the arrows in the corresponding tree. We conclude that the set $\mathcal{C}$ of such limits can be naturally identified with the space $\{g_0, g_1\}^\omega$. Denote by $\mathcal{C}_i$ the subset of $\mathcal{C}$ consisting of sequences with the first symbol equal to g_i.

If the sequence v_n belongs to the "haris" and stays on a fixed distance d from the binary subtree of Γ, then the limit will be a graph obtained from an element of $\mathcal{C}$ by moving the root on distance d to a vertex on the "hair".

It follows that $\overline{\Gamma}$ is a union of a dense set of isolated points (the set of rooted graphs (Γ, v)), the space $\mathcal{C} \times \{0, 1, 2, \ldots\}$ and two points L_{g_0} and L_{g_1}, where the sets $\mathcal{C}_i \times \{n, n+1, n+2, \ldots\}$ form a basis of neighborhoods of L_{g_i}. The hull $\overline{\Gamma_0}$ is obtained from $\overline{\Gamma}$ by removing all isolated points. In other words, $\overline{\Gamma_0}$ is homeomorphic to the direct product of a Cantor set and the subspace $\{-\infty, \ldots -2, -1, 0, 1, 2, \ldots, +\infty\}$ of the two-point compactification of the real line. See Exercises 2.21 and 2.22 for an interpretation of the action of the Thompson group on the real line in terms of $\overline{\Gamma_0}$.

2.1.7. Chabauty space of a group. The above construction of the hull of an action does not depend on the generating set S, and it is more natural to define it without using any generating sets. In particular, it can be generalized to the case of arbitrary (i.e., not necessarily finitely generated) groups.

Let G be a (discrete) group. Consider the set 2^G of subsets of G with the direct product topology (coming from the identification of the set of all sets with the set of maps $G \longrightarrow \{0, 1\}$).

We leave the proof of the next lemma as an exercise.

Lemma 2.1.26. *The set of subgroups and the set of normal subgroups of G are closed subsets of 2^G.*

Denote by $\mathcal{S}_G$ the set of all subgroups of G with the topology induced from 2^G. By definition, a basis of topology on $\mathcal{S}_G$ consists of the set of the form

$$C_{A,B} = \{H \leqslant G \ : \ A \subset H, B \cap H = \varnothing\},$$

where A and B are finite subsets of G. The defined topology on $\mathcal{S}_G$ is a particular case of a more general *Chabauty topology*, see...

Suppose that G is generated by a finite set S. Then for every positive integer n and for every $H \leqslant G$ the isomorphism class of the rooted ball in the Schreier graph $\Gamma(G/H, S)$ of the radius n with the center $1H$ is uniquely determined by the conditions of the form $g_1 \cdot g_2 \in H$ or $g_1 \cdot g_2 \notin H$, where g_1 and g_2 are elements of length n of the group G. It follows that the space of all Schreier graphs of G with topology induced from the space $\mathcal{S}_S$ of all perfectly S-labeled graphs is naturally homeomorphic to the space $\mathcal{S}_G$.

The space $\mathcal{S}_S$ of perfectly labeled graphs is naturally isomorphic to the space $\mathcal{S}_{F_S}$ of subgroups of the free group F_S generated by S. Every rooted graph $(\Gamma, v) \in \mathcal{S}_S$ defines an action of the free group F_S on the set of its vertices, and the corresponding point of $\mathcal{S}_{F_S}$ is the stabilizer of v, which is naturally identified with the fundamental group $\pi_1(\Gamma, v)$, since every element of the stabilizer corresponds to a loop based at v.

If $G \curvearrowright \mathcal{X}$ is a group action, then we have a natural map $x \mapsto G_x$ from $\mathcal{X}$ to $\mathcal{S}_G$. The proof of the following proposition is essentially the same as the proof of Proposition 2.1.11.

Proposition 2.1.27. *The map $\Delta : x \mapsto G_x$ is continuous at G-regular points of $\mathcal{X}$. The map $x \mapsto G_{(x)}$ is continuous at all regular points and all Hausdorff singularities.*

Recall that $G_{(x)}$ is the subgroup of elements of G acting trivially on a neighborhood of x.

Proof. Let $C_{A,B}$ be a neighborhood of G_x in $\mathcal{S}_G$. Then we have $g(x) = x$ for every $g \in A$ and $g(x) \neq x$ for every $g \in B$. Since x is G-regular, there exists a neighborhood U of x such that $g(y) = y$ for every $g \in A$ and $y \in U$, and $g(y) \neq y$ for every $g \in B$ and $y \in U$. Consequently, $G_y \in C_{A,B}$ for every $y \in U$. We showed that Δ is continuous at x. The statement about the map $x \mapsto G_{(x)}$ is proved in a similar way, see Proposition 2.1.14. □

In general, the map $x \mapsto G_x$ is only *upper semi-continuous* on $\mathcal{X}$: if x_i is a net of points converging to $x \in \mathcal{X}$, then all partial limits of the net G_{x_i} are contained in G_x. The map $x \mapsto G_{(x)}$ is *lower semi-continuous*: all partial limits of $G_{(x_i)}$ contain $G_{(x)}$. Moreover, $G_{(x)}$ can be reconstructed from the partial limits of G_{x_i} in the following way.

Proposition 2.1.28. *Let $G \curvearrowright \mathcal{X}$ be an action of a countable group on a compact Hausdorff space, and let x be an arbitrary non-isolated point of $\mathcal{X}$. Let $\mathcal{L}$ be the set of the limits of all convergent nets $G_{(x_i)}$, where $x_i \in \mathcal{X} \setminus \{x\}$ converges to x. Then $G_{(x)} = \bigcap_{H \in \mathcal{L}} H$.*

Proof. Denote $K = \bigcap_{H \in \mathcal{L}} H$. Suppose that $g \in G_{(x)}$. Then g acts trivially on a neighborhood of U of x. Then for every net x_n converging to x we have $x_n \in U$ for all n big enough, which implies $g \in G_{(x_n)}$. It follows that $g \in H$ for every $H \in \mathcal{L}$, hence $g \in K$.

Suppose now that $g \notin G_x$. Then $g(x) \neq x$, and hence there exists a neighborhood U of x such that $g(U) \cap U = \varnothing$. Then for every $x_i \in U$ we have $g(x_i) \neq x_i$, hence $g_i \notin H$ for every $H \in \mathcal{L}$, hence $g \notin K$.

Suppose now that $g \in G_x \setminus G_{(x)}$. Then $g(x) = x$, but for every neighborhood U of x there exists $y \in U$ such that $g(y) \neq y$. It follows, by compactness of the space of subgroups, that there exists a net $y_i \in \mathcal{X} \setminus \{x\}$ such that $G_{(y_i)}$ is convergent and $g(y_i) \neq y_i$ for every i. Then g does not belong to the limit of $G_{(y_i)}$, hence $g \notin K$. We have shown that $g \in G_{(x)}$ if and only if $g \in K$. □

2.1.8. Minimal invariant subsets of the Chabauty space. Following [**GW15**], we define a *uniformly recurrent subgroup* of G (a URS) as a closed subset $\mathcal{C}$ of $\mathcal{S}_G$ such that action of G on $\mathcal{C}$ by conjugation is minimal. Similarly, it is a *topologically transitive subgroup* if the action is topologicaly transitive.

URS is a generalization of the notion of a normal subgroup. Namely, a singleton is a URS if and only if it is consists of a normal subgroup. More generally, if a subgroup H has a finite number of conjugates (i.e., if index of the normalizer of H in G is finite), then the set of conjugates of H is an example of a URS.

The notion of a URS is a topological analog the notion of an *invariant random subgroup*, which is defined as a G-invariant probability measure on $\mathcal{S}_G$....

The following theorem from [**GW15**] shows that every minimal action of a countable group defines a URS.

Theorem 2.1.29. *Let $G \curvearrowright \mathcal{X}$ be a minimal action of a countable group on a compact topological space. Let $\mathcal{C}$ be the closure in $\mathcal{S}_G$ of the set of stabilizers G_x of G-regular points of $\mathcal{X}$. Then $\mathcal{C}$ is a URS. Moreover, it is a unique URS contained in the closure of the set $\Delta(\mathcal{X}) = \{G_x : x \in \mathcal{X}\}$.*

Proof. Let us reprove Proposition 2.1.18 in terms of the Chabauty space by dropping the condition that G is finitely generated and talking about the stabilizers instead of orbital graphs. Namely, we want to prove that if x is a G-regular point, then the closure of the set $\{G_{g(y)} : g \in G\}$ for any $y \in \mathcal{X}$ contains G_x. Equivalently, we want to prove that every neighborhood of G_x contains $G_{g(y)}$ for some $g \in G$. A basis of neighborhoods of G_x is formed by the sets of the form $\{H \leqslant G : A \subset H, B \cap H = \varnothing\}$, where A and B are finite subsets of G. If A and B are finite subsets such that $A \subset G_x$ and $B \cap G_x = \varnothing$, then, by the definition of a regular point, there exists a neighborhood $U \subset \mathcal{X}$ of x such that $A \subset G_z$ and $B \cap G_z = \varnothing$ for every

$z \in U$. By minimality, there exists $g \in G$ such that $g(y) \in U$, which finishes the proof of the claim.

We have shown that the closure of the every G-orbit of G_y contains $\mathcal{C}$, which implies that $\mathcal{C}$ is the unique minimal subset in the closure of $\Delta(\mathcal{X})$. $\square$

Note that if $\mathcal{C}$ is a uniformly recurrent subgroup, then $\Delta(\mathcal{C})$ is in general different from $\mathcal{C}$, since the stabilizer for the action by conjugation of a subgroup $H \leqslant G$ is its *normalizer* and it can be different from H (i.e., H may not be *self-normalizing*. So, it is not immediately clear if all uniformly recurrent subgroups of G can be obtained using Theorem 2.1.29. The fact that it is true (that for every URS $\mathcal{C}$ of G there exists a minimal action $G \curvearrowright \mathcal{X}$ such that $\mathcal{C}$ is the closure of the set of stabilizers of G-regular points of $\mathcal{X}$) was shown for finitely generated groups by G. Elek [**Ele18**] and in general (even for locally compact groups) by N. Matte Bon and T. Tsankov [**MBT17**].

Note also that $\Delta(\mathcal{X})$ in general erases a lot of information about the action $G \curvearrowright \mathcal{X}$. In fact, a group may have many minimal free actions on compact spaces, when $\Delta(\mathcal{X})$ is a singleton......

2.1.9. Space of marked groups. By Lemma 2.1.26, the set of normal subgroups of a group G is a closed subset of 2^G. Let us denote it $\mathcal{Q}_G$. We can identify $\mathcal{Q}_G$ with the set of all epimorphisms $G \longrightarrow H$.

An interpretation in terms of marked Cayley graphs, different historical remarks and overview...

2.2. Localizable actions and Rubin's theorem

We present here a result from the paper [**Rub89**] by M. Rubin. The paper contains several theorems describing classes of group actions such that if $G_1 \curvearrowright \mathcal{X}_1$ and $G_2 \curvearrowright \mathcal{X}_2$ belong to such a class, then for every isomorphism $\phi : G_1 \longrightarrow G_2$ there exists a homeomorphism $F : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ conjugating the actions, i.e., such that $F(g \cdot x) = \phi(g) \cdot F(x)$ for all $x \in \mathcal{X}_1$ and $g \in G$. In fact, the theorems show how to reconstruct the action $G \curvearrowright \mathcal{X}$ from the algebraic structure of an abstract group G. Such theorems make it possible to distinguish abstract groups using invariants of dynamical systems such as quasi-isometry classes of orbital graphs, entropy, groupoids of germs, etc.. This will be useful for us in many instances.

M. Rubin's paper does not define one big class of group actions, rather several closely related classes. Finding the most general "Rubin's theorem" is an interesting open problem. We will not present all results of [**Rub89**]. Moreover, we will make some substantial simplifications. But most examples that are of interest for us will be covered.

Theorems similar to [**Rub89**] were proved in different generality in several other papers. For example, Fremlin shows in [Theorem 383D] that if a group of automorphisms of a complete Boolean algebra "contains many involutions" then the Boolean algebra can be reconstructed from the group structure. Giordano, Putnam, and Skau proved a rigidity theorem for *topological full groups* of minimal homeomorphisms, see .... K. Medynets proved in... that isomorphisms *full groups* of actions on Cantor sets are realized by homeomorphisms. In the context of full groups it was proved in... Check also results of Matui... We will also prove a reconstruction theorem from [**LN02**] for groups acting on rooted trees, see Theorem 2.4.39, which is very similar to Rubin's theorems, though does not follow directly from them.

2.2.1. Localizable actions. Let G be a subgroup of the homeomorphism group of a Hausdorff topological space $\mathcal{X}$.

Definition 2.2.1. Denote, for an open subset $U \subset \mathcal{X}$, by $G[U]$ the set of all elements of G acting trivially on $\mathcal{X} \setminus U$.

We say that the action $G \curvearrowright \mathcal{X}$ is *localizable* if $\mathcal{X}$ for every non-empty open set U the subgroup $G[U]$ is non-trivial.

Note that if $G \curvearrowright \mathcal{X}$ is localizable, then $\mathcal{X}$ has no isolated points.

Denote, for $g \in G$, by $var(g)$ the interior of the closure of the set of points $x \in \mathcal{X}$ such that $g(x) \neq x$. The set of points moved by g is contained and is dense in $var(g)$.

We start with a property of localizable actions which is often used to study normal structure of groups of homeomorphisms. Analogs of this proposition appeared in many papers as a lemma for proving simplicity of just-infiniteness of groups acting on topological spaces, see...

Lemma 2.2.2. *Let $N \lhd G$ be a normal subgroup. If $g \in N$ and an open set $U \subset \mathcal{X}$ are such that $g(U) \cap U = \emptyset$, then the derived subgroup $G[U]' = [G[U], G[U]]$ of $G[U]$ is contained in N.*

Proof. Let $h_1, h_2 \in G[U] \setminus \{1\}$. The element $h_1 g h_1^{-1} g^{-1}$ acts trivially on $\mathcal{X} \setminus (U \cup g(U))$, as h_1 on U, and as $g h_1^{-1} g^{-1}$ on $g(U)$. Therefore, $[h_1 g h_1^{-1} g^{-1}, h_2] = [h_1, h_2]$. But $h_1 g h_1^{-1} \cdot g^{-1} \in N$, hence $[h_1 g h_1^{-1} g^{-1}, h_2] \in N$. We have proved that $[h_1, h_2] \in N$ for all $h_1, h_2 \in G[U]$. □

Let us prove a simple lemma, which will be used later several times.

Lemma 2.2.3. *Let $g_1, \ldots, g_n \in G$ and $x \in \mathcal{X}$ be such that $g_i(x) \neq g_j(x)$ for all $i \neq j$. Then there exists an open neighborhood U of x such that $g_i(U)$ are pairwise disjoint.*

Proof. For every pair $i \neq j$ there exist neighborhoods $U_{i,j} \ni g_i(x)$ and $V_{i,j} \ni g_j(x)$ of x such that $g_i(U_{i,j}) \cap g_j(V_{i,j}) = \varnothing$. Consider the intersection U of the neighborhoods $U_{i,j}$ and $V_{i,j}$ for all $i \neq j$. Then $g_i(U) \subset g_i(U_{i,j})$ and $g_j(U) \subset g_j(V_{i,j})$, hence $g_i(U) \cap g_j(U) = \varnothing$. □

As a corollary of Lemmas 2.2.2 and 2.2.3, we get the following proposition.

Proposition 2.2.4. *Let G be a group acting on a Hausodorff topological space $\mathcal{X}$. If $N \lhd G$ is a non-trivial normal subgroup of G, then there exists a non-empty open subset $U \subset \mathcal{X}$ such that the normal closure of $G[U]'$ in G is contained in N.*

It is clear that not every group admits a localizable action (e.g., such a map must contain many commuting elements). Results of M. Abért [**Abé05**], for instance, imply that no group adimiting a localizable action can satisfy a non-trivial group law. A *group law* is a word $w(x_1, x_2, \ldots, x_n)$ in the free group generated by $x_1, x_2, \ldots, x_n$ such that $w(g_1, g_2, \ldots, g_n)$ for all $g_i \in G$.

Theorem 2.2.5. *If $G \curvearrowright \mathcal{X}$ is a localizable action, then G satisfies no non-trivial group law.*

Proof. We approximately follow the proof of [**Abé05**, Theorem 1.1]. Let $w(x_1, x_2, \ldots, x_n) = g_m g_{m-1} \cdots g_1$ be a word in the free group generated by $\{x_1, x_2, \ldots, x_n\}$, where $g_i \in \{x_1, x_2, \ldots, x_n\} \cup \{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$. We assume that it is *reduced*, i.e., that $g_{i+1}g_i$ is not of the form xx^{-1} or $x^{-1}x$ for any $i = 1, 2, \ldots, n-1$. Denote by $w_k(x_1, x_2, \ldots, x_n) = g_k g_{k-1} \cdots g_1$ its suffix of length k.

Let us prove by induction on m that there exist elements $h_1, h_2, \ldots, h_n \in G$ and a point $p \in \mathcal{X}$ such that all the points $p_i = w_i(h_1, h_2, \ldots, h_n)(p)$, for $i = 1, 2, \ldots, m$ are pairwise different. This, of course, will imply that the law $w(x_1, x_2, \ldots, x_n)$ is not satisfied in G. The statement is obviously true for $m = 1$: if $g_1 = x_i$ or $g_1 = x_i^{-1}$, choose $h \in G$, and $x \in \mathcal{X}$ such that $h(x) \neq x$. Then the statement is true for any collection $h_1, h_2, \ldots, h_n \in G$ such that $h_i = h$.

Suppose that the statement is true for $m-1$, let us prove it for m. By the hypothesis, there exists a point $p \in \mathcal{X}$ and a collection $(h_1, h_2, \ldots, h_n) \in G^n$ such that $p_i = w_i(h_1, h_2, \ldots, h_n)(p)$ are pairwise distinct for all $i = 1, 2, \ldots, m-1$. By Lemma 2.2.3, there exists a neighborhood U of p such that the sets $U_i = w_i(h_1, h_2, \ldots, h_n)(U)$ are pairwise disjoint for all $i = 1, 2, \ldots, m-1$.

If $w_m(h_1, h_2, \ldots, h_n)(p) \notin \{p_i \ : \ 1 \leqslant i \leqslant m-1\}$, then we are done. Suppose that $w_m(h_1, h_2, \ldots, h_n)(p) = p_{i_0}$ for some $1 \leqslant i_0 \leqslant m-1$.

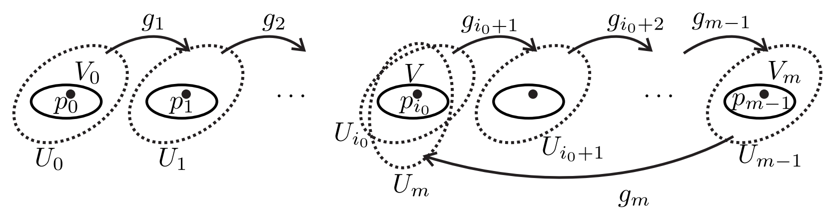

Figure 2.10. Proof of Theorem 2.2.5

Then the intersection $U_m \cap U_{i_0}$ contains p_{i_0}, hence it is non-empty, and there exists a neighborhood $V \subset U_{i_0}$ of p_{i_0} such that $g_m g_{m-1} \cdots g_{i_0+1}(V) \subset U_{i_0} \cap U_m$. If $g_m g_{m-1} \cdots g_{i_0+1}|_V$ is not the identity map, then there exists $p'_{i_0} \in V$ such that $g_m g_{m-1} \cdots g_{i_0+1}(p'_{i_0}) \neq p'_{i_0}$. Let $p' = (g_{i_0} g_{i_0-1} \cdots g_1)^{-1}(p'_{i_0})$. Then $p' \in U$, and the points $p'_k = (g_k g_{k-1} \cdots g_1)(p')$, for $k = 1, 2, \ldots, m-1$, belong to pairwise disjoint sets U_k, hence are pairwise distinct. The last point p'_m belongs to $V \subset U_{i_0}$, hence can not be equal to p'_i for $i \neq i_0$, but it is also different from p'_{i_0}. It follows that all points p'_k are distinct.

Suppose now that $g_m g_{m-1} \cdots g_{i_0+1}|_V$ is identical. Denote $V_0 = (g_{i_0} g_{i_0-1} \cdots g_1)^{-1}(V)$, and $V_i = g_i g_{i+1} \cdots g_1(V_0)$. Note that $V = V_{i_0}$.

Choose $f \in G[V] \setminus \{1\}$. Let j_0 be such that $g_m = x_{j_0}$ or $g_m = x_{j_0}^{-1}$. Replace then h_{j_0} by $h'_{j_0} = f h_{j_0}$ if $g_m = x_{j_0}$ and by $h'_{j_0} = h_{j_0} f^{-1}$ if $g_m = x_{j_0}^{-1}$, so that g_m is replaced by $f g_m$. Denote by $(h'_1, h'_2, \ldots, h'_n)$ the new collection of the values of the variables (so that $h'_i = h_i$ for $i \neq j_0$). Consider the restriction of $w_k(h'_1, h'_2, \ldots, h'_n)$ to $V_0 = w_{i_0}(h_1, h_2, \ldots, h_n)^{-1}(V)$, and denote $V_i = g_i g_{i-1} \cdots g_1(V_0)$. The sets V_i are pairwise disjoint.

If $i_0 + 1 = m - 1$, then $g_{i_0+1} \neq g_m^{-1}$, since the word $g_m g_{m-1} \cdots g_2 g_1$ is reduced. If $i_0+1 < m-1$, then $g_{i_0+1}(U_{i_0}) \cap U_{m-1} = \varnothing$ but $U_{i_0} \cap g_m(U_{m-1}) \neq \varnothing$. Hence, we always have $g_{i_0+1} \neq g_m^{-1}$.

If $g_m = x_{j_0}$, then the map h_{j_0} was modified only on V_m. If $g_m = x_{j_0}^{-1}$, then the map h_{j_0} was modified only on V_{i_0}. Since the sets V_i are pairwise disjoint, and $g_{i_0+1} \neq g_m^{-1}$, restrictions of the maps g_i to V_{i-1} for $i = 1, \ldots, m-1$ were not changed, see Figure 2.10. Consequently, $w_k(h'_1, h'_2, \ldots, h'_n)|_{V_0} = w_k(h_1, h_2, \ldots, h_n)|_{V_0}$ for $k = 1, 2, \ldots, m-1$, and $w_m(h'_1, h'_2, \ldots, h'_n)|_{V_0} = f \cdot w_m(h_1, h_2, \ldots, h)|_{V_0}$. Since $f \neq 1$, there exists $p'' \in V_0$ such that $f \cdot w_m(h_1, h_2, \ldots, h)(p'') \neq w_m(h_1, h_2, \ldots, h_n)(p'')$ and then the points $w_k(h'_1, h'_2, \ldots, h'_n)(p'')$ are pairwise distinct. □

Corollary 2.2.6. *If $G \curvearrowright \mathcal{X}$ is localizable, then $G' \curvearrowright \mathcal{X}$ is localizable.*

Proof. If $G \curvearrowright \mathcal{X}$ is localizable, then $G[U] \curvearrowright U$ is localizable for every non-empty open subset $U \subset \mathcal{X}$. Since $G[U] \curvearrowright U$ is localizable, it is not commutative, by Theorem 2.2.5. It follows that $G[U]'$ is non-trivial for every non-empty open subset U. We obviously have $G[U]' \leqslant G'[U]$, hence $G' \curvearrowright \mathcal{X}$ is localizable. $\square$

Examples of localizable actions... (full homeomorphism groups of manifolds, and the Cantor set, Thompson group

2.2.2. Boolean algebras. Here we present a very short overview of the theory of Boolean algebras. For more, see (Sikorski, Koppelberg)...

A *Boolean algebra* is a set $\mathcal{A}$ with two binary operations $\vee, \wedge$ and one unary operation $\sim$ satisfying the following axioms for all $a, b, c \in \mathcal{A}$:

(1) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$;
(2) $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;
(3) $(a \wedge b) \vee b = b$, $(a \vee b) \wedge b = b$;
(4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$;
(5) $(a \wedge \sim a) \vee b = b$, $(a \vee \sim a) \wedge b = b$.

One can show that the axioms imply the following properties.

(6) For every $a \in \mathcal{A}$ we have $a \vee a = a$, $a \wedge a = a$.
(7) Write $a \subset b$ if $a \wedge b = a$. Then $a \subset b$ if and only if $a \vee b = b$, and the relation $a \subset b$ is a partial order on $\mathcal{A}$.
(8) The elements $a \wedge \sim a$ and $a \vee \sim a$ do not depend on a. We will denote them by O and I, respectively. The elements O and I are the minimal and the maximal element of $\mathcal{A}$ with respect to $\subset$.
(9) For every $a \in \mathcal{A}$ we have $\sim\sim a = a$.
(10) For every $a, b \in \mathcal{A}$ we have $\sim (a \wedge b) = (\sim a) \vee (\sim b)$ and $\sim (a \vee b) = (\sim a) \wedge (\sim b)$.

Basically, any algebraic statement which is true for the usual operations $\cap, \cup, X \setminus A$ on the set 2^X of all subsets of a set X (on the *Boolean* of X) is true for every Boolean algebra. More precisely, one has the following *Stone Representation Theorem*, see (Sikorski Theorem 8.2).

Theorem 2.2.7. *Every Boolean algebra $\mathcal{A}$ is isomorphic to the Boolean algebra of clopen subsets of a compact totally disconnected space $\mathfrak{S}$ (with respect to the operations $A \wedge B = A \cap B$, $A \vee B = A \cup B$, $\sim A = \mathfrak{S} \setminus A$. The space $\mathfrak{S}$ is called the* Stone space *of the algebra and it is the space of all ultrafilters of $\mathcal{A}$.*

The Stone space $\mathfrak{S}$ of ultrafilters is defined in the following way.

Definition 2.2.8. A *filter* on a Boolean algebra $\mathcal{A}$ is a set $\delta \subset \mathcal{A}$ such that

(1) if $a, b \in \delta$, then $a \wedge b \in \delta$;

(2) if $b \in \alpha$ and $a \supset b$, then $a \in \delta$.

For example, for a given $a \in \mathcal{A}$ the set of elements $b \in \mathcal{A}$ such that $a \subset b$ is a filter. Such filters are called *principal*. A filter is called *proper* if it does not coincide with $\mathcal{A}$, i.e., if it does not contain O.

An *ultrafilter* is a maximal (with respect to inclusion) proper filter. An easy application of Zorn's lemma shows that every proper filter is contained in an ultrafilter.

A set $\alpha \subset \mathcal{A}$ is an ultrafilter if and only if it is equal to the preimage of I under a homomorphism $h : \mathcal{A} \longrightarrow \{O, I\}$ onto the two-element Boolean algebra (which is isomorphic to the Boolean $2^{\{\cdot\}}$ of a one-point set). In particular, if α is an ultrafilter, then for every $a \in \mathcal{A}$ either $a \in \alpha$ or $\sim a \in \alpha$.

Let $\mathfrak{S}$ be the set of all ultrafilters of the algebra $\mathcal{A}$. For an element $a \in \mathcal{A}$, let U_a be the set of ultrafilters $\alpha \in \mathfrak{S}$ such that $a \in \alpha$. The set of all sets U_a is a basis of topology on $\mathfrak{S}$. We have $\mathfrak{S} \setminus U_a = U_{\sim a}$, hence the sets of the form U_α are clopen. The space $\mathfrak{S}$ is the *Stone space* of the algebra, and it is the space from Theorem 2.2.7.

Example 2.2.9. Let X be a discrete set, and let 2^X be the Boolean algebra of all subsets of X. Then the space of ultrafilters $\mathfrak{S}$ of the algebra 2^X is the *Stone-Čech* compactification βX of X. Here X is naturally identified with the set of *principal ultrafilters* of the form $\{U \subset X \ : \ x \in U\}$ for $x \in X$.

For a subset $A \subset \mathcal{A}$ an *upper bound* (resp. *lower bound*) of A is an element $b \in \mathcal{A}$ such that $a \subset b$ (resp. $b \subset a$) for every $a \in A$. The *supremum* $\bigvee A$ (resp. *infimum* $\bigwedge A$) of A is the smallest (resp. largest) upper (resp. lower) bound of A. Suprema and infima do not always exist.

Definition 2.2.10. A Boolean algebra $\mathcal{A}$ is said to be *complete* if for every set $A \subset \mathcal{A}$ the supremum $\bigvee A$ exists.

If the algebra is complete, then for every set $A \subset \mathcal{A}$ the infimum $\bigwedge A$ exists.

2.2.3. Reconstructing the Boolean algebra of regular sets. Let $\mathcal{X}$ be a topological space. A subset $U \subset \mathcal{X}$ is said to be a *regular open set* (or just a *regular set*) if it is equal to the interior of its closure. Denote by $\mathcal{R}(\mathcal{X})$ the set of all regular subsets of $\mathcal{X}$.

For $A, B \in \mathcal{R}(\mathcal{X})$, denote by $A \vee B$ the interior of the closure of $A \cup B$, by $A \wedge B$ the intersection $A \cap B$, and by $\sim A$ the interior of $\mathcal{X} \setminus A$. We denote $A \sim B = A \cap (\sim B)$. This defines a structure of a Boolean algebra on $\mathcal{R}(\mathcal{X})$.

The Boolean algebra $\mathcal{R}(\mathcal{X})$ is complete, see [**Fre04**, Theorem 314P]. If $\mathcal{U}$ is a subset of $\mathcal{R}(\mathcal{X})$, then its *supremum* is the set $\bigvee_{U \in \mathcal{U}} U$ equal to the interior of the closure of $\bigcup_{U\in\mathcal{U}} U$, and its *infimum* $\bigwedge_{U\in\mathcal{U}} U$ is the interior of $\bigcap_{U\in\mathcal{U}} U$. Note that if $\mathcal{U}$ is finite, then $\bigwedge_{U\in\mathcal{U}} U = \bigcap_{U\in\mathcal{U}} U$.

Our first goal is to show that if an action $G \curvearrowright \mathcal{X}$ is sufficiently rich (if $G[U]$ are sufficiently big), then the structure of G as an abstract group uniquely determines the Boolean algebra $\mathcal{R}(\mathcal{X})$ of regular open subsets of $\mathcal{X}$.

Definition 2.2.11. We say that an action $G \curvearrowright \mathcal{X}$ is *locally transitive* if for every open set $W \subset \mathcal{X}$ there exists an open subset $U \subset W$ such that the action $G[U] \curvearrowright U$ is topologically transitive.

The following theorem is proved in [**Rub89**, Theorem 0.2]. We have simplified it a bit, by imposing a stronger condition on $G \curvearrowright \mathcal{X}$.

Theorem 2.2.12. *If the action $G \curvearrowright \mathcal{X}$ is locally transitive and $\mathcal{X}$ is Hausdorff, then the Boolean algebra $\mathcal{R}(\mathcal{X})$ and the action of G on it are uniquely determined by G.*

In particular, if $G_1 \curvearrowright \mathcal{X}_1$ and $G_2 \curvearrowright \mathcal{X}_2$ are locally transitive actions on Hausdorff spaces, and $\phi : G_1 \longrightarrow G_2$ is an isomorphism of groups, then there exists an isomorphism of Boolean algebras $\Phi : \mathcal{R}(\mathcal{X}_1) \longrightarrow \mathcal{R}(\mathcal{X}_2)$ such that $\Phi(g(U)) = \phi(g)(\Phi(U))$ for all $U \in \mathcal{R}(\mathcal{X}_1)$ and $g \in G_1$.

Proof. The main idea of the proof is to model regular sets $U \in \mathcal{R}(\mathcal{X})$ by the subgroups $G[U]$. The Boolean operations in $\mathcal{R}(\mathcal{X})$ can be modeled by group-theoretic operations on subgroups of G in the following way.

We denote by $\mathcal{Z}_G(A)$ the centralizer of $A \subset G$, i.e., the subgroup of all elements $g \in G$ commuting with every element of A.

Proposition 2.2.13. *a) For different $U_1, U_2 \in \mathcal{R}$ the subgroups $G[U_1]$ and $G[U_2]$ are different.*

b) For every $U \in \mathcal{R}(\mathcal{X})$ we have

$$G[\sim U] = \mathcal{Z}_G(G[U]).$$

c) For every set $\mathcal{U} \subset \mathcal{R}(\mathcal{X})$ we have $G\left[\bigwedge_{U\in\mathcal{U}} U\right] = \bigcap_{U\in\mathcal{U}} G[U]$.

Proof. Let us prove at first the following two lemmas.

Lemma 2.2.14. *If U is regular, then*

$$G[U] = \{g \in G \ : \ var(g) \subset U\}$$

and

$$G[\sim U] = \{g \in G \ : \ var(g) \cap U = \varnothing\}.$$

Proof. Let D_g be the set of points moved by g. Then $var(g)$ is the interior of the closure of D_g. Since the set D_g is open, $D_g \subset var(g)$. We defined

$G[U]$ as the set of elements $g \in G$ such that $D_g \subset U$. If $var(g) \subset U$, then $D_g \subset U$. On the other hand, if $D_g \subset U$, then the interior of the closure of D_g is a subset of the interior of the closure of U, which is equal to U. It follows that $D_g \subset U$ is equivalent to $var(g) \subset U$ for regular U. □

Lemma 2.2.15. *If U is open and $U \cap var(g) \neq \varnothing$, then there exists $h \in G[U]$ such that $[g, h] \neq 1$.*

Proof. There exists $x \in U$ such that $g(x) \neq x$. Then, by Lemma 2.2.3, there exists an open neighborhood N such that N and $g(N)$ are disjoint. Let h be any non-trivial element of $G[N]$. Then $ghg^{-1} \in G[g(N)]$, and $G[N] \cap G[g(N)] = \{1\}$, hence h and ghg^{-1} are different, i.e., g and h do not commute. □

Let us prove statement (a) of the proposition. If $U_1 \neq U_2$, then one of the sets $U_1 \sim U_2$, $U_2 \sim U_1$ is non-empty. Suppose that $V = U_1 \sim U_2$ is non-empty. Then $G[V] \leqslant G[U_1]$ and $G[V] \cap G[U_2] = \{1\}$, which implies that $G[U_1] \neq G[U_2]$, thus proving (a).

Let us prove (b). We obviously have $G[\sim U] \leqslant \mathcal{Z}_G(G[U])$. Suppose that $g \notin G[\sim U]$. Then g moves a point in the complement of $\sim U$, i.e., in the closure of U. Since the set of points moved by g is open, it follows that g moves a point of U, and by Lemma 2.2.15, there exists $h \in G[U]$ such that g and h do not commute, i.e., $g \notin \mathcal{Z}_G(G[U])$.

Let us prove (c). The set $\bigwedge_{U \in \mathcal{U}} U$ is, by definition, the interior of $\bigcap_{U \in \mathcal{U}} U$. It follows that $G\left[\bigwedge_{U \in \mathcal{U}} U\right]$ is equal to the set of elements $g \in G$ such that $var(g) \subset U$ for every $U \in \mathcal{U}$, i.e., to $\bigcap_{U \in \mathcal{U}} G[U]$, see Lemma 2.2.14. □

Proposition 2.2.13 shows that if we can describe subgroups of the form $G[U]$ for $U \in \mathcal{R}(\mathcal{X})$ in purely group-theoretic terms, then we can reconstruct the Boolean algebra $\mathcal{R}(\mathcal{X})$ from the abstract group G. Moreover, it is enough to find some subset $\mathcal{U} \subset \mathcal{R}(\mathcal{X})$ such that the groups of the form $G[U]$ for $U \in \mathcal{U}$ have a group-theoretic characterization, and $\mathcal{R}(\mathcal{X})$ is the smallest complete Boolean subalgebra of $\mathcal{R}(\mathcal{X})$ containing $\mathcal{U}$.

In the original proof by M. Rubin [**Rub89**] the groups $G[U]$ were constructed in the form $\mathcal{Z}_G(g^{\mathcal{Z}_G(h)})$ for pairs $g, h \in G$ satisfying a rather complicated condition. We simplify his construction by formulating it in terms of subgroups rather than group elements. (We will loose, however, some model-theoretic properties of the interpretation of $\mathcal{R}(\mathcal{X})$ in G.)

Definition 2.2.16. We say that a non-trivial proper subgroup $H < G$ is *flexible* if its center is trivial and the following two conditions are satisfied.

(1) If $g_1, g_2 \in G \setminus \mathcal{Z}_G(H)$ then there exists $f \in H$ such that $[g_1^f, g_2] \neq 1$.

(2) If $g \in G \setminus H$ then there exist $f_1, f_2 \in \mathcal{Z}_G(H)$ such that $[f_1, f_2] \neq 1$ and $[f_1^g, f_2] = 1$.

Conditions (1) and (2) of Definition 2.2.16 are analogous to the predicates ψ_1 and ψ_2 of [**Rub89**], respectively.

Note that the first condition of Definition 2.2.16 is equivalent to the condition that for every $g \in G \setminus \mathcal{Z}_G(H)$ we have $\mathcal{Z}_G\left(g^H\right) \leqslant \mathcal{Z}_G(H)$. In particular, this implies that for every non-trivial normal subgroup $N \lhd H$ we have $\mathcal{Z}_G(N) = \mathcal{Z}_G(H)$.

We leave it as an exercise to the reader to check that in any group the equality $[f_1^g, f_2] = 1$ is equivalent to $[[g, f_1], f_2] = [f_1, f_2]$.

Proposition 2.2.17. *A subgroup $H < G$ is flexible if and only if there exists a proper non-empty regular subset $U \subset \mathcal{X}$ such that $H = G[U]$ and $G[U] \curvearrowright U$ is topologically transitive.*

Proof. Let us prove at first a series of lemmas.

Lemma 2.2.18. *If H is flexible, then $\mathcal{Z}_G(\mathcal{Z}_G(H)) = H$.*

Proof. We obviously have $\mathcal{Z}_G(\mathcal{Z}_G(H)) \geqslant H$. Suppose that there exists $g \in \mathcal{Z}_G(\mathcal{Z}_G(H)) \setminus H$. By the second condition of Definition 2.2.16, there exist $f_1, f_2 \in \mathcal{Z}_G(H)$ such that $[f_1, f_2] \neq 1$ and $[f_1^g, f_2] = 1$. The latter equality is equivalent to $[[g, f_1], f_2] = [f_1, f_2]$, but we have $g \in \mathcal{Z}_G(\mathcal{Z}_G(H))$, so that $[g, f_1] = 1$, hence $[[g, f_1], f_2] = 1$, which is a contradiction. □

Note that, as a part of Definition 2.2.16, we assume that for every flexible H we have $H \cap \mathcal{Z}_G(H) = \{1\}$.

Lemma 2.2.19. *If U is a regular non-empty set such that $G[U] \curvearrowright U$ is topologically transitive, then $G[U]$ satisfies the first condition of Definition 2.2.16.*

Proof. Recall that $\mathcal{Z}_G(G[U]) = G[\sim U]$. Let $g \in G \setminus G[\sim U]$. We have to prove that $\mathcal{Z}_G(g^{G[U]}) \subset G[\sim U]$. Suppose that, on the contrary, there exists $h \in \mathcal{Z}_G(g^{G[U]})$ such that $h \notin G[\sim U]$.

We have $g, h \notin G[\sim U]$, hence $var(g)$ and $var(h)$ have non-empty intersections with U. Then there exist open sets $W_g, W_h \subset U$ such that $g(W_g) \cap W_g = h(W_h) \cap W_h = \varnothing$. For any $h_1, h_2 \in G[W_g]$ we have $[h_1, h_2] = [[g, h_1], h_2] \in \langle g^{G[W_g]} \rangle$, as in Lemma 2.2.2. Since $G[W_g] \leqslant G[U]$ and $h \in \mathcal{Z}_G(g^{G[U]})$, we conclude that $G[W_g]'$ commutes with h. Moreover, since the action of $G[U]$ on U is topologically transitive, there exists a non-empty open subset $W' \subset W_g$ and $f \in G[U]$ such that $f(W') \subset W_h$. The group $G[W']'$ is non-trivial by Lemma 2.2.15. We have $G[f(W')] \leqslant \langle g^{G[U]} \rangle$, hence $G[f(W')]$ commutes with h, but this is a contradiction with the fact that the sets $h(f(W'))$ and $f(W')$ are subsets of $h(W_h)$ and W_h, and therefore are disjoint, see Lemma 2.2.15. $\square$

Let us denote, for $H \subset G$, by $var(H)$ the set $\bigvee_{h\in H} var(h)$, i.e., the interior of the closure of the set $\bigcup_{h\in H} var(h)$.

Lemma 2.2.20. *If H satisfies the first condition of Definition 2.2.16, then it is topologically transitive on $var(H)$.*

Proof. If H is not transitive on $var(H)$, then there exist disjoint non-empty H-invariant subsets U_1, U_2 of $var(H)$. By Lemma 2.2.15, there exist $g_i \in G[U_i]$ such that g_i do not commute with some elements of H, hence do not belong to $\mathcal{Z}_G(H)$. Consider the subgroup $\mathcal{Z}_G(g_1^H)$. By the first condition of Definition 2.2.16, it must be contained in $\mathcal{Z}_G(H)$. We have then $\mathcal{Z}_G(H) \geqslant \mathcal{Z}_G(g_1^H) \geqslant G[\sim U_1]$. But it implies $H \leqslant G[U_1]$, which is a contradiction with $U_2 \subset var(H)$. $\square$

Lemma 2.2.21. *For every proper open subset W the group $G[W]$ satisfies the second condition of Definition 2.2.16.*

Proof. Let $g \in G \setminus G[W]$. Then $var(g) \cap \sim W \neq \varnothing$. It follows that there exists a non-empty open set $V \subset \sim W$ such that $g^{-1}(V) \cap V = \varnothing$. Let $h_1, h_2 \in G[V] \leqslant G[\sim W] = \mathcal{Z}_G(G[W])$ be arbitrary elements such that $[h_1, h_2] \neq 1$. They exist by Lemma 2.2.15. Then, in the same way as in Lemma 2.2.2, the element $[g, h_1] = g^{-1}h_1^{-1}gh_1 = g^{-1}h_1^{-1}g \cdot h_1$ acts as $g^{-1}h_1^{-1}g$ on $g^{-1}(V)$, as h_1 on V, and trivially everwhere else. It follows that $[[g, h_1], h_2]$ acts as $[h_1, h_2]$ on V and trivially everywhere else, i.e., that $[[g, h_1], h_2] = [h_1, h_2]$. $\square$

It remains to prove that if $H \leqslant G$ is flexible then $H = G[var(H)]$. We know that if H is flexible, then it acts topologically transitively on $var(H)$. Let us show that $\mathcal{Z}_G(H)$ acts identically on $var(H)$, i.e., that $var(\mathcal{Z}_G(H))$ and $var(H)$ are disjoint.

Lemma 2.2.22. *An element $g \in \mathcal{Z}_G(H)$ can not act non-trivially on $var(H)$ but trivially on a non-empty open subset U of $\mathcal{Z}_G(H)$.*

Proof. Suppse that g acts non-trivially on $var(H)$ and trivially on a non-empty open subset $U \subset \mathcal{Z}_G(H)$. Then for every $h \in H$ the element $g^h = g$ acts trivially on $h^{-1}(U)$, which, by topological transitivity of $H \curvearrowright var(H)$, implies that g is trivial. $\square$

Suppose that there exists $x \in var(H) \cap var(\mathcal{Z}_G(H))$. Since H acts topologically transitively on $var(H)$, the H-orbit of x is infinite, and therefore there exist four elements $h_1, \ldots, h_4 \in H$ such that $x, h_1(x), h_2(x), \ldots, h_4(x)$

are pairwise different. Then by Lemma 2.2.3, there exists a non-empty open neighborhood W of x such that $W, h_1(W), h_2(W), \ldots, h_4(W)$ are pairwise different and contained in $var(H)$. Since x is moved by an element of $\mathcal{Z}_G(H)$, there exists a neighborhood $W' \subset W$ of x such that every non-trivial element of $G[W']$ does not commute with $\mathcal{Z}_G(H)$ (see Lemma 2.2.15). Then, by the second condition of Definition 2.2.16, for any non-trivial element $g \in G[W']$ there exist elements $f_1, f_2 \in \mathcal{Z}_G(H)$ such that $1 \neq [f_1, f_2] = [[g, f_1], f_2]$. We have $var([[g, f_1], f_2]) \subseteq var(g) \cup f_1^{-1}(var(g)) \cup f_2^{-1}(var(g)) \cup f_2^{-1}f_1^{-1}(var(g)) \subseteq W \cup f_1^{-1}(W) \cup f_2^{-1}(W) \cup f_2^{-1}f_1^{-1}(W)$. If $W \cup f_1^{-1}(W) \cup f_2^{-1}(W) \cup f_2^{-1}f_1^{-1}(W) \neq var(H)$, (note $var(H)$ is $\mathcal{Z}_G(H)$-invariant, since an element of $\mathcal{Z}_G(H)$ can not move a global fixed point of H to a point of $var(H)$), then $[f_1, f_2]$ acts identically on an open subset of $var(H)$ and is supported inside $var(H)$, which is a contradiction with Lemma 2.2.22. It follows that $W \cup f_1^{-1}(W) \cup f_2^{-1}(W) \cup f_2^{-1}f_1^{-1}(W) = var(H)$.

Note that $[f_1, f_2] = [f_1, f_2]^{h_i^{-1}} = [[g^{h_i^{-1}}, f_1], f_2]$ for $i = 1, 2, \ldots, 4$. It follows then by the same argument as above that $h_i(W) \cup f_1^{-1}(h_i(W)) \cup f_2^{-1}(h_i(W)) \cup f_2^{-1}f_1^{-1}(h_i(W)) = var(H)$. Consider an arbitrary point $y \in var(H)$. It belongs to at most one of the sets $h_i(W)$, to at most one of the sets $f_1^{-1}(h_i(W))$, to at most one of the sets $f_2^{-1}(h_i(W))$, and to at most one of the sets $f_2^{-1}f_1^{-1}(h_i(W))$. But then it follows that it belongs to at most four of the sets $h_i(W) \cup f_1^{-1}(h_i(W)) \cup f_2^{-1}f_1^{-1}(h_i(W))$, for $i = 0, 1, \ldots, 4$ (where $h_0 = 1$), which is a contradiction. □

Let $\mathcal{F}$ be the set of all open regular subsets of $\mathcal{X}$ such that $G[U] \curvearrowright U$ is topologically transitive. It remains to show, in order to finish the proof of Theorem 2.2.12, that $\mathcal{R}(\mathcal{X})$ is generated by $\mathcal{F}$ as a complete Boolean algebra.

Let $W \in \mathcal{R}(\mathcal{X})$, and consider the supremum

$$W' = \bigvee_{U \subset W,\ G[U] \curvearrowright U \text{ topologically transitive}} U.$$

Suppose that $W \neq W'$. Then $W \sim W'$ is a non-empty open subset of W, and there exists a subset $U \subset W \sim W'$ such that $G[U] \curvearrowright U$ is topologically transitive. But this is a contradiction with the choice of W'. Consequently, $W = W'$, and $\mathcal{F}$ generates $\mathcal{R}(\mathcal{X})$.

Note that we actually proved that $H \leqslant G$ satisfies $H = G[var(H)]$ if and only if H is equal to the intersection of a collection of centralizers of flexible subgroups of G. □

Another, in some cases more general, way of reconstructing the Boolean algebra of regular open sets is given by the following theorem of D.H. Fremlin, see [**Fre04**, Theorem 384.D].

Definition 2.2.23. Let G be an automorphism group of a Boolean algebra $\mathcal{A}$. We say that G has *many involutions* if for every non-zero $a \in \mathcal{A}$ there exists an involution $g \in G$ such that $g(b) = b$ for all $b \subset\sim a$. (We say then that *a supports g*.)

Theorem 2.2.24. *Suppose that $\mathcal{A}_i$ are complete Boolean algebras, and $G_i \curvearrowright \mathcal{A}_i$ are faithful actions by automorphisms with many involutions. Then every isomorphism $\phi : G_1 \longrightarrow G_2$ is induced by an isomorphism of Boolean algebras $\Phi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$.*

2.2.4. Reconstructing $\mathcal{X}$ from G. The next step is to show how one can reconstruct the action $G \curvearrowright \mathcal{X}$ from the Boolean algebra $\mathcal{R}(\mathcal{X})$ of regular subsets of $\mathcal{X}$ and the action of G on it. Again, we are not formulating the most general condition, but a condition sufficient for all our examples.

Consider the following condition for an action $G \curvearrowright \mathcal{X}$.

(R) For every $x \in \mathcal{X}$ and every neighborhood U of x the $G[U]$-orbit of x is somewhere dense.

Theorem 2.2.25. *Suppose that the actions $G_i \curvearrowright \mathcal{X}_i$ on locally compact Hausdorff spaces are locally transitive and satisfy condition (R). Then every isomorphism $\phi : G_1 \longrightarrow G_2$ is induced by a homeomorphism $\mathcal{X}_1 \longrightarrow \mathcal{X}_2$.*

For example, if for the action $G \curvearrowright \mathcal{X}$ on a locally compact Hausdorff space there exists a basis of neighborhoods of $\mathcal{X}$ consisting of sets U such that $G[U] \curvearrowright U$ is minimal, then $G \curvearrowright \mathcal{X}$ is uniquely determined by G.

Local transitivity is not enough, see Exercise 2.24.

Proof. Let $G \curvearrowright \mathcal{X}$ be a locally transitive action satisfying Condition (R), and $\mathcal{X}$ is a locally compact Hausdorff space. We are going to show how the space $\mathcal{X}$ and the action of G on it can be reconstructed from the Boolean algebra $\mathcal{R}(\mathcal{X})$ and the action of G on it.

Let $\mathfrak{S}$ be the Stone space of $\mathcal{R}(\mathcal{X})$. The group G acts on it naturally by homeomorphisms. This action and the subgroups $G[U]$ for $U \in \mathcal{R}(\mathcal{X})$ are uniquely determined by the algebraic structure of G (see Theorem 2.2.12 and its proof).

For every $\alpha \in \mathfrak{S}$ the set $\bigcap_{U \in \alpha} \overline{U}$ is either empty, or consists of a single point, which we will denote by x_α (if it exists). Note that if U is a regular set such that $x_\alpha \in U$, then $U \in \alpha$, since $x_\alpha \notin\sim U$. Similarly, if $U \in \alpha$, then x_α must belong to the closure of U. Also note that if α contains an element U such that $\overline{U}$ is compact, then x_α exists.

Definition 2.2.26. We say that an element U of an ultrafilter $\alpha \in \mathfrak{S}$ is an *$\mathcal{X}$-neighborhood* of α if there exists $V \in \mathcal{R}(\mathcal{X})$ such that $V \subseteq U$ and for every $V' \in \mathcal{R}(\mathcal{X})$ such that $V' \subset V$ there exists $g \in G[U]$ such that $g(V') \in \alpha$.

Note that the condition of U being an $\mathcal{X}$-neighborhood of $\alpha \in \mathfrak{S}$ is formulated in purely group-theoretic terms.

Lemma 2.2.27. *An element $U \in \mathcal{R}(\mathcal{X})$ is an $\mathcal{X}$-neighborhood of $\alpha \in \mathfrak{S}$ if and only if x_α exists and belongs to U.*

Proof. If x_α exists and belongs to U, then there exists an open regular set $V \subset U$ such that the $G[U]$-orbit of x_α is dense in V. It is easy to see that then the conditions of Definition 2.2.26 are satisfied.

Conversely, suppose that U is an $\mathcal{X}$-neighborhood of an ultrafilter α, and let V be as in Definition 2.2.26. Then there exists an open regular set $V' \subset V$ such that $\overline{V'}$ is compact. It follows that α contains an element $g(V')$ with compact closure, hence x_α exists. It follows then from Definition 2.2.26 that the $G[U]$-orbit of x_α is somewhere dense. We have $x_\alpha \in \overline{U}$, but since the interior U of $\overline{U}$ is $G[U]$-invariant, this actually means that x_α belongs to the interior of $\overline{U}$, i.e., to U. □

Note that for every $x \in \mathcal{X}$ the set of all elements $U \in \mathcal{R}(\mathcal{X})$ containing x is a filter, hence it is contained in an ultrafilter α. Since there exist sets $U \in \mathcal{R}(\mathcal{X})$ such that $x \in U$ and $\overline{U}$ is compact, the point x_α exists. But it can be equal only to x, since for every point y different from x there exists $U \in \mathcal{R}(\mathcal{X})$ such that $x \in U$ and $y \in \sim U$.

Note that $x_{\alpha_1} \neq x_{\alpha_2}$ if and only if there exist $U_1, U_2 \in \mathcal{R}(\mathcal{X})$ such that U_i is a neighborhood of α_i, and $U_1 \sim U_2 = \varnothing$.

We have described in group-theoretic terms all ultrafilters α for which x_α exists, when two points $x_{\alpha_1}, x_{\alpha_2}$ are different, and when a point x_α belongs to a given regular open set. Since the set of regular open sets is a basis of topology of $\mathcal{X}$ (see the conditions of the theorem), we get a complete description of the action $G \curvearrowright \mathcal{X}$ in group-theoretic terms. □

Corollary 2.2.28. *Let $G \curvearrowright \mathcal{X}$ be an action satisfying the conditions of Theorem 2.2.25. Then every automorphism of G is induced by a conjugation by a homeomorphism of $\mathcal{X}$. In other words, the automorphism group of G coincides with its normalizer in the homeomorphism group of $\mathcal{X}$.*

Example 2.2.29. Let G be the group of all homeomorphisms of the circle $\mathbb{R}/\mathbb{Z}$. It is easy to see that if U is an open arc of the circle, then $G[U]$ acts transitively on U. It follows that $G \curvearrowright \mathbb{R}/\mathbb{Z}$ satisfies the conditions of Theorem 2.2.25, hence all automorphisms of G are inner.

Example 2.2.30. Another example of a group satisfying the conditions of Theorem 2.2.25 is the Thompson group F, see...

2.3. Automata

2.3.1. Definitions. Let X and Y be finite alphabets, and let $f : \mathsf{X}^\omega \longrightarrow \mathsf{Y}^\omega$ be a continuous map. Then for every $x_1x_2\ldots \in \mathsf{X}^\omega$ and every n there exists m such that the first n letters of $f(x_1x_2\ldots x_m a_{m+1}a_{m+2}\ldots)$ do not depend on $a_{m+1}a_{m+2}\ldots \in \mathsf{X}^\omega$. If we interpret the transformation f as a work of a machine that reads $x_1x_2\ldots$ and prints $y_1y_2\ldots$, then it has to read only a finite beginning of $x_1x_2\ldots$ in order to be able to write a beginning of arbitrary length of $f(x_1x_2\ldots)$. This can be formalized in the following way.

Definition 2.3.1. An *automaton* is a tuple $\mathcal{A} = (\mathsf{X}, \mathsf{Y}, Q, q_0, \pi, \lambda)$, where

- X, Y are finite alphabets (called the *input* and the *output* alphabets, respectively),
- Q is a set (called the *set of states* of the automaton,
- $q_0 \in Q$ (called the *initial state*),
- $\pi : Q \times \mathsf{X} \longrightarrow Q$ is a map (called the *transition function*),
- $\lambda : Q \times \mathsf{X} \longrightarrow \mathsf{Y}^*$ is a map (called the *output funciton*).

We interpret $\mathcal{A}$ as a machine that being in state q and reading $x \in \mathsf{X}$ on the input prints the word $\lambda(q, x)$ on the output, and then changes its state to $\pi(q, x)$. According to this interpretation, we extend the maps π and λ to $Q \times \mathsf{X}^*$ by the inductive rules

$$\begin{aligned} \pi(q, x_1x_2\ldots x_n) &= \pi(\pi(q, x_1), x_2x_3\ldots x_n), \\ \lambda(q, x_1x_2\ldots x_n) &= \lambda(q, x_1)\lambda(\pi(q, x_1), x_2x_3\ldots x_n). \end{aligned}$$

Then $\pi(q, x_1x_2\ldots x_n)$ is the state of the automaton after reading the word $x_1x_2\ldots x_n$ and $\lambda(q, x_1x_2\ldots x_n)$ is the total output word.

If $w = x_1x_2\ldots$, then since the word $\lambda(q, x_1x_2\ldots x_n)$ is a beginning of the word $\lambda(q, x_1x_2\ldots x_nx_{n+1})$, we can define the word $\lambda(q, w)$ as the limit of the words $\lambda(q, x_1x_2\ldots x_n)$. Note that since $\lambda(q, x)$ can be an empty word, the word $\lambda(q, w)$ is infinite or finite.

Definition 2.3.1 describes what is sometimes called *asynchronous* automata, which refers to the fact that the length of the output can be different from the length of the input, so that the input and output “tapes” are not synchronized. We will study later in 2.4.6 the class of *synchronous automata* for which all values of $\lambda : Q \times \mathsf{X} \longrightarrow \mathsf{Y}$ are one-letter words. bibliography...

Definition 2.3.2. The transformation *defined by the automaton* $\mathcal{A} = (\mathsf{X}, \mathsf{Y}, Q, q_0, \pi, \lambda)$ is the map $w \mapsto \lambda(q_0, w) : \mathsf{X}^* \cup \mathsf{X}^\omega \longrightarrow \mathsf{Y}^* \cup \mathsf{Y}^\omega$. We say that $\mathcal{A}$ is *non-degenerate* (or *almost positive* [**Eil74**]) if for every $w \in \mathsf{X}^\omega$ the word $\lambda(q_0, w)$ is infinite.

Proposition 2.3.3. *Every transformation $f : \mathsf{X}^\omega \longrightarrow \mathsf{Y}^\omega$ defined by a non-degenerate automaton is continuous. Every continuous map $f : \mathsf{X}^\omega \longrightarrow \mathsf{Y}^\omega$ is defined by a non-degenerate automaton.*

Proof. Continuity of the maps defined by automata is straightforward. Conversely, let $f : \mathsf{X}^\omega \longrightarrow \mathsf{Y}^\omega$ be a continuous map. Let $P(A)$ for a subset $A \subset \mathsf{Y}^\omega$ denote the longest common prefix of the words in A. Consider the following automaton with the set of states X^*, initial state $\varnothing$, the transition function $\pi(v, x) = vx$, and the output function given by the condition that if $P(f(v\mathsf{Y}^\omega))\lambda(v, x) = P(vx\mathsf{Y}^\omega)$. Note that $\lambda(v, x)$ may be infinite in this definition. It is easy to see that then $\lambda(\varnothing, w) = f(w)$ for all $w \in \mathsf{Y}^\omega$.

The constructed automaton can give infinite outputs in one step. This happens only when f is constant on $v\mathsf{Y}^\omega$. But then $\lambda(vw, x)$ is empty for all $w \in \mathsf{Y}^\omega$, so we can modify the automaton so that it produces the word $\lambda(v, x)$ letter by letter independently of the input after the state v. $\square$

Definition 2.3.4. An automaton $\mathcal{A} = (\mathsf{X}, \mathsf{Y}, Q, q_0, \pi, \lambda)$ is said to be *finite* if the set of states Q is finite.

It is obvious that the set of all maps $f : \mathsf{X}^\omega \longrightarrow \mathsf{Y}^\omega$ defined by finite automata is countable.

Composition of two maps defined by finite automata is defined by a finite automaton that can be constructed in the following way.

Proposition 2.3.5. *Let $\mathcal{A}_1 = (\mathsf{X}_1, \mathsf{X}_2, Q, q_0, \pi_1, \lambda_1)$ and $\mathcal{A}_2 = (\mathsf{X}_2, \mathsf{X}_3, P, p_0, \pi_2, \lambda_2)$ be automata. Consider the automaton $\mathcal{A}_2 \circ \mathcal{A}_1$ with the input and output alphabets $\mathsf{X}_1, \mathsf{X}_3$, respectively, the set of states $P \times Q$, the initial state (p_0, q_0), and the transition and output functions given by*

$$\pi((p, q), x) = (\pi_2(p, \lambda_1(q, x)), \pi_1(q, x)), \quad \lambda((p, q), x) = \lambda_2(p, \lambda_1(q, x)).$$

Then the transformation defined by $\mathcal{A}_2 \circ \mathcal{A}_1$ is equal to the composition of the transformation $\mathsf{X}_1^\omega \longrightarrow \mathsf{X}_2^\omega$ defined by $\mathcal{A}_1$ with the transformation $\mathsf{X}_2^\omega \longrightarrow \mathsf{X}_3^\omega$ defined by $\mathcal{A}_2$.

We leave the proof as an exercise. Note that in the definition of the maps π and λ we use extensions of the maps π_i, λ_i to finite words.

The set of maps defined by finite automata can be described in the following way.

Proposition 2.3.6. *Let $f : \mathsf{X}^\omega \longrightarrow \mathsf{Y}^\omega$ be a continuous map. For a word $v \in \mathsf{X}^*$, let W_v be the longest common beginning of all words belonging to $f(v\mathsf{X}^\omega)$. If W_v is finite, then $f(vw) = W_v f_v(w)$ for some continuous map $f_v : \mathsf{X}^\omega \longrightarrow \mathsf{Y}^\omega$. Otherwise, we say that f_v is* empty*. The map f is defined by a finite automaton if and only if the set $\{f_v : v \in \mathsf{X}^*\}$ is finite and every infinite word W_v is eventually periodic.*

Proof. We can modify the definition of a finite automaton by allowing it to give on output an infinite eventually periodic word w and moving after that to a state q_t such that $\pi(q_t, x) = q_t$ and $\lambda(q_t, x) = \varnothing$ for all $x \in \mathsf{X}$. Namely, every such an automaton can be transformed to a finite automaton in the usual sense by adding a loop producing the eventually periodic word w independently of the input letters. We will use this modified definition in this proof.

Suppose that f is defined by a finite automaton $\mathcal{A} = (\mathsf{X}, \mathsf{Y}, Q, q_0, \pi, \lambda)$. It follows directly from the definitions that the map f_v is uniquely determined by the state $\pi(q_0, v)$. Consequently, the set $\{f_v : v \in \mathsf{X}^*\}$ is finite. Note that if the output $w \in \mathsf{Y}^\omega$ of a finite initial automaton does not depend on the input, then w is eventually periodic.

Suppose now that the set $\{f_v : v \in \mathsf{X}^*\}$ is finite. Take it as the set of states Q, set the initial state $q_0 = f = f_\varnothing$, the transition function $\pi(f_v, x) = (f_v)_x = f_{vx}$, and the output $\lambda(f_v, x)$ equal to the longest common beginning of the words in the set $f_v(x\mathsf{X}^\omega)$ (it will be infinite if f_{vx} is empty). It is easy to see that the constructed automaton will define f. □

If $\mathcal{A} = (\mathsf{X}, \mathsf{Y}, Q, q_0, \pi, \lambda)$ is a finite automaton, then we depict it using its *Moore diagram.* It is a rooted labeled directed graph with the set of vertices Q, in which for every $q \in Q$ and $x \in \mathsf{X}$ there is an arrow starting in q, ending in $\pi(q, x)$, labeled by $x|\lambda(q, x)$. The root is q_0. Given such a Moore diagram Γ, the image of a word $x_1x_2\ldots$ under the action of the automaton is computed by finding an oriented path $e_1e_2\ldots$ such that e_1 starts in the root q_0, and e_i is labeled by $x_i|v_i$ for some $v_i \in \mathsf{Y}^*$. Then the image of $x_1x_2\ldots$ is equal to the concatenation $v_1v_2\ldots$ of the second halves of the labels of the arrows in the path.

Example 2.3.7. Consider the automaton with the Moore diagram shown on on the left-hand side of Figure 2.11. The initial state is marked by a double circle. It is easy to see that this automaton defines the one-sided shift $x_1x_2\ldots \mapsto x_2x_3\ldots$ over the alphabet $\mathsf{X} = \{1, 2, 3\}$.

The automaton on the right hand side of Figure 2.11 appends the letter 1 to every infinite word. change the figure...

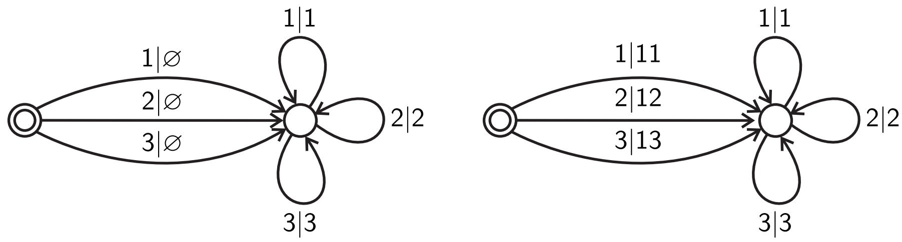

Figure 2.11. The one-sided shift

It follows from Proposition 2.3.5 that composition of two maps defined by finite automata is defined by a finite automaton. Consequently, the set of all maps $f : \mathsf{X}^\omega \longrightarrow \mathsf{X}^\omega$ defined by finite automata is a semigroup.

The following is proved in [**GNS00**].

Theorem 2.3.8. *The set of all homeomorphisms $f : \mathsf{X}^\omega \longrightarrow \mathsf{X}^\omega$ defined by finite automata is a group. There are algorithms which given a finite automaton decide if the map defined by it is identity, and if it is invertible. In particular, the word problem is solvable for every finitely-generated subgroup of the group of homeomorphisms defined by finite automata.*

The algorithms for computation with asynchronous automata are implemented in the GAP package...

The following proposition from ... shows that the group of homeomorphisms defined by finite automata does not depend on the alphabet.

Proposition 2.3.9. *For any two finite alphabets X, Y there exists a finite automaton defining a homeomorphism $\mathsf{X}^\omega \longrightarrow \mathsf{Y}^\omega$.*

Proof. It is enough to prove the proposition for $\mathsf{X} = \{0, 1\}$ and $\mathsf{Y} = \{0, 1, 2, \ldots, d\}$ for every $d \geqslant 2$. Consider the homomorphism $\phi : \mathsf{Y}^* \longrightarrow \mathsf{X}^*$ of free monoids given by $\phi(k) = \underbrace{11\ldots1}_{k \text{ times}}0$ for $k = 0, 1, \ldots, d-1$ and $\phi(d) = \underbrace{11\ldots1}_{d \text{ times}}$. Denote also by ϕ its extension to $\mathsf{X}^\omega \longrightarrow \mathsf{X}^\omega$. It is defined on infinite sequences by $\phi(x_1x_2\ldots) = \phi(x_1)\phi(x_2)\ldots$. The map ϕ is defined by a finite automaton with one state q_0 and output function $\lambda(q_0, x) = \phi(x)$. It is easy to see that ϕ is a homeomorphism and that the inverse ϕ^{-1} is given by the automaton with the Moore diagram shown on Figure 2.12. $\square$

Definition 2.3.10. The group $\mathcal{Q}(\mathsf{X})$ of all homeomorphisms $f : \mathsf{X}^\omega \longrightarrow \mathsf{X}^\omega$ defined by finite automata is called the *group of rational homeomorphisms of the Cantor set.*

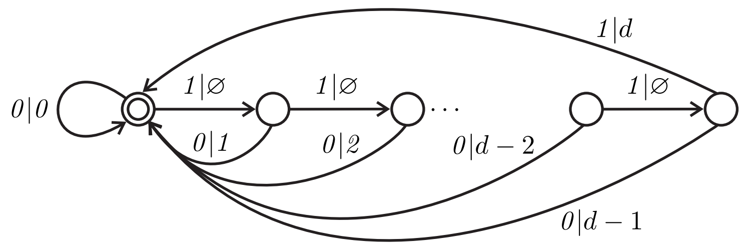

Figure 2.12. A homeomorphism $\{0, 1, \ldots, d\}^\omega \longrightarrow \{0, 1\}^\omega$

It follows from Proposition 2.3.9 that the group $\mathcal{Q} = \mathcal{Q}(\mathsf{X})$ of rational homeomorphisms does not depend (up to a conjugacy of the action, and hence up to an isomorphism) on the size of the alphabet.

The group $\mathcal{R}$ was introduced in [**GNS00**]. We know (see Theorem 2.3.8) that every finitely generated subgroup of $\mathcal{R}$ has solvable word problem. On the other hand, it is proved in [**BB17**] that the *order problem* for $\mathcal{R}$ is unsolvable, i.e., that there is no algorithm deciding whether a given element of $\mathcal{R}$ has finite or infinite order. It is shown in [**BMH17**] that the group of rational homeomorphisms is simple (see also Theorem...) and not finitely generated.

2.3.2. Examples of subgroups of $\mathcal{Q}$.

2.3.2.1. *The Higman-Thompson group.*

Definition 2.3.11. For any two sequences $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ such that $\{v_i\mathsf{X}^\omega\}$ and $\{u_i\mathsf{X}^\omega\}$ are partitions of X^ω define a homeomorphism f, denoted $f = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \\ u_1 & u_2 & \ldots & u_n \end{pmatrix}$, by

$$f(v_i w) = u_i w$$

for all $i = 1, \ldots, n$ and $w \in \mathsf{X}^\omega$. The set of all such homeomorphisms is a group called the *Higman-Thompson group* $V(\mathsf{X})$.

It is easy to check that $V(\mathsf{X}) \curvearrowright \mathsf{X}^\omega$ satisfies the conditions of Theorem 2.2.25. In fact, for every $v \in \mathsf{X}^*$ the action $V(\mathsf{X})[v\mathsf{X}^\omega] \curvearrowright v\mathsf{X}^\omega$ is conjugated by the map $vw \mapsto w$ with the whole action $V(\mathsf{X}) \curvearrowright \mathsf{X}^\omega$, and is minimal. In particular, it follows that the automorphism group of $V(\mathsf{X})$ is naturally isomorphic to the normalizer of $V(\mathsf{X})$ in the homeomorphism group of X^ω. It is shown in [**BCM$^+$16**] that the normalizer is naturally embedded into $\mathcal{R}(\mathsf{X})$, and the image of the embedding has a nice automata-theoretic description.

2.3.2.2. *Partial automata and adic transformations.* It is convenient in some cases to consider *partial automata*, i.e., automata $(\mathsf{X}, \mathsf{Y}, Q, q_0, \pi, \lambda)$, where the maps π, λ are defined on a (common for both maps) subset of $Q \times \mathsf{X}$. Such an automaton *accepts* an infinite word $x_1x_2\ldots \in \mathsf{X}^\omega$ if all the maps values of π in the formula

$$\pi(\ldots\pi(\pi(\pi(q_0, x_1), x_2), x_3), \ldots, x_n)$$

for computation of $\pi(q_0, x_1x_2\ldots x_n)$ are defined for all n. In other words, the word $x_1x_2\ldots$ is accepted by the automaton if there exists a path in the Moore diagram of the automaton starting in q_0 and such that the word $x_1x_2\ldots$ is read on the first halves of the labels $x|v$ of the arrows. We say that a subset $L \subset \mathsf{X}^\omega$ is *rational* if there exists a finite automaton such that the set of all infinite words accepted by it is equal to L. The following is proved in [**GNS00**, Proposition 2.11].

Proposition 2.3.12. *A set $L \subset \mathsf{X}^\omega$ is rational if and only if there exists a map $f : \{0,1\}^\omega \longrightarrow \mathsf{X}^\omega$ defined by a finite automaton such that $f(\{0,1\}^\omega) = L$. If L has no isolated points, then we can choose f to be a homeomorphism onto its range.*

In particular, if L is a rational set without isolated points, and G is a group of homeomorphisms of L defined by (partial) finite automata, then the action $G \curvearrowright L$ can be conjugated by a map defined by a finite automaton to an action $G \curvearrowright \{0,1\}^\omega$ by rational homeomorphisms. Consequently, G can be embedded into $\mathcal{R}$.

Let us consider some examples coming from dynamics. Let $\sigma : \mathsf{X} \longrightarrow \mathsf{X}^*$ be a substitution, and let B_σ be the associated Vershik-Bratteli diagram, see 1.3.7, and suppose that B_σ is properly ordered, see Proposition 1.3.31.

Let E be the set of edges of one level of B_σ. For every $x \in \mathsf{X}$ the set $\mathbf{r}^{-1}(x)$ is in a bijection with the letters of the word $\sigma(x) = x_1x_2\ldots x_n$. The ordering of the edges $e_1, e_2, \ldots, e_n \in E$ corresponding to the letters $x_1x_2\ldots x_n$ is the natural one: $e_1 < e_2 < \ldots < e_n$, and we have $\mathbf{s}(e_i) = x_i$.

We identify the set of vertices V with X. The space $\mathcal{P}(\mathsf{B}_\sigma)$ is a Markov subshift of the full one-sided shift E^ω consisting of all sequences $(e_1, e_2, \ldots)$ such that $\mathbf{r}(e_i) = \mathbf{s}(e_{i+1})$.

It follows from Proposition 1.3.31 that after replacing σ by σ^k for some $k \geqslant 1$ we may assume that there exist $x_0, x_1 \in \mathsf{X}$ such that $\sigma(x)$ starts with x_0 and ends with x_1 for all $x \in \mathsf{X}$.

Let $\tau \circlearrowleft \mathcal{P}(\mathsf{B}_\sigma)$ be the associated adic transformation. If $e_1 \in E$ is not maximal, then

$$\tau(e_1e_2e_3\ldots) = e_1'e_2, e_3\ldots, \tag{2.1}$$

where e' is the next edge after e.

If e_1 is maximal, but e_2 is not, then $\tau(e_1e_2e_3\ldots) = (e_1''e_2'e_3\ldots$, where e_2' is the next edge after e_2 in the ordering, and e_1' is the unique minimal path ending in the beginning of e_2'. Note that we have

$$\tau(e_1e_2e_3\ldots) = e_1''\tau(e_2e_3\ldots) \tag{2.2}$$

in this case, and that e_1'' depends only on e_2.

If both e_1 and e_2 are maximal, then

$$\tau(e_1e_2e_3\ldots) = e_1''\tau(e_2e_3\ldots), \tag{2.3}$$

where e_1'' is the unique minimal edge ending in x_0, since the first edge of $\tau(e_1e_2e_3\ldots)$ will be minimal.

We see that the adic transformation can be described by the following automaton.

Proposition 2.3.13. *Let $\sigma : \mathsf{X} \longrightarrow \mathsf{X}^*$ be a primitive substitution and $x_0 \in \mathsf{X}$ are such that $\sigma(x)$ starts with x_0 for all $x \in \mathsf{X}$. Let B_σ be the associated Vershik-Bratteli diagram. Then the adic on the space of paths $\mathcal{P}(\mathsf{B}_\sigma)$ is equal to the transformation defined by the initial state q_0 of the following automaton.*

The states of the automaton are q_0, 1_x, and τ_x, where x are letters of X. The output function λ and the transition function π are given by:

If e is not maximal, then $\lambda(q_0, e)$ is the next edge after e and $\pi(q_0, e) = 1_{\mathbf{r}(e)}$. If e is maximal, then $\lambda(q_0, e) = \varnothing$ and $\pi(q_0, e) = \tau_{\mathbf{r}(e)}$.

If e is not maximal, then $\lambda(\tau_{\mathbf{s}(e)}, e) = e_1e_2$, $\pi(\tau_{\mathbf{s}(e)}, e) = 1_{\mathbf{r}(e)}$, where e_2 is the next edge after e, and e_1 is the minimal edge ending in $\mathbf{s}(e_2)$.

If e is maximal, then $\lambda(\tau_{\mathbf{s}(e)}, e)$ is the minimal edge ending in x_0 and $\pi(\tau_{\mathbf{s}(e)}, e) = \tau_{\mathbf{r}(e)}$.

We have $\lambda(1_{\mathbf{s}(e)}, e) = e$ and $\pi(1_x, e) = 1_{\mathbf{r}(e)}$.

The automaton does not accept the input in the cases not covered by the above rules.

See, for example, the automaton on Figure 2.13 defining the adic transformation τ for the substitution $\sigma(a) = ab, \sigma(b) = abb$. The corresponding diagram B_σ is shown on Figure 2.14. The edges corresponding to the letters a, a, b of $\sigma(a)$ are denoted a_1, a_2, a_3, and the edges corresponding to the letters a, b of $\sigma(b)$ are denoted b_1, b_2, see Figure 2.14. The green states form an automaton acting identically on the sequences and accepting only the sequences belonging to the space of paths of the Bratteli diagram.

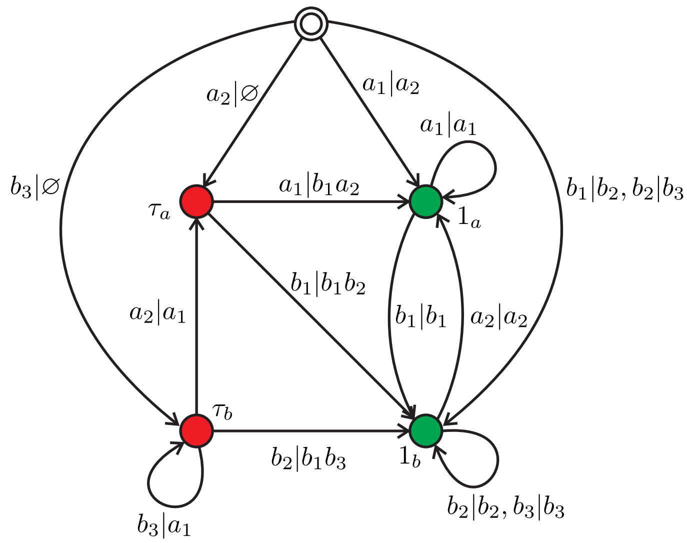

Figure 2.13. Adic transformation

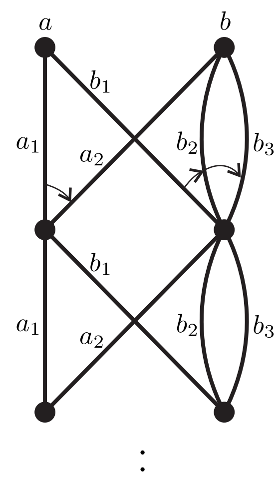

Figure 2.14.

The corresponding recursive definition of τ is

$$\begin{aligned}
\tau(a_1 w) &= a_2 w,\\
\tau(a_2 w) &= b_1 \tau(w),\\
\tau(b_1 w) &= b_2 w,\\
\tau(b_2 w) &= b_3 w,\\
\tau(b_3 a_2 w) &= a_1 \tau(a_2 w),\\
\tau(b_3 b_2 w) &= b_1 \tau(b_2 w),\\
\tau(b_3 b_3 w) &= a_1 \tau(b_3 w).
\end{aligned}$$

Compare the recursive definition of τ with the automaton on Figure 2.13. Note that the output of the automaton is sometimes delayed by one symbol comparing to the input, since the first letter of $\tau(x_1x_2\ldots)$ depends not only on x_1 but also on x_2.

Definition 2.3.14. Let $\tau \circlearrowright \mathcal{X}$ be a homeomorphism of the Cantor set. Its *topological full group* $[[\tau]]$ is the group of all homeomorphisms $f \circlearrowright \mathcal{X}$ such that for every $x \in \mathcal{X}$ there exists a neighborhood U of x and an integer $n \in \mathbb{Z}$ such that $f|_U = \tau^n|_U$.

It follows directly from Proposition 2.3.6 that if $\tau \circlearrowright \mathsf{X}^\omega$ is rational, then $[[\tau]]$ is a subgroup of the group $\mathcal{Q}(\mathsf{X})$ of rational homeomorphisms of X^ω. This gives us the following interesting class of subgroups of $\mathcal{Q}$. We will study the properties of topological full groups later in ...

Theorem 2.3.15. *Let τ be the adic transformation defined by a stationary properly ordered Vershik-Bratteli diagram. Then its topological full group $[[\tau]]$ can be embedded into $\mathcal{Q}$. In particular, the word problem in $[[\tau]]$ is solvable.*

2.3.2.3. *Transformations associated with Smale spaces.* From my paper on Smale spaces....

2.3.2.4. *Hyperbolic groups.* J. Belk, C. Bleak, and F. Matucci proved the following theorem, see [**BBM17**]. (For a definition of Gromov hyperbolic groups and their boundaries see...)

Theorem 2.3.16. *If G is a Gromov hyperbolic group acting faithfully on its boundary ∂G, then G can be embedded into $\mathcal{R}$.*

One can show that for any non-elementary Gromov hyperbolic group G the kernel of the action of G on ∂G is finite. In particular, every torsion-free Gromov hyperbolic group is a subgroup of $\mathcal{R}$.

The proof of 2.3.16 is based on a G-equivariant symbolic encoding of ∂G by one-sided sequences.

2.3.3. Non-deterministic automata and dual Moore diagrams. Another formalism for describing rational homeomorphisms of the Cantor set, is given by the notion of *non-deterministic finite automata.* Here we allow several initial states and several arrows starting in the same state with labels $x|v_1$ and $x|v_2$ for the same x, so that the output $\lambda(q, x)$ is not unique. We require, however, that for every infinite sequence $x_1x_2\ldots$ there exists at most one oriented path starting in an initial state with the labels of the form $x_1|v_1, x_2|v_2, \ldots$. We say that such automata are *ω-deterministic*). The formal definition is as follows.

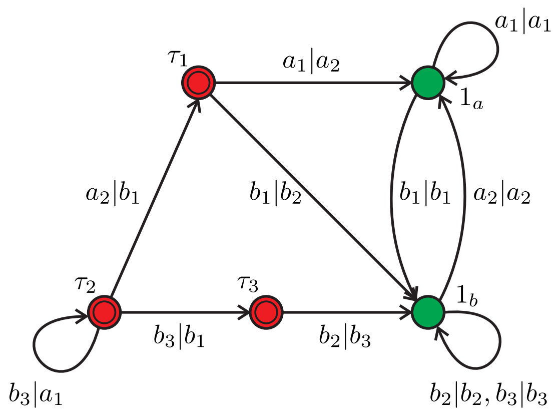

Figure 2.15. An ω-deterministic automaton

Definition 2.3.17. A *non-deterministic automaton* is a set $T \subset Q \times Q \times \mathsf{X} \times \mathsf{Y}^*$ of *transitions* and a subset $I \subset Q$ of *initial states*. Here Q is the set of internal states, X and Y are input and output alphabets. If $t = (q_1, q_2, x, v)$, then we say that t is a transition from q_1 to q_2 with input x and output v. The automaton is said to be *ω-deterministic* if for every sequence $x_1x_2 \ldots \in \mathsf{X}^\omega$ there exists at most one sequence of transitions of the form $(q_0, q_1, x_1, v_2), (q_1, q_2, x_2, v_2), \ldots$ such that $q_0 \in I$. The concatenation $v_1v_2 \ldots$ is the image of $x_1x_2 \ldots$ under the map *defined by the automaton*. The automaton is *synchronous* if T is a subset of $Q \times Q \times \mathsf{X} \times \mathsf{Y}$.

We also represent non-deterministic automata by their Moore diagrams. Its set of vertices is Q, set of edges T, where $(p, q, x, v) \in T$ is an arrow from p to q labeled by $x|v$. All non-deterministic automata in our book will be synchronous.

As an example, consider again the adic transformation τ defined on the space of paths of the diagram B_σ for $\sigma : a \mapsto aab, b \mapsto ab$. We can interpret the recurrent formulas in Proposition 2.3.13 as the work of a non-deterministic automaton with the Moore diagram shown on Figure 2.15. Note that it has three initial states (shown red) and that it preserves the length of finite words, unlike the asynchronous automaton on Figure 2.13.

The state τ_2 of the automaton is non-deterministic: if it gets b_3 on the input, the automaton can go either to τ_2 or to τ_3. But the next letter of the input will be accepted only by one of these two states: a_2 and b_3 by τ_2, and b_2 by τ_3. So, the first letter of the output is unique after reading a two-letter word. It is easy to check that this non-deterministic automaton defines the transformation τ.

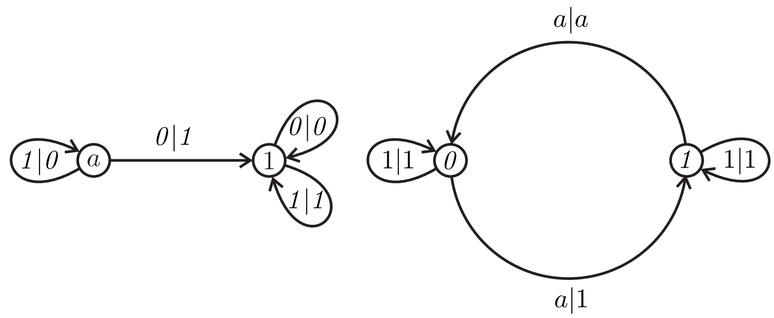

Figure 2.16. The binary adding machine and its dual Moore diagram

Sometimes it is more convenient to draw the *dual Moore diagrams* of automata instead of the usual Moore diagrams. Suppose we have a finite ω-deterministic automaton, and suppose that it is synchronous. We also assume that the input and the output alphabets coincide. Then the dual Moore diagram is obtained by switching the role of the alphabet and the set of states: for every transition from q_1 to q_2 with input and output $x_1|x_2$ we have an arrow from x_1 to x_2 labeled by $q_1|q_2$ in the dual Moore diagram. As the arrows of the dual Moore diagram describe the action of the automaton on the letters, the dual Moore diagram is often more natural than the usual Moore diagram.

See the Moore diagram of the binary adding machine and its dual Moore diagram on Figure 2.16.

The dual Moore diagram of the automaton from Figure 2.15 is shown on Figure 2.17.

We can compose non-deterministic automata, in a way similar to composition of deterministic automata. The composition is formulated in terms of the dual Moore diagrams in the following way. If Γ_1 and Γ_2 are dual Moore diagrams of synchronous automata with the same set of states Q, then their composition $\Gamma_1 \otimes \Gamma_2$ is the graph with the set of vertices equal to the direct product of the sets of vertices of Γ_1 and Γ_2, where we have an arrow from (x_1, x_2) to (y_1, y_2) labeled by $q_1|q_2$ if and only if there exists an arrow from x_1 to y_1 labeled by $q_1|p$ and an arrow from x_2 to y_2 labeled by $p|q_2$ for some state $q \in Q$.

In particular, if Γ is the dual diagram of an automaton over the alphabet X, then $\Gamma^{\otimes n} = \Gamma \otimes \Gamma \otimes \cdots \otimes \Gamma$ is the dual Moore diagram of the same automaton over the alphabet X^n. The associated action on infinite sequences will be the same. The automaton over X^n interprets a sequence $x_1x_2\ldots \in \mathsf{X}^\omega$ as the sequence $(x_1x_2\ldots x_n)(x_{n+1}x_{n+2}\ldots x_{2n})\ldots \in (\mathsf{X}^n)^\omega$.

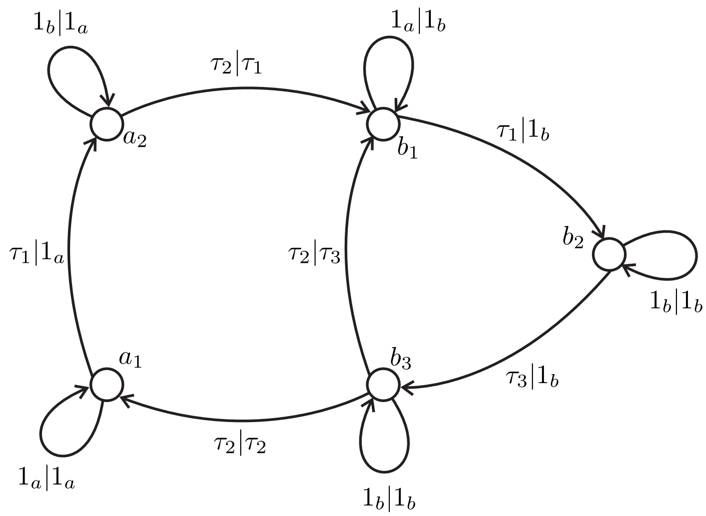

Figure 2.17. Dual Moore diagram of an adic transformation

Note that we have natural maps $\Gamma^{\otimes(n+1)} \longrightarrow \Gamma^{\otimes n}$ erasing the last letter in $\mathsf{X}^{n+1} \in \Gamma^{\otimes(n+1)}$ inducing a map of the oriented graphs, preserving the first half of the edge labels (see the definition of $\Gamma_1 \otimes \Gamma_2$ above). It follows from the definitions that the inverse limit of the graphs $\Gamma^{\otimes n}$ with respect to these maps is the graph of the action of the states on X^ω.

There is a natural topological interpretation of dual Moore diagrams. Let Γ be a dual Moore diagram of an automaton over the alphabet X and with the set of states Q. Let Γ_0 be the graph with one vertex and the set of loops X. The labels of Γ define two maps $\pi_{1,\Gamma}, \pi_{2,\Gamma} : \Gamma \longrightarrow \Gamma_0$ mapping an arrow labeled by $q_1|q_2$ to the loop q_i by π_i. Then the product $\Gamma_1 \otimes \Gamma_2$ of dual Moore diagrams is interpreted as the *fiber product*: it is the subset of the direct product $\Gamma_1 \times \Gamma_2$ consisting of points (x_1, x_2) such that $\pi_{2,\Gamma_1}(x_1) = \pi_{1,\Gamma_2}(x_2)$. The corresponding maps are $\pi_{1,\Gamma_1\otimes\Gamma_2}(x_1, x_2) = \pi_{1,\Gamma_1}(x_1)$ and $\pi_{2,\Gamma_1\otimes\Gamma_2}(x_1, x_2) = \pi_{2,\Gamma_2}(x_2)$.

Note that the automaton is deterministic if and only if the the map π_{1,Γ_1} is a covering of graphs.

The pair of maps $\pi_{1,\Gamma}, \pi_{2,\Gamma} : \Gamma_0 \longrightarrow \Gamma$ is called the *correspondence* associated with the automaton. In general, a *topological correspondence* is a pair of maps $f_1, f_2 : \mathcal{M} \longrightarrow \mathcal{M}_0$ between topological spaces. Correspondences can be iterated in the same way as dual Moore diagrams of automata. Define $\mathcal{M}_n$ to be the subspace of the direct power $\mathcal{M}^n$ consisting of all sequences $(x_1, x_2, \ldots, x_n)$ such that $f_2(x_i) = f_1(x_{i+1})$ for all $i = 1, 2, \ldots, n-1$. We have then the maps $(x_1, x_2, \ldots, x_n) \mapsto f_1(x_1)$ and $(x_1, x_2, \ldots, x_n) \mapsto f_2(x_n)$ from $\mathcal{M}_n$ to $\mathcal{M}_0$. We also have the natural *erasing maps* $(x_1, x_2, \ldots, x_{n+1}) \mapsto$

$(x_1, x_2, \ldots, x_n)$ and $(x_1, x_2, \ldots, x_{n+1}) \mapsto (x_2, x_3, \ldots, x_{n+1})$ from $\mathcal{M}_{n+1}$ to $\mathcal{M}_n$...

Example 2.3.18. The adding machine and the circle doubling map as a dual Moore diagram...

2.3.4. Time-varying automata and topological Bratteli diagrams. It is convenient in some cases to consider automata whose set of states and input-output alphabet change with time. Transformations and groups defined by such automata were considered in ... Namely, consider a sequence of *alphabets* $X_1, X_2, \ldots$, a sequence of *sets of states* $Q_1, Q_2, \ldots$, and a sequence of *transitions* $T_1, T_2, \ldots$, where $T_n \subset Q_n \times Q_{n+1} \times X_n \times X_n$. The definition of ω-deterministic automata is analogous to Definition 2.3.17.

Note that each time the output is from the same alphabet as the input, while the next state is from the next set of states. Every initial state $q \in Q_1$ defines a partial transformation of the set of sequence $X_1 \times X_2 \times X_3 \times \cdots$.

We will usually describe such time-varying automata by the corresponding sequence of dual Moore diagrams. It is a sequence of graphs $\Gamma_1, \Gamma_2, \ldots$, where Γ_n is the graph with the set of vertices X_n and the set of edges T_n in which every $(p, q, x, y) \in T_n$ is an arrow from x to y labeled by $p|q$. If Δ_n is a bouquet of loops labeled by Q_n, then the labels define a sequence of morphisms of graphs

$$\Delta_1 \xleftarrow{\mathbf{s}_1} \Gamma_1 \xrightarrow{\mathbf{r}_1} \Delta_2 \xleftarrow{\mathbf{s}_2} \Gamma_2 \xrightarrow{\mathbf{r}_2} \Delta_3 \ldots,$$

where an arrow of Γ_n labeled by $p|q$ is mapped by $\mathbf{s}_n, \mathbf{r}_n$ to p and q, respectively.

We may relax the definition of the dual Moore diagram by considering more general graphs Δ_n and more general maps $\mathbf{s}_n, \mathbf{r}_n$ (for example, by allowing arrows to be mapped to vertices, and considering vertices as states defining partial identity transformations)....

Example 2.3.19. Consider the constant sequence of alphabets $X_n = \{0, 1\}$ and sets of states $Q = \{a, b, c\}$, and a sequence of (deterministic) automata $\mathcal{A}_1, \mathcal{A}_2, \ldots$, where each of $\mathcal{A}_i$ is given by one of the two dual Moore diagrams $\mathcal{R}$ or $\mathcal{A}$ shown of Figure **??**.

Every sequence $\mathcal{A}_i$ will define three transformations a, b, c of the space $\{0, 1\}^\omega$ generating a group. We get an uncountable set of group actions. We will study this and similar constructions in ...

Example 2.3.20. Vershik-Bratteli diagrams are naturally interpreted as time-varying automata. Let B be a Vershik-Bratteli diagram. Consider the time-varying automaton with sequence of alphabets equal to the sequence E_n of sets of edges of B. For every vertex $v \in V_n$ we have the corresponding *trivial state* $1_v \in Q_n$. For every pair of vertices $v_1, v_2 \in V_n$ we have the corresponding state τ_{v_1,v_2}. The states τ_{v_1,v_2} for $v_1, v_2 \in V_0$ are the initial states defining the adic transformation.

The set of transitions consists of the *trivial transitions* $(1_{\mathbf{s}(e)}, 1_{\mathbf{r}(e)}, e, e)$ for all $e \in E_n$, and the set of *active transitions* of two types:

- $(\tau_{\mathbf{s}(e),\mathbf{s}(e')}, 1_{\mathbf{r}(e)}, e, e')$, where e is a non-maximal edge, and e' is the next edge in the ordering;
- $(\tau_{\mathbf{s}(e),\mathbf{s}(f)}, \tau_{\mathbf{r}(e),\mathbf{r}(f)}, e, f)$, where e is a maximal edge, and f is a minimal edge.

Note that some of the states τ_{v_1,v_2} will not accept any infinite sequences, so we may remove them from the automaton and all the transitions involving them.

We leave it to the reader as an exercise to show that this automaton defines the adic transformation.

The corresponding dual Moore diagrams can be described in the following way. The graphs Δ_n have V_n as the set of vertices, where each vertex v also represents the state 1_v. For every state τ_{v_1,v_2} we have the corresponding arrow from v_1 to v_2.

The graph Γ_n has E_n as the set of vertices. If e is a non-maximal edge, then we have an arrow from e to the next edge e' mapped by $\mathbf{s}_n$ to $\tau_{\mathbf{s}(e),\mathbf{s}(e')}$ and by $\mathbf{r}_n$ to $1_{\mathbf{r}(e)}$. If e is a maximal, and f is a minimal edge, then we have an arrow from e to f mapped by $\mathbf{s}_n$ to $\tau_{\mathbf{s}(e),\mathbf{s}(f)}$ and to $\tau_{\mathbf{r}(e),\mathbf{r}(f)}$ by $\mathbf{r}_n$. Note that we may also remove the arrows corresponding to states τ_{v_1,v_2} that do not accept infinite sequences.

As an example, consider again the Vershik-Bratteli diagram shown on Figure 2.14 and the corresponding adic transformation. According to the above, it is given by the dual Moore diagram shown on Figure 2.18. Compare it with the diagram on Figure 2.17. We have labeled the states τ_{v_1,v_2} according to their labels on Figure 2.17. Namely, we have $\tau_1 = \tau_{a,b}$, $\tau_2 = \tau_{b,a}$, and $\tau_3 = \tau_{b,b}$. Note that the state $\tau_{a,a}$ is removed, as it will not accept any letter.

The automaton (and the Vershik-Bratteli diagram) are *stationary* in this case, i.e., do not depend on the level.

Definition 2.3.21. A *topological Bratteli diagram* B is a sequence of compact spaces and continuous maps

$$V_0 \xleftarrow{\mathbf{s}_1} E_1 \xrightarrow{\mathbf{r}_1} V_1 \xleftarrow{\mathbf{s}_2} E_2 \xrightarrow{\mathbf{r}_2} V_2 \ldots.$$

Its *space of paths* $\mathcal{P}(\mathsf{B})$ is the subspace of $\prod_{n\geqslant 1} E_n$ consisting of all sequences $(e_1, e_2, \ldots)$ such that $\mathbf{r}(e_n) = \mathbf{s}(e_{n+1})$ for all $n \geqslant 1$.

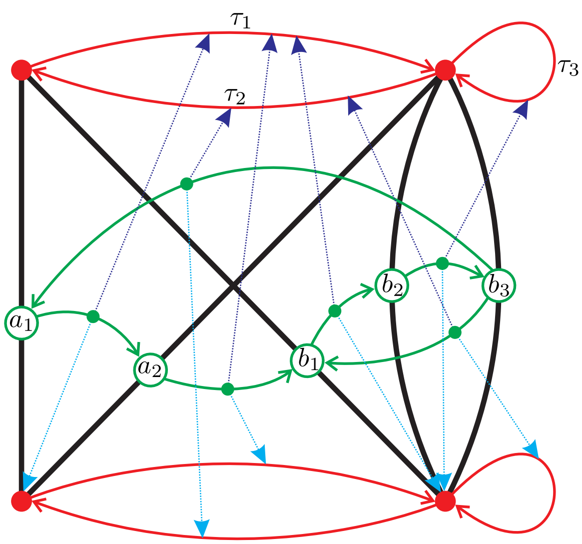

Figure 2.18. Dual Moore and Vershik-Bratteli diagrams

If B is the topological Bratteli diagram consisting of dual Moore diagrams of a time-varying automaton, then the space of paths is the graph of the action of its states on the space of infinite sequence.

Example 2.3.22. Note that the map $\mathbf{r}$ in the stationary diagram on Figure 2.18 is a homotopy equivalence... The space of paths can be therefore identified with the inverse limit of .... reference to...

2.4. Groups acting on rooted trees

2.4.1. Rooted trees. Let T be a rooted tree with the root v_0. Its *nth level* is the set L_n of vertices on distance n from the root. Every automorphism of the rooted tree T preserves the levels, since it preserves the root and is an isometry of T.

If v_1 and v_2 are vertices of T such that the unique simple path from the root to v_2 passes through v_1, then we write $v_1 \preceq v_2$. It is easy to check that $\preceq$ is a partial order on the set of vertices of T. In fact, the tree T is the Hasse diagram of the order $\preceq$.

For a vertex v of T, we denote by T_v the subtree consisting of all vertices w such that $v \preceq w$. We consider T_v to be a rooted tree with the root v, see Figure 2.19. The *branching index* of the vertex v is the number of vertices in the first level of T_v.

The *boundary* ∂T of the tree T is the set of all infinite simple paths starting in the root. In other words, it is the inverse limit of the levels L_n

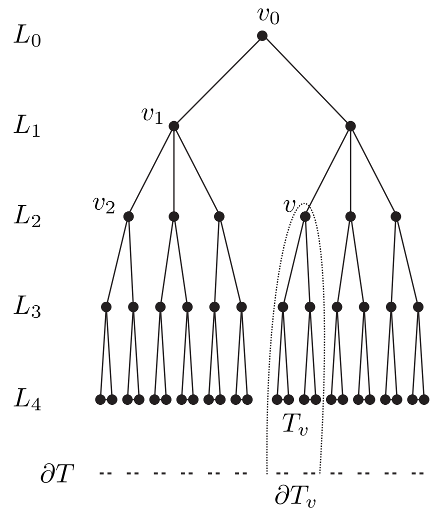

Figure 2.19. Rooted tree

with respect to the natural maps $L_{n+1} \longrightarrow L_n$ mapping a vertex $v \in L_{n+1}$ to the unique vertex $v' \in L_n$ (its *parent*) such that $v' \preceq v$.

The boundary ∂T_v is naturally identified with the set of paths $(v_0, v_1, \ldots) \in \partial T$ containing v. The collection of all subsets of ∂T of the form ∂T_v is a basis of the natural topology on ∂T. This topology obviously coincides with the inverse limit topology of the discrete sets L_n with respect to the natural maps defined above.

The space ∂T is compact and totally disconnected. A path $(v_0, v_1, \ldots) \in \partial T$ is an isolated point if and only if the branching index n_i of the vertex v_i is equal to 1 for all i big enough. So, if the branching indices of all vertices of T are greater than 1, then ∂T is homeomorphic to the Cantor set.

A rooted tree T is *spherically homogeneous* (or *level-transitive*) if the automorphism group of T acts transitively on each level L_n, or, equivalently, if for every L_n the branching indices of all vertices $v \in L_n$ are equal.

Let T be a spherically homogeneous tree, and let $\kappa = (m_1, m_2, m_3, \ldots)$ be the sequence of numbers such that m_k is the branching index of points of L_{k-1}. Then the rooted tree is uniquely determined, up to an isomorphism by the sequence κ.

Namely, let $\mathsf{X} = (X_1, X_2, \ldots)$ be a sequence of finite sets. Denote $\mathsf{X}^n = X_1 \times X_2 \times \cdots \times X_n$ for $n \geqslant 1$, and $\mathsf{X}^0 = \{\varnothing\}$. Let $\mathsf{X}^* = \bigcup_{n=0}^{\infty} \mathsf{X}^n$. The set X^* has a natural structure of a rooted tree, where a vertex $x_1 x_2 \ldots x_n \in \mathsf{X}^n$ is connected to the vertices of the form $x_1 x_2 \ldots x_n a$ for $a \in X_{n+1}$. The vertex $\varnothing$ is the root. Every spherically homogeneous tree of branching index $(|X_1|, |X_2|, \ldots)$ is isomorphic to X^*.

Note that for every vertex $v \in \mathsf{X}^n$ the tree $(\mathsf{X}^*)_v$ is isomorphic to $(\mathsf{X}_n)^*$, where $\mathsf{X}_n = (X_{n+1}, X_{n+2}, \ldots)$, and that the boundary $\partial\mathsf{X}^*$ is naturally homeomorphic to the direct product $\mathsf{X}^\omega = \prod_{n=1}^\infty X_n$, where the homeomorphism maps a sequence $x_1x_2\ldots \in \mathsf{X}^\omega$ to the path $(\varnothing, x_1, x_1x_2, x_1x_2x_3, \ldots) \in \partial\mathsf{X}^*$. The space X^ω has a unique $\mathrm{Aut}\,\mathsf{X}^*$-invarian probability measure equal to the direct product of the uniform distributions on X_n.

Special examples of spherically homogeneous trees are *regular rooted trees*, i.e., trees in which the branching indices are the same for all vertices. Every regular rooted tree is isomorphic to the tree X^* for a constant sequence $\mathsf{X} = (X, X, \ldots)$. In this case we identify X with X, and consider X^* as the set of all finite words over the alphabet X (i.e., the free monoid generated by X). A vertex $v \in \mathsf{X}^*$ is connected to every vertex of the form vx for $x \in \mathsf{X}$.

Let g be an automorphism of the spherically homogeneous tree X^*, where $\mathsf{X} = (X_1, X_2, \ldots)$, and let $v \in X_1 \times X_2 \times \cdots \times X_n$ be a vertex of the nth level. Then there exists a unique automorphism $g|_v$ of the tree X_n^* satisfying

$$g(vw) = g(v)g|_v(w)$$

for every $w \in \mathsf{X}_n^*$. We call $g|_v$ the *section* of g in v. It is easy to see that sections satisfy the following conditions

$$g|_{v_1v_2} = g|_{v_1}|_{v_2}, \qquad (g_1g_2)|_v = g_1|_{g_2(v)}g_2|_v. \tag{2.4}$$

Every automorphism $g \in \mathrm{Aut}\,\mathsf{X}^*$ can be uniquely described by the map $x \mapsto g|_x$ and the permutation $\alpha \in \mathsf{S}(X_1)$ it defines on the first level. Namely, we have

$$g(xw) = \alpha(x)g|_x(w)$$

for every $w \in \mathsf{X}_1^*$. The map $x \mapsto g|_x$ is an element of the direct power $(\mathrm{Aut}\,\mathsf{X}_1^*)^{X_1}$, and we get a map $g \mapsto \alpha \cdot (g|_x)_{x\in X_1}$ from $\mathrm{Aut}\,\mathsf{X}^*$ to the semidirect product $\mathsf{S}(X_1) \ltimes (\mathrm{Aut}\,\mathsf{X}_1^*)^{X_1}$. Properties (2.4) imply that this map is a homomorphism. It easily follows from the definitions that it is a bijection, i.e., an isomorphism. This isomorphism is called the *wreath recursion*, since the semidirect product $\mathsf{S}(X_1) \ltimes (\mathrm{Aut}\,\mathsf{X}_1^*)^{X_1}$ is, by definition, the (permutational) wreath product of $\mathsf{S}(X_1)$ with $\mathrm{Aut}\,\mathsf{X}_1^*$.

If the alphabet X_1 is identified with the set $\{1, 2, \ldots, |X_1|\}$ (or, sometimes, $\{0, 1, \ldots, |X_1| - 1\}$, then we write the elements of the wreath product as $\alpha(g_1, g_2, \ldots, g_{|X_1|})$, where $g_i = g|_i$.

Example 2.4.1. The wreath recursion notation can be used to give recurrent definitions of automorphisms of trees. For example, let $\mathsf{X} = \{\mathsf{0}, \mathsf{1}\}$, and let σ be the transposition $(\mathsf{0}, \mathsf{1})$. Then there exists a unique automorphism a of the rooted tree X^* such that its image under the wreath recursion is $\sigma(Id, a)$. By definition, it acts on the words $v \in \mathsf{X}^*$ by the recurrent rules:

$$a(\mathsf{0}w) = \mathsf{1}w, \qquad a(\mathsf{1}w) = \mathsf{0}a(w).$$

We will write such recurrent definition just $a = \sigma(Id, a)$ or $a = \sigma(1, a)$, identifying automorphisms of trees with their images under the wreath recursion isomorphism.

2.4.2. Group actions on rooted trees. Let G be a group acting on a locally finite rooted tree T. Its *nth level stabilizer* is the subgroup of elements acting trivially on the nth level L_n of the tree. We denote it $\mathsf{Stab}_n(G)$. The quotient $G/\mathsf{Stab}_n(G)$ is naturally isomorphic to the subgroup of $\mathsf{S}(L_n)$ consisting of permutations defined by elements of G on L_n. If the action $G \curvearrowright T$ is faithful, then every element g is uniquely determined by the sequence $(\alpha_1, \alpha_2, \ldots)$ of permutations it defines on the levels of the tree T, i.e., the natural homomorphism $G \longrightarrow \prod_{n\geqslant 1} \mathsf{S}(L_n)$ is injective. In particular, G is residually finite.

Every group element $g \in G$ maps a point $(v_0, v_1, \ldots) \in \partial T$ to the point $(g(v_0), g(v_1), \ldots) \in \partial T$, thus we get a natural action of G on ∂T. This action is an action by homeomorphisms, since $g(T_v) = T_{g(v)}$.

An action $G \curvearrowright T$ is said to be *level-transitive* if its is transitive on every level L_n of the tree T. If the action is level-transitive, then the tree T is spherically homogeneous, hence isomorphic to the tree X^* for some sequence $\mathsf{X} = (X_1, X_2, \ldots)$ of finite sets.

Proposition 2.4.2. *Let $G \curvearrowright T$ be an action by automorphisms on a rooted tree. Then the following conditions are equivalent:*

(1) *The action $G \curvearrowright T$ is level-transitive.*
(2) *The action $G \curvearrowright \partial T$ is topologically transitive.*
(3) *The action $G \curvearrowright \partial T$ is minimal.*

Proof. Suppose that $G \curvearrowright T$ is level transitive. Let $\xi \in \partial T$ and let v be a vertex of T. Then the path ξ contains a vertex v' on the same level as v, so there exists $g \in G$ such that $g(v') = v$. But then $g(\xi) \in \partial T_v$, which shows that the orbit of ξ intersects every set of the form ∂T_v, hence it is dense in ∂T, and $G \curvearrowright \partial T$ is minimal.

Minimality implies topological transitivity, so it is enough to show that topological transitivity implies level-transitivity. Let v_1, v_2 be two vertices of the same level. By topological transitivity, there exists $g \in G$ such that $g(\partial T_{v_1}) \cap \partial T_{v_2} \neq \varnothing$. We have $g(\partial T_{v_1}) = \partial T_{g(v_1)}$. Since $g(v_1)$ and v_2 belong to the same level, the sets $\partial T_{g(v_1)}$ and ∂T_{v_2} are either disjoint or coincide. Since they have a non-empty intersection, they coincide, but this implies $g(v_1) = v_2$, which shows that $G \curvearrowright T$ is level-transitive. $\square$

Proposition 2.4.3. *Let $G \curvearrowright T$ be a level-transitive action (not necessarily faithful), and let $w \in \partial T$. Then the kernel of the action is equal to $\bigcap_{g\in G} g^{-1} G_w g$.*

Proof. We have $g^{-1}G_wg = G_{g^{-1}(w)}$, hence the elements of $\cap_{g\in G}g^{-1}G_wg$ fix the G-orbit of w pointwise. Since the action $G \curvearrowright \partial T$ is minimal, this implies that they act trivially on ∂T, hence trivially on T. $\square$

If the action $G \curvearrowright T$ is level-transitive, then for every $v \in L_n$ the stabilizer G_v has index $|L_n|$ in G. Note also that if $w = (v_0, v_1, \ldots) \in \partial T$ is a simple path starting in the root, then we have $G = G_{v_0} \geqslant G_{v_1} \geqslant G_{v_2} \geqslant \ldots$, and $[G_{v_n} : G_{v_{n+1}}]$ is equal to the branching index of vertices of the nth level. The intersection $\bigcap_{n\geqslant 1} G_{v_n}$ is equal to the stabilizer of w.

Conversely, if $G_0 = G \geqslant G_1 \geqslant G_2 \geqslant \cdots$ is a sequence of subgroups of finite index, then we can construct a spherically homogeneous tree T and an action of G on it such that each subgroup G_n is the stabilizer of a vertex v_n of a path $(v_0, v_1, \ldots) \in \partial T$. Namely, define the nth level L_n of the tree T as the set of cosets G/G_n, and connect two cosets gG_n and hG_{n+1} by an edge if and only if $gG_n \geqslant hG_{n+1}$. We leave it as an exercise for the reader to prove that we really get a rooted tree and that the natural actions of G on the sets of cosets G/G_n define a level transitive action of G on T, and that G_n is the stabilizer of the vertex v_n equal to the coset $1G_n$. We call T the *coset tree* for the chain $G \geqslant G_1 \geqslant G_2 \geqslant \ldots$.

Theorem 2.4.4. *Let G be a countable group. Then the following conditions are equivalent.*

(1) *There exists a faithful action of G on a rooted tree.*

(2) *There exists a faithful level-transitive action of G on a rooted tree.*

(3) *The group G is residually finite.*

Proof. If there exists a faithful action of G on a rooted tree, then the level stabilizers $\mathsf{Stab}_n(G)$ are finite index subgroups such that the intersection $\bigcap_{n=0}^{\infty} \mathsf{Stab}_n(G)$ is trivial, which proves that G is residually finite. This shows that (1) or (2) implies (3).

If G is residually finite and countable, then there exists a descending sequence $G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \ldots$ of finite index normal subgroups with trivial intersection. Consider the action of G on the associated coset tree. It is level-transitive, and G_i is equal to the ith level stabilizer, hence the action is faithful. We proved that (3) implies (1) and (2). $\square$

2.4.3. Action of cyclic groups of tree automorphisms. According to Proposition 2.4.2 an action of an automorphism g of a rooted tree T on the boundary ∂T is minimal if and only if the action of g is transitive on the levels of T.

The following is straightforward.

Lemma 2.4.5. *Let X^* be a level-homogeneous tree defined by a sequence $\mathsf{X} = (X_1, X_2, \ldots)$. An automorphism g of T is level-transitive if and only if for every n the section $g^{|X_1 \times \cdots \times X_n|}|_v$ acts transitively on X_{n+1}, where $v \in X_1 \times X_2 \times \cdots X_n$ is arbitrary.*

There is a particular class of automorphisms of rooted trees for which it is easy to decide if they are level transitive or not.

Consider the sequence $\mathsf{X} = (\mathbb{Z}/d_1\mathbb{Z}, \mathbb{Z}/d_2\mathbb{Z}, \ldots)$ of alphabets identified with cyclic groups. We say that an automorphism g of the rooted tree X^* *acts by cyclic permutations at vertices* if for every $v \in \mathsf{X}^*$ the action of $g|_v$ on the first level $\mathbb{Z}/d_{|v|+1}\mathbb{Z}$ is by $x \mapsto x + a$ for some $a \in \mathbb{Z}/d_{|v|+1}\mathbb{Z}$. The set of all such automorphisms is a group isomorphic (basically, by definition) to the infinite wreath product $\wr_{n \geqslant 1}\mathbb{Z}/d_n\mathbb{Z}$ see....

If $g \in \wr_{n \geqslant 1}\mathbb{Z}/d_n\mathbb{Z}$, then we denote by $\alpha_n(g)$, for $n = 0, 1, \ldots$, the sum $\sum_{v \in \mathsf{X}^n} a_v$, where $a_v \in \mathbb{Z}/d_{n+1}\mathbb{Z}$ are such that $g|_v(x) = x + a_v$ for every $x \in \mathbb{Z}/d_{n+1}\mathbb{Z}$. Denote $\alpha(g) = (\alpha_0(g), \alpha_1(g), \ldots)$. It is easy to check that $\alpha : \wr_{n \geqslant 1}\mathbb{Z}/d_n\mathbb{Z} \longrightarrow \prod_{n \geqslant 1}(\mathbb{Z}/d_n\mathbb{Z})$ is a homomorphism of groups. In fact, it is the abelianization homomorphism.

Proposition 2.4.6. *An automorphism $g \in \wr_{n \geqslant 1}\mathbb{Z}/d_n\mathbb{Z}$ of X^* is level-transitive if and only if $\alpha_n(g)$ is a generator of $\mathbb{Z}/d_{n+1}\mathbb{Z}$ for every $n \geqslant 0$.*

Proof. A direct corollary of Lemma 2.4.5. □

Let g be an automorphism of an arbitrary locally finite rooted tree T. Let $\langle g \rangle \backslash T$ be the graph whose vertices are g-orbits of the action on the set of vertices of T, and where two vertices are connected by an edge if and only if the corresponding orbits contain vertices connected by an edge in T. Note that every g-orbit belongs to one level of T, hence the vertices of $\langle g \rangle \backslash T$ are also naturally partitioned into levels, and vertices connected by an edge belong to neighboring levels. Since every vertex of T except for the root is connected to a unique vertex of the previous level, the same is true for the graph $\langle g \rangle \backslash T$, hence it is also a tree. Let us label the vertices of $\langle g \rangle \backslash T$ by the cardinalities of the corresponding orbits. The obtained rooted labeled tree is called the *tree of orbits* of g.

The following is proved in ...

Theorem 2.4.7. *Two automorphisms g_1, g_2 of a rooted tree T are conjugate in $\operatorname{Aut} T$ if and only if their trees of orbits are isomorphic. In particular, any two level-transitive automorphisms of T are conjugate.*

Each point of the boundary of $\langle g \rangle \backslash T$ is an infinite rooted path in the tree of orbits, and its preimage in T is a g-invariant rooted subtree of T on which g acts level-transitively. The boundary of this subtree is a minimal closed

g-invariant subset of ∂T. We see that ∂T is decomposed into a disjoint union of closed minimal g-invariant subsets, and the boundary of the tree of orbits $\langle g\rangle\backslash T$ can be interpreted as the set of minimal closed g-invariant subsets of ∂T.

The action of $\mathbb{Z}$ on the coset tree of a sequence $\mathbb{Z} \geqslant d_1\mathbb{Z} \geqslant d_1d_2\mathbb{Z} \geqslant d_1d_2d_3\mathbb{Z} \geqslant \ldots$ is level-transitive for every sequence d_i of positive integers. The branching index of a vertex of the nth level of this tree is equal to d_{n+1}. It follows that every level-transitive cyclic group of automorphisms of a rooted tree is conjugate by an isomorphism of rooted trees with the action of $\mathbb{Z}$ on some of its coset trees.

2.4.4. Residually finite actions. Actions of groups on boundaries of rooted trees can be characterized in purely topological terms in the following way.

Definition 2.4.8. A group action $G \curvearrowright \mathcal{X}$ on a Cantor set $\mathcal{X}$ is said to be *residually finite* if the G-orbit of every clopen subset of $\mathcal{X}$ is finite.

The following is proved in [**GNS00**, Proposition 6.4].

Theorem 2.4.9. *An action $G \curvearrowright \mathcal{X}$ on a Cantor set is residually finite if and only if it is topologically conjugate to the action $G \curvearrowright \partial T$ for some action of G by automorphisms of a rooted tree T.*

Proof. Let us prove at first the following lemma.

Lemma 2.4.10. *Let $G \curvearrowright \mathcal{X}$ be a residually finite action. Then for every finite clopen cover of $\mathcal{X}$ there exists a subordinate finite G-invariant clopen partition of $\mathcal{X}$.*

Proof. Since the G-orbit of every clopen set is finite, every finite clopen cover $\mathcal{F}$ is contained in a G-invariant finite clopen cover $\mathcal{F}_1$. Consider the Boolean algebra generated by $\mathcal{F}_1$ (for the usual set-theoretic operations). It is finite, and is equal to the set of all unions of its atoms. The set of all atoms will be a finite clopen partition of $\mathcal{X}$. It is G-invariant, since the algebra is G-invariant. $\square$

Let $\mathcal{U} = \{U_1, U_2, \ldots,\}$ be a countable basis of topology on $\mathcal{X}$. Define inductively G-invariant partitions L_n of $\mathcal{X}$ into clopen subsets in the following way. Set $L_0 = \{\mathcal{X}\}$. If L_n, for $n \geqslant 0$, is defined, consider the set $L'_{n+1} = L_n \cup \{U_{n+1}\}$, and find a finite clopen G-invariant partition L_{n+1} subordinate to L'_{n+1}, which exists by Lemma 2.4.10.

The partition L_{n+1} is a refinement of L_n, and the set U_{n+1} is a union of elements of L_{n+1}. It follows that $\bigcup_{n=0}^{\infty} L_n$ is a basis of topology on $\mathcal{X}$. Consider the ordering of $\bigcup_{n=0}^{\infty} L_n$ by inclusion, and let T be the Hasse diagram of the ordering. Since L_{n+1} is a refinement of L_n for every n, the diagram T is a rooted graph. We have $V_1 \preceq V_2$ if and only if $V_1 \supseteq V_2$.

The group G acts on T, since the sets L_n are G-invariant. The action is faithful, since the union of the sets L_n is a basis of topology on $\mathcal{X}$, and the action $G \curvearrowright \mathcal{X}$ is faithful. Every point $w \in \partial T$ is a sequence $(U_0, U_1, \ldots)$ of elements of L_i such that $U_{i+1} \subset U_i$. Since each cover L_n is disjoint, and their union is a basis of topology, the intersection $\bigcap_{n \geqslant 0} U_n$ is a singleton, which we will denote $\psi(w)$. We get a map $\psi : \partial T \longrightarrow \mathcal{X}$. It is easy to see that it is G-equivariant. We leave it to the reader as an exercise to show that ψ is a homeomorphism. □

Example 2.4.11. Let G be a profinite group, i.e., a compact group with a basis of neighborhood of the identity consisting of subgroups of finite index. A clopen subset $U \subset G$ is hence a union of cosets $g_i H$ of a subgroup $H \leqslant G$ of finite index. It follows that every set of the form gU, $g \in G$, is a union of cosets of H. Consequently, the orbit of U is finite. It follows that if G is homeomorphic to the Cantor set, then the action of G on itself by left multiplication is residually finite, i.e., conjugate to an action of G on the boundary of a rooted tree. Similarly, the action of any subgroup of G on G is residually finite.

Example 2.4.12. A particular instance of Example 2.4.11 is the *odometer* or *adding machine* action defined in 1.1.4. It is the action of $\mathbb{Z}$ on the profinite group $\mathbb{Z}_2$ of dyadic integers. The corresponding action on the tree can be defined as the action on the coset tree defined by the sequence $\mathbb{Z} > 2\mathbb{Z} > 2^2\mathbb{Z} > 2^3\mathbb{Z} > \ldots$.

Another classical description of residually finite actions uses the notion of an equicontinuous action.

Definition 2.4.13. An action $G \curvearrowright \mathcal{X}$ of a group on a metric space is said to be *equicontinuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ for $x, y \in \mathcal{X}$, then $d(g(x), g(y)) < \epsilon$ for all $g \in G$.

Note that equicontinuity depends only on the uniformity defined by the metric. In particular, if $\mathcal{X}$ is compact, then it does not depend on the choice of the metric. Any action by isometries is obviously equicontinuous. On the other hand, an expansive action is not equicontinuous.

Proposition 2.4.14. *An action $G \curvearrowright \mathcal{X}$ of a group on a Cantor set is residually finite if and only if it is equicontinuous.*

Proof. Since the Cantor set is compact, equicontinuity does not depend on the choice of the metric. In particular, we may assume that the metric d is an *ultrametric*, i.e., that it satisfies $d(x, z) \leqslant \max(d(x, y), d(y, z))$ for any

$x, y, z \in \mathcal{X}$ (for instance, the classical metric ... is an ultrametric). Define $d_G(x, y) = \sup_{g \in G} d(g(x), g(y))$. Since $\mathcal{X}$ is compact, $d_G(x, y)$ is bounded. We also have for any $x, y, z \in \mathcal{X}$:

$$\begin{aligned} d_G(x, z) &= \sup_{g \in G} d(g(x), g(z)) \leqslant \sup_{g \in G} \max(d(g(x), g(y)), d(g(y), g(z))) \\ &= \max\left(\sup_{g \in G} d(g(x), g(y)), \sup_{g \in G} d(g(y), g(z))\right) = \max(d_G(x, y), d_G(y, z)), \end{aligned}$$

i.e., d_G is an ultrametric.

For every $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $d_G(x, y) < \epsilon$. We also have $d_G(x, y) \geqslant d(x, y)$ for all $x, y \in \mathcal{X}$. It follows that d and d_G define the same topologies on $\mathcal{X}$.

It follows from the definition of an ultrametric that if $d_G(x, y) < \epsilon$, then the open balls of radius ϵ with centers in x and y coincide. In other words, two open balls of radius ϵ either coincide or are disjoint. Moreover, two balls of different radii either are disjoint or one is a subset of the other. In particular, the set of all open balls of radius ϵ is finite. The metric d_G is G-invariant, so G permutes the balls of a given radius. Also not that every clopen subset of $\mathcal{X}$ is a finite union of open balls. Consequently, the G-orbit of every clopen set is finite. □

2.4.5. Graphs of action. Let G be a group acting on a rooted tree T, and let S be a finite generating set of G. The group G acts by permutations on each of the levels L_n of the tree T. Denote by Γ_n the graph of the action. If the action is level-transitive, then Γ_n is the Schreier graph of G modulo the stabilizer of a point of L_n.

Let $p_n : L_{n+1} \longrightarrow L_n$ be the natural map defined by the condition that $p_n(v)$ is the parent of v (i.e., the unique vertex of L_n such that the path connecting the root with v passes through $p_n(v)$). The following is straightforward.

Lemma 2.4.15. *The map $p_n : L_{n+1} \longrightarrow L_n$ extended to the sets of edges of Γ_n by the rule $p_n(s, v) = (s, p_n(v))$ is a covering of labeled graphs.*

We get thus an inverse sequence of coverings of finite graphs

$$\Gamma_0 \xleftarrow{p_0} \Gamma_1 \xleftarrow{p_1} \Gamma_2 \xleftarrow{p_2} \cdots$$

The inverse limit of this sequence is the graph of the action of G on the boundary ∂T of the tree.

Example 2.4.16. Consider the adding machine action of $\mathbb{Z}$ on the coset tree of the sequence $\mathbb{Z} > 2\mathbb{Z} > 2^2\mathbb{Z} > 2^3\mathbb{Z} > \ldots$, and the generating set $S = \{1\}$ of $\mathbb{Z}$. Then the graphs Γ_n are cycles of 2^n vertices coinciding with the Cayley

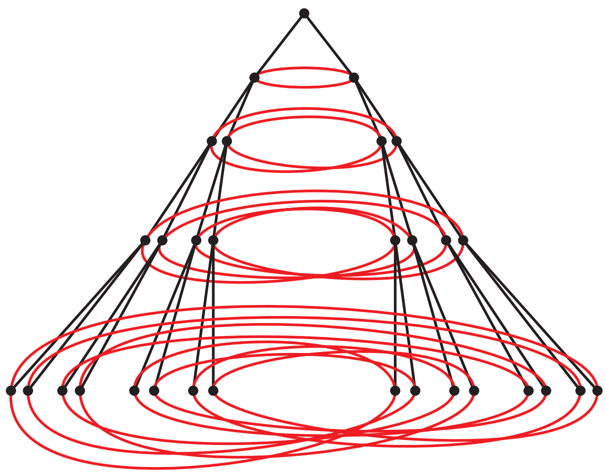

Figure 2.20. Odometer graphs

graphs of the cyclic groups $\mathbb{Z}/2^n\mathbb{Z}$. The coverings $p_n : \Gamma_{n+1} \longrightarrow \Gamma_n$ are the natural double covering maps $x \mapsto x : \mathbb{Z}/2^{n+1}\mathbb{Z} \longrightarrow \mathbb{Z}/2^n\mathbb{Z}$, i.e., the natural epimorphisms from $\mathbb{Z}/2^{n+1}\mathbb{Z}$ to $\mathbb{Z}/2^n\mathbb{Z}$. See the graphs Γ_n for $n = 1, 2, 3, 4$ on Figure 2.20.

We see that the inverse limit of the graphs Γ_n, i.e., the graph of the action of $\mathbb{Z}$ on the boundary $\mathbb{Z}_2$ of the coset tree coincides with the Smale-Williams solenoid described in 1.1.4.

The abstract connected components of the inverse limit are the orbital graphs of the action of G on ∂T.

Proposition 2.4.17. *Let $(v_0, v_1, \ldots)$ be a path representing a point $w \in \partial T$, where $v_n \in L_n$. Then the rooted graph Γ_w of the action of G on the orbit of w is isomorphic to the limit of the rooted orbital graphs Γ_{v_n} of the action of G on the orbit of v_n.*

Proof. If $g \in G$ fixes w, then it fixes v_n for every n. On the other hand, if $g \in G$ moves w to a different point of the boundary, then there exists n such that $g(v_n) \neq v_n$. $\square$

Conversely, suppose that

$$\Gamma_0 \xleftarrow{p_0} \Gamma_1 \xleftarrow{p_1} \Gamma_2 \xleftarrow{p_2} \cdots$$

is a sequence of perfectly labeled by a set S graphs and covering maps, such that Γ_0 has one vertex. Let G be the group defined by the disjoint union of the graphs Γ_n (see 2.1.1). Consider the tree T whose set of vertices is the disjoint union of the sets of vertices of the graphs Γ_n, and where a vertex $v \in \Gamma_{n+1}$ is connected to the vertex $p_n(v)$. Then T is a tree and G acts on it by automorphisms. The graph Γ_n is the graph of the action of G on the nth level of the tree.

Example 2.4.18. Consider the punctured plain $\mathcal{M} = \mathbb{C}\backslash\{0, -1\}$. Check that for $f(z) = z^2 - 1$ we have $f^{-1}(\mathcal{M}) \subset \mathcal{M}$, and that $f^n : f^{-n}(\mathcal{M}) \longrightarrow \mathcal{M}$ are covering maps. Choose a point $t \in \mathcal{M}$, and consider two generators a and b of the fundamental group $\pi_1(\mathcal{M}, t)$, which are loops going around 0 and -1. Let Γ_0 be the graph with one vertex t and two loops a and b, labeled accordingly. Then $f : f^{-n}(\mathcal{M}) \longrightarrow f^{-(n-1)}(\mathcal{M})$ restricts to a covering map of labeled graphs $\Gamma_n \longrightarrow \Gamma_{n-1}$, where $\Gamma_n = f^{-n}(\Gamma_0)$. The obtained sequence of graphs and maps defines a group acting on a binary rooted tree, called the *iterated monodromy group* of $z^2 - 1$. We will study iterated monodromy groups in Chapter 4.

Example 2.4.19. This is an example from ... Consider the left Cayley graph K of the free product $G = \langle a, b, c \mid a^2 = b^2 = c^2 = 1\rangle \cong C_2 * C_2 * C_2$ with the labeling of the edges by the generators a, b, c. Choose a bijection $n \leftrightarrow e_n$ of the set of edges of K with natural numbers. We will construct a sequence of subtrees of K spanned by finite subsets $A_n \subset T$ in the following recurrent way. Let $A_0 = \{1\}$. If A_n is defined, then let $e_{m_n} = (g_n, x_n g_n)$ for $g \in G$ and $x \in \{a, b, c\}$ be the edge connecting a vertex of A_n with a vertex not in A_n with the smallest possible m. Then define $A_{n+1} = A_n \cup A_n^{-1} x_n g_n$, see Figure... Let Γ_n be the graph obtained from the tree spanned by A_n by attaching the necessary loops to the leaves, so that we get a graph perfectly labeled by $\{a, b, c\}$. We have then natural covering maps $p_n : \Gamma_{n+1} \longrightarrow G_n$ folding the edge e_{m_n} into a loop, and mapping to the vertex $h \in A_n$ the vertices h and $h^{-1} x_n g_n$ of A_{n+1}.

We get hence an action of G on a binary rooted tree defined by the sequence of graphs and coverings $p_{n+1} : \Gamma_{n+1} \longrightarrow \Gamma_n$. One of the ends of this rooted tree is the constant sequence $1, 1, \ldots$ of identity element of G. Since all edges of K are eventually included into Γ_n, the orbital graph of this end coincides with the Cayley graph K of the group G. In particular, the action of G on the binary rooted tree is faithful. Note that this proves that G is residually finite.

2.4.6. Finite-state automorphisms of rooted trees.

Definition 2.4.20. An automaton $\mathcal{A} = (\mathsf{X}, \mathsf{Y}, Q, q_0, \pi, \lambda)$ is said to be *synchronous* if the values of the output function $\lambda(q, x)$ is always a single-letter word, i.e., if λ is a map from $Q \times \mathsf{X}$ to Y.

It is easy to see that composition of two synchronous automata, as defined in Proposition 2.3.5 is synchronous.

If the automaton is synchronous, then for every input word $x_1 x_2 \ldots x_n$ and every state $q \in Q$ the output word $\lambda(q, x_1 x_2 \ldots x_n)$ is a word $y_1 y_2 \ldots y_n$

of the length equal to the length of the input word. Moreover, the beginning of length k of $y_1 y_2 \dots y_n$ depends only on q and the beginning of the length k of the input word $x_1 x_2 \dots x_n$. It follows that the map $\lambda(q, *) : \mathsf{X}^* \longrightarrow \mathsf{Y}^*$ is level-preserving morphism of rooted trees. If it is invertible, then it is an isomorphism of the rooted trees.

Conversely, it is easy to see that every level-preserving morphism of rooted trees $\mathsf{X}^* \longrightarrow \mathsf{Y}^*$ is defined by an initial synchronous automaton. The group of all automorphisms of the rooted tree X^* is therefore isomorphic to the group of all invertible transformations defined by synchronous automata with the input and output alphabet X. A synchronous automaton $(\mathsf{X}, Q, q_0, \pi, \lambda)$ defines an invertible transformation (of X^* or, equivalently, of X^ω) if and only if for every state q accessible from q_0 (i.e., such that $q = \pi(q_0, v)$ for some $v \in \mathsf{X}^*$) the transformation $x \mapsto \lambda(q, x)$ is a permutation of X, see...

The set of all automorphisms $\mathsf{X}^* \longrightarrow \mathsf{X}^*$ defined by *finite* synchronous automata is a group, which we call the group of *finite-state automorphisms of the tree*, or the *group of finite synchronous automata* over the alphabet X.

Examples of groups that can be embedded into the group of finite synchronous automata....

Example 2.4.21. Free abelian groups...

Example 2.4.22. Linear groups...

Example 2.4.23. Free groups (Aleshin and Belaterra examples)...

2.4.7. Self-similar groups. Refer to [**Nek05**]...

Definition 2.4.24. Let X be a finite alphabet. A faithful action of a group $G \curvearrowright \mathsf{X}^*$ on the rooted tree X^* is *self-similar* if for every $g \in G$ and $x \in \mathsf{X}$ the element $g|_x$ belongs to G.

Recall that $g|_x$ is the automorphism of X^* uniquely determined by the condition that

$$g(xv) = g(x) g|_x(v)$$

for all $v \in \mathsf{X}^*$, see 2.4.1.

Suppose that $G \curvearrowright \mathsf{X}^*$ is self-similar. We can interpret then G as the set of states of an automaton with the output function $\lambda(g, x) = g(x)$ and the transition function $\pi(g, x) = g|_x$. Then the action of the automaton with the initial state g on finite words coincides with the original action of $g \in G$. We call this automaton the *full automaton* of the action.

Example 2.4.25. The full automorphism group $\mathrm{Aut}\,\mathsf{X}^*$ of the rooted tree X^* is obviously self-similar. In particular, the section $g|_v$ is defined for any $g \in \mathrm{Aut}\,\mathsf{X}^*$.

Example 2.4.26. A subset $S \subset \mathrm{Aut}\,\mathsf{X}^*$ is *self-similar* if $g|_x \in S$ for every $g \in S$ and $x \in \mathsf{X}$. If S is self-similar, then the group generated by S is self-similar.

Self-similar sets are basically the same as invertible non-initial automata. So, if $\mathcal{A} = (\mathsf{X}, Q, \pi, \lambda)$ is an invertible non-initial automaton, then the group generated by the initial automata $\mathcal{A}_q = (\mathsf{X}, Q, q, \pi, \lambda)$ for all $q \in Q$ is self-similar. This is a standard method of defining self-similar groups, especially if the automaton $\mathcal{A}$ is finite. Self-similar groups obtained this way are called sometimes *automaton groups*.

Example 2.4.27. Aleshin free group?...

2.4.8. Wreath recursion. Let $H \curvearrowright \mathsf{X}$ be a group acting on a set, and let G be a group. The *permutational wreath product* of the action $H \curvearrowright \mathsf{X}$ and the group G is the semidirect product $H \ltimes G^{\mathsf{X}}$, where H acts on G^{X} by permuting the coordinates by the action $H \curvearrowright \mathsf{X}$. It is usually denoted $G \wr_{\mathsf{X}} H$, though sometimes there is inconsistency in the order of the factors.

In particular, if H is the symmetric group $\mathsf{S}(\mathsf{X})$, then we have the natural permutational wreath product $\mathsf{S}(\mathsf{X}) \ltimes G^{\mathsf{X}}$. We leave the next proposition as an exercise for the readers.

Proposition 2.4.28. *Let $G \curvearrowright \mathsf{X}^*$ be a self-similar action. For $g \in G$, let $\sigma_g \in \mathsf{S}(\mathsf{X})$ be the action of g on the first level $\mathsf{X} \subset \mathsf{X}^*$ of the tree. Then the map*

$$\psi : g \mapsto (\sigma_g, (g|_x)_{x \in \mathsf{X}})$$

is a homomorphism $\psi : G \mapsto \mathsf{S}(\mathsf{X}) \ltimes G^{\mathsf{X}}$. Here $(g|_x)_{x \in \mathsf{X}}$ is considered to be an element of G^{X} (as the function $x \mapsto g|_x$). The homomorphism ψ is injective.

Definition 2.4.29. The homomorphism from Proposition 2.4.28 is called the *wreath recursion* associated with the self-similar group.

Wreath recursions are convenient compact ways of describing the automaton generating a self-similar group. The values of the wreath recursion on the generators uniquely determine the wreath recursion, hence they uniquely determine the action of the group on the first level, and the sections of the elements, i.e., they completely determine the output and transitions in the full automaton of the action. Therefore, every finitely generated self-similar group is uniquely determined by a finite collection of equalities of the form

$$\begin{cases} \psi(s_1) &= \sigma_1(g_{1,1}, g_{2,1}, \dots, g_{d,1}) \\ \psi(s_2) &= \sigma_2(g_{1,2}, g_{2,2}, \dots, g_{d,2}) \\ \quad\vdots & \\ \psi(s_k) &= \sigma_k(g_{1,k}, g_{2,k}, \dots, g_{d,k}), \end{cases}$$

where $\{s_1, s_2, \dots, s_k\}$ is a generating set of the group, $\sigma_i \in \mathsf{S}(d)$, and $g_{i,j}$ are elements of the group, i.e., products of the generators s_i and their inverses. Here we identify X with $\{1, 2, \dots, d\}$, and write the elements of G^{X} as ordered strings of elements of G. Sometimes we identify X with $\{0, 1, \dots, d-1\}$.

2.4.9. Weakly branch groups.

Definition 2.4.30. Let $G \curvearrowright T$ be an action on a rooted tree. For a vertex $v \in T$, the *rigid stabilizer* $G[v]$ is the group of elements $g \in G$ acting trivially on the complement of T_v. The nth level *rigid stabilizer* $\mathsf{Rist}_n(G)$ is the group generated by $\bigcup_{v \in L_n} G[v]$.

Note that $\mathsf{Rist}_n(G)$ is naturally isomorphic to the direct product $\prod_{v \in L_n} G[v]$, since $G[v_1]$ and $G[v_2]$ commute if v_1 and v_2 are incomparable. Note also that $G[v] = G[\partial T_v]$, where $G[\partial T_v]$ is defined for the action $G \curvearrowright \partial T$ as in Definition 2.2.1. The nth level rigid stabilizer $\mathsf{Rist}_n(G)$ is a normal subgroup of G.

Definition 2.4.31. A subgroup $G \leqslant \operatorname{Aut} T$ is *weakly branch* if it is level-transitive and $G[v]$ is non-trivial for every vertex v. It is called *branch* if $\mathsf{Rist}_n(G)$ has finite index in G for every n.

A level-transitive action is weakly branch if and only if it is level transitive and localizable in the sense of Definition 2.2.1.

Example 2.4.32. The group of all automorphisms of a level-transitive tree is obviously branch, since in this case the level stabilizers coincide with the rigid level stabilizers.

Example 2.4.33. This example is a particular case of a construction by Peter Neumann... Let A be a transitive subgroup of $\mathsf{S}(\mathsf{X})$. Suppose that the stabilizers A_x of points $x \in \mathsf{X}$ are *perfect*, i.e., that $[A_x, A_x] = A_x$. Also suppose that the union of the subgroups $[A_{x_1} \cap A_{x_2}, A_{x_1} \cap A_{x_2}]$ for $x_1 \neq x_2$ generates A. For instance, we can take $A = \mathsf{A}(\mathsf{X})$ for $|\mathsf{X}| \geqslant 6$.

For every $g \in A$ and $a \in \mathsf{X}$ such that $g(a) = a$, define an automorphism $t_{(a,g)}$ of the tree X^* by the recurrent rules

$$t_{(a,g)}(xv) = \begin{cases} a t_{(a,g)}(v) & \text{if } x = a, \\ g(x)v & \text{otherwise.} \end{cases}$$

Let $\mathcal{P}_A$ be the group generated by all the elements $t_{(a,g)}$.

Let us show that $\mathcal{P}_A$ is branch. Let $h_1, h_2 \in A_{x_1} \cap A_{x_2}$ for $x_1 \neq x_2$. Then it is easy to check that $[h_1, h_2]$ changes only the first letter of every word $v \in \mathsf{X}^*$. Since the derived subgroups of $A_{x_1} \cap A_{x_2}$ for all $x_1 \neq x_2$ generate A, we conclude that $\mathcal{P}_A$ contains the group acting as A on the first letter, and not changing the remaining letters. We identify this group with A. It follows that $\mathcal{P}_A$ contains the elements $g^{-1}t_{(a,g)}$, which act identically on the first level and all subtrees $x\mathsf{X}^*$ for $x \neq a$, and as $t_{(a,g)}$ on the subtree $a\mathsf{X}^*$. Conjugating the elements $g^{-1}t_{(a,g)}$ by elements of A, we conclude that the first level stabilizer of $\mathcal{P}_A$ coincides with the first level rigid stabilizer, and is naturally identified with $\mathcal{P}_A^{\mathsf{X}}$. This proves by induction that all rigid level stabilizers in $\mathcal{P}_A$ coincide with the corresponding level stabilizers, hence $\mathcal{P}_A$ are brach.

Example 2.4.34. As an example of a weakly branch group, consider the group G (isomorphic to IMG $\left(z^2 - 1\right)$, see...) generated by two automorphisms a, b of the binary tree $\mathsf{X}^* = \{\mathsf{0}, \mathsf{1}\}^*$ satisfying the conditions

$$a = \sigma(1, b), \qquad b = (1, a),$$

see Example 2.4.1 and a comment before it for an explanation of this notation.

Note that as $a^2 = (b, b), b = (1, a), a^{-1}ba = (a, 1)$, the restriction of the first level stabilizer to the subtrees $\mathsf{0}\mathsf{X}^*$ and $\mathsf{1}\mathsf{X}^*$ contains G, and hence coincides with G. We have $[a^2, b] = (1, [b, a])$. Conjugating $[a^2, b]$ by all elements of the first level stabilizer, we will get all elements of the form $(1, [b, a]^g)$ for $g \in G$. It follows that $[G, G]$ contains $(1, [G, G])$. Conjugating by a, we conclude that $[G, G]$ contains $([G, G], 1)$, hence $[G, G]$ contains the subgroup $[G, G] \times [G, G]$ of the first level rigid stabilizer. We conclude by induction that the derived subgroup $[G, G]$ contains the subgroup $[G, G]^{\mathsf{X}^n}$ of the nth level rigid stabilizers, hence all rigid stabilizers are non-trivial, and the group G is weakly branch.

Example 2.4.35. Let us analyze in a similar way the group G (isomorphic to IMG $\left(z^2 + i\right)$, see...) generated by the elements

$$a = \sigma, \quad b = (a, c), \quad c = (b, 1).$$

We have the following elements of the first level stabilizer: $b = (a, c), c = (b, 1), c^a = (1, b)$, which, in the same way as in the previous example shows that restrictions of the first level stabilizer to the subtrees $\mathsf{0}\mathsf{X}^*$ and $\mathsf{1}\mathsf{X}^*$ coincide with G. Let N be the normal closure of $\{[a, b], [b, c]\}$, i.e., the group generated by the union of the conjugacy classes of these two elements. We have $[b, c] = ([a, b], 1)$, $[b^a, c] = ([c, b], 1)$. Note that $[b^a, c]$ belongs to N, since commutation of b with a and c implies $[b^a, c] = [b, c] = 1$. In the same way as in the previous example, we conclude that N contains $N \times N$, and hence the nth level rigid stabilizer contains N^{X^n}. Let us show that G/N is finite, which will imply that G is branch. It is easy to see that $a^2 = b^2 = c^2 = 1$. As b commutes with a and c, we can write every element of G modulo N as a product of the form gb or g for $g \in \langle a, c\rangle$. But it is checked directly that $(ac)^4 = 1$ in G, hence the group $\langle a, c\rangle$ is of order 8. It follows that G/N is at most of order 16.

We will see more examples of branch and weakly brach groups later...

2.4.10. Normal subgroups. The following is a direct corollary of Proposition 2.2.4.

Proposition 2.4.36. *Suppose that $G \leqslant \mathrm{Aut}\, T$ is weakly branch, and let N be a non-trivial normal subgroup of G. Then there exists n such that $[\mathsf{Rist}_n(G), \mathsf{Rist}_n(G)] \leqslant N$.*

Note that it follows that if $[\mathsf{Rist}_n(G), \mathsf{Rist}_n(G)]$ has finite index in G (equivalently, in $\mathsf{Stab}_n(G)$) for every n, then every proper quotient of G is finite. Groups which have only finite proper quotients are called *just-infinite*.

We will see later that, for example, the Grigorchuk group ... is an example of a branch just-infinite group.

Example 2.4.37. Let us show that the group from Example 2.4.35 is just-infinite. We know that $\mathsf{Rist}_n(G)$ contains the subgroup N^{X^n}, where N is the normal closure of $[b, a]$ and $[b, c]$ in G. We know that N has finite index in G, so it is enough to show that $[N, N]$ has finite index in N. The group N is generated by conjugates of the elements $[a, b] = abab$ and $[b, c] = bcbc$. It is checked directly that $(bc)^4 = ((ab)^4, c^4) = ((ab)^4, 1)$ and $(ab)^4 = (\sigma(a, c))^4 = (ca, ac)^2 = ((ca)^4, (ac)^4)$, and $(ac)^4 = (\sigma(b, 1))^4 = (b, b)^2 = 1$, hence $[a, b]^2 = [b, c]^2 = 1$, so that N is generated by elements of order 2, which implies that all elements of the abelian group $N/[N, N]$ are of order 2. The group N is finitely generated, since it has a finite index in a finitely generated group G. Consequently, $N/[N, N]$ is finitely generated abelian in which all elements are of order two, hence it is finite.

Example 2.4.38. Not every branch group is just-infinite. For example,...

For more on just-infinite and branch groups, see [**BGŠ03**].

2.4.11. Rigidity. The following theorem is proved in [**LN02**, Proposition 6.2]. We use a shorter argument from [**Röv99**, Lemma 5.7]. See its exposition in [**Nek05**, Theorem 2.10.1]. We also rewrite it in the spirit of the proof of Theorem 2.2.12, and instead of just finding the homeomorphism conjugating the actions, we show how the topological space (the boundary of the tree) can be reconstructed from the algebraic structure of the group. We hope that this makes the original proofs more natural.

Theorem 2.4.39. *Let $G_i \leqslant \operatorname{Aut} T_i$ be weakly branch groups. Then every isomorphism $\phi : G_1 \longrightarrow G_2$ is induced by a homeomorphism $\partial T_1 \longrightarrow \partial T_2$.*

Note that, contrary to what is said in [**Nek05**, p. 51], it actually seems that Theorem 2.4.39 in its full generality does not follow from Rubin's theorems [**Rub89**] directly.

Proof. Let $G \curvearrowright \mathcal{X}$ be a minimal localizable residually finite action on a Cantor set.

Definition 2.4.40. Let $G \curvearrowright \mathcal{X}$ be a residually finite action on a Cantor set. A *basic clopen set* is a subset $U \subset \mathcal{X}$ such that for every $g \in G$ either $g(U) = U$ or $g(U) \cap U = \varnothing$.

For example, if $G \curvearrowright T$ is an action on a rooted tree, then for every vertex v the subset ∂T_v of ∂T is a basic clopen set. Note that it follows from Lemma 2.4.10 that every clopen subset of $\mathcal{X}$ is a disjoint union of a finite number of basic sets. In particular, the set of basic sets is a basis of topology of $\mathcal{X}$.

We want to characterize in purely group-theoretic terms all subgroups of the form $G[U]$, where U is a basic clopen subset of $\mathcal{X}$. Here, as in 2.2, $G[U]$ denotes the set of elements of G acting trivially on the complement of U. If $\mathcal{X} = \partial T$ and G acts on the rooted tree T by automorphisms, then we have $G[v] = G[\partial T_v]$ for vertices v of T.

Definition 2.4.41. We say that a subgroup $H \leqslant G$ is a *basic subgroup* if the following conditions are satisfied.

(1) The number of subgroups of G conjugate to H is finite.
(2) Two subgroups conjugate to H commute if and only if they do not coincide, if and only if their intersection is trivial.
(3) If $h \notin H$, then there exists $g \in G$ such that $H^g \neq H$ and $[H_1^g, h] \neq 1$ for every finite index subgroup H_1 of H.

Equivalently, we can define an *abstract rigid stabilizer* as a non-abelian normal subgroup $R \lhd G$ together with a decomposition $R = H_1 \times H_2 \times \cdots \times H_n$ into a direct product of subgroups such that

(1) The group G acts transitively on the set of factors H_i by conjugation.
(2) For every $j \in \{1, 2, \ldots, n\}$ and every finite index subgroups $\tilde{H}_i < H_i$ the centralizer $\mathcal{Z}_G(\prod_{i \neq j} \tilde{H}_i)$ is equal to H_j.

It is easy to see that $H \leqslant G$ is a basic subgroup if and only if it is a factor of an abstract rigid stabilizer.

Lemma 2.4.42. *For every basic clopen subset $V \subset \partial T$ the subgroup $G[V]$ is a basic subgroup.*

Proof. We have $gG[V]g^{-1} = G[g(V)]$ for every $V \subset \partial T$ and $g \in G$. Consequently, the number of subgroups conjugate to $G[V]$ is not more than the size of the orbit of V, which is finite for every clopen subset $V \subset \partial T$. This proves condition (1) of Definition 2.4.41.

The subgroups $G[V]$ and $G[g(V)]$ obviously coincide if $V = g(V)$. If $V \neq g(V)$, then by definition of basic clopen subsets, $V \cap g(V) = \varnothing$, so $G[V]$ and $G[g(V)]$ act on disjoint sets, hence they commute and their intersection is trivial. On the other hand, for every non-trivial element $g \in G[V]$ there exists a basic clopen set $U \subset V$ such that $g(U) \cap U = \varnothing$, since the set of basic sets is a basis of the topology of $\mathcal{X}$. Let $h \in G[U] \leqslant G[V]$ be any non-trivial element (which exists, since G is weakly branch). Then $[g, h] \neq 1$, see the proof of Lemma 2.2.15. It follows that $[G[v], G[v]] \neq 1$, which finishes the proof of condition (2).

Suppose that $h \notin G[V]$. Then h moves a point of $\partial T \setminus V$, hence there exists a basic clopen set U such that $h(U) \neq U$, and $(U \cup h(U)) \cap V = \varnothing$. Let V' be an element of the orbit of V such that $V' \cap U \neq \varnothing$ (which exists, since we assume that the action of G on ∂T is level-transitive, hence topologically transitive). Let $g \in G$ be such that $g(V') = V$. Then $G[V]^g = G[V']$. Let H_1 be any finite index subgroup of $G[V]$. Then H_1^g is a finite index subgroup of $G[V']$. We want to show that $[H_1^g, h] \neq 1$. Let f be a non-trivial element of $G[V' \cap U] \cap H_1^g$. It exists, since $G[V' \cap U] \leqslant G[V']$ is infinite, and H_1^g has finite index in $G[V']$. Then $[f, h] \neq 1$, hence $[H_1^g, h] \neq 1$. □

Let $H \leqslant G$ be an arbitrary basic subgroup. Denote by L_H the set of subgroups of G conjugate to H. It is a finite set, by condition (1) of Definition 2.4.41. Denote $m_H = |L_H|$. The group G acts on L_H by conjugation. Denote by Stab_H the kernel of the action. It is the intersection of the normalizers of the elements of L_H. Denote by Rist_H the subgroup of G generated by all conjugates of H. We have $\mathsf{Rist}_H \cong H^{m_H}$. If $H = G[v]$, then L_H is the level of v, Stab_H is the level stabilizer, m_H is the number of the vertices in the level of v, and Rist_H is the rigid stabilizer of the level of v.

We say that a subgroup H_2 *moves* a subgroup H_1 if there exists $g \in H_2$ such that $H_1^g \neq H_1$.

Lemma 2.4.43. *Suppose that H_1 and H_2 are basic subgroups. If H_2 moves H_1, then*

$$H_1 \cap \mathsf{Stab}_{H_2} \leqslant H_2.$$

Proof. Suppose that, on the contrary, there exists $g \in (H_1 \cap \mathsf{Stab}_{H_2}) \setminus H_2$. Since $g \notin H_2$, there exists $f \in G$ such that $H_2^f \neq H_2$ and $[H_2^f \cap \mathsf{Stab}_{H_1}, g] \neq 1$ (see condition (3) of Definition 2.4.41). Let $h_1 \in H_2^f \cap \mathsf{Stab}_{H_1}$ be such that $[g, h_1] \neq 1$.

We have $[g, h_1] = g^{-1} \cdot h_1^{-1} g h_1$, and $h_1 \in \mathsf{Stab}_{H_1}$, $g \in H_1$, hence $[g, h_1] \in H_1$.

Let $h_2 \in H_2$ be such that $H_1^{h_2} \neq H_1$. Then $H_1^{h_2} \cap H_1 = \{1\}$, hence $[[g, h_1], h_2] \neq 1$.

On the other hand, $[h_1, H_2] = 1$, since $h_1 \in H_2^f$ and $H_2^f \neq H_2$. We also have $g \in \mathsf{Stab}_{H_2}$, hence $H_2^{fg} = H_2^f$, hence $h_1^g \in H_2^f$ and $[h_1^g, H_2] = 1$. It follows that $[g, h_1] = g^{-1} h_1^{-1} g \cdot h_1$ commutes with H_2. But $h_2 \in H_2$, so $[[g, h_1], h_2] = 1$, which is a contradiction. $\square$

Lemma 2.4.44. *Let H_1 and H_2 be basic subgroups of G. If $m_{H_1} \geqslant m_{H_2}$, then $\mathsf{Rist}_{H_1} \leqslant \mathsf{Stab}_{H_2}$.*

Proof. Consider the actions of the conjugates H_1^h of H_1 on L_{H_2}. If $H_2^g \in L_{H_2}$ is moved by H_1^h, then, by Lemma 2.4.43, we have

$$H_2^g \cap \mathsf{Stab}_{H_1^h} \leqslant H_1^h.$$

Note that $\mathsf{Stab}_{H_1^h} = \mathsf{Stab}_{H_1}$ and conjugates of H_1 are disjoint when they are not equal. It follows that if H_2^g is moved by some conjugate of H_1, it is fixed by the other conjugates of H_1. Since $m_{H_1} \geqslant m_{H_2}$, it follows that there exists a conjugate H_1^h of H_1 such that it fixes all subgroups H_2^g, i.e., is such that $H_1^h \leqslant \mathsf{Stab}_{H_2}$. But since Stab_{H_2} is normal, we get $H_1^h \leqslant \mathsf{Stab}_{H_2}$ for all $h \in G$, hence $\mathsf{Rist}_{H_1} \leqslant \mathsf{Stab}_{H_2}$. $\square$

Corollary 2.4.45. *Suppose that H_1, H_2 are basic subgroups, and suppose that $m_{H_1} \geqslant m_{H_2}$. If H_1 is moved by H_2, then $H_1 \leqslant H_2$.*

We are ready now to show how to reconstruct $G \curvearrowright \partial T$ from the group structure of G. Consider the set of all infinite chains $H_1 > H_2 > H_3 > \ldots$ of basic subgroups of G such that $\bigcap_{n=1}^{\infty} \mathsf{Stab}_{H_n} = \{1\}$. (Note that we have then $m_n \to \infty$.) We introduce an equivalence relation on such chains, saying that two chains $H_1 > H_2 > H_3 > \ldots$ and $\tilde{H}_1 > \tilde{H}_2 > \tilde{H}_3 > \ldots$ are equivalent if and only if for every n there exists m such that $H_n \geqslant \tilde{H}_m$ and $\tilde{H}_n \geqslant H_m$. It is easy to see that this is an equivalence relation. Let $\mathcal{X}$ be the set of equivalence classes. For a basic subgroup H, let $\mathcal{C}_H \subset \mathcal{X}$ be the set of all equivalence classes of chains containing H.

The group G acts on chains by conjugation. This action obviously agrees with the equivalence relation, so that we get an action of G on $\mathcal{X}$. We also have $g(\mathcal{C}_H) = \mathcal{C}_{H^{g^{-1}}}$.

Proposition 2.4.46. *There is a G-equivariant bijection $\phi : \partial T \longrightarrow \mathcal{X}$ such that for every vertex v of T we have $\phi(\partial T_v) = \mathcal{C}_{G[v]}$, and for every vertex subgroup $H \leqslant G$ the set $\phi^{-1}(\mathcal{C}_H)$ is open.*

Proof. Let $\xi = (v_0, v_1, v_2, \ldots)$ be a point of ∂T. Define $\phi(\xi)$ as the equivalence class of the chain $G[v_0] > G[v_1] > G[v_2] > \ldots$. Note that not two such chains are equivalent to each other, hence we get an injective map $\phi : \partial T \longrightarrow \mathcal{X}$.

Suppose that $H_1 > H_2 > H_3 > \ldots$ be a chain representing an element of $\mathcal{X}$. Let v_n be a vertex of nth level of the tree T, and let $g \in G[v_n]$ be a non-trivial element. Then there exists m such that $g \notin St_{H_m}$. We will have then $g \notin St_{H_k}$ for all $k \geqslant m$, so we may assume that m_{H_m} is bigger than the number of vertices of the nth level of T. Then there exists $h \in G$ such that H_m^h is moved by $g \in G[v_n]$, hence H_m is moved by $G[h(v_n)]$. Then Corollary 2.4.45 implies that $H_m \leqslant G[h(v_n)]$. We proved that there exists a vertex u_n of the nth level such that $H_m \leqslant G[u_n]$ for all m big enough. Note that the vertex u_n is uniquely defined by this condition, since $G[u_n] \cap G[u'_n] = \{1\}$ for any two different vertices u_n, u'_n of the nth level. It is also clear that $u_{n+1} \in T_{u_n}$, so that we get a path $(u_0, u_1, u_2, \ldots) \in \partial T$.

Let us show that the chains $H_1 > H_2 > H_3 > \ldots$ and $G[u_0] > G[u_1] > G[u_2] > \ldots$ are equivalent. By construction, we already have that for every n there exists m such that $H_m \leqslant G[u_n]$.

By the same argument as above, there exists a unique chain $\tilde{H}_1 > \tilde{H}_2 > \tilde{H}_3 > \ldots$ of basic subgroups $\tilde{H}_n \in L_{H_n}$ such that $G[u_m] \leqslant \tilde{H}_n$ for all m big enough. Then for all n and all m_1, m_2 big enough we have $H_{m_2} \leqslant G[u_{m_1}] \leqslant \tilde{H}_n$. Let k be the first index such that $H_k \neq \tilde{H}_k$. Then $[H_k, \tilde{H}_k] = 1$, and for all $n_1, n_2 \geqslant k$ we have $[H_{n_1}, \tilde{H}_{n_2}] = 1$, since $H_{n_1} \leqslant H_k$ and $\tilde{H}_{n_2} \leqslant H_{n_2}$. But this implies, by condition (1) of Definition 2.4.41, that $H_{n_1} \cap \tilde{H}_{n_2} = \{1\}$ for all $n_1, n_2 \geqslant k$. This is a contradiction with the condition that for all n and all m_2 big enough we have $H_{m_2} \leqslant \tilde{H}_n$. It follows that $H_n = \tilde{H}_n$ for all n, and since $G[u_m] \leqslant H_n$ for all n and all m big enough, we proved that $H_1 > H_2 > H_3 > \ldots$ and $G[u_0] > G[u_1] > G[u_2] > \ldots$ are equivalent.

We proved that $\phi : \partial T \longrightarrow \mathcal{X}$ is a bijection. Its equivariance is straightforward.

According to the given above description of equivalence of elements of $\mathcal{X}$ to points of ∂T, the set $\phi^{-1}(\mathcal{C}_H) \subset \partial T$ consists of sequences $(u_0, u_1, \ldots)$ such that $G[u_m] \leqslant H$ for all m big enough. Note that if $G[u_m] \leqslant H$, then $G[v] \leqslant H$ for all $v \in T_{u_m}$, hence the set $\mathcal{C}_H$ is open. It follows from the same description that $\phi(\partial T_v) = \mathcal{C}_{G[v]}$, since $G[v_1] \leqslant G[v_2]$ is equivalent to $v_1 \in T_{v_2}$. $\square$

It follows from Proposition 2.4.46 that the action $G \curvearrowright \partial T$ can be reconstructed from the structure of G. Namely, we consider the set $\mathcal{X}$ with the topology given by the basis of open sets of the form $\mathcal{C}_H$ for all basic subgroups H. Then $G \curvearrowright \mathcal{X}$ is topologically conjugate to $G \curvearrowright \partial T$. This finishes the proof of the theorem. □

Let us describe now a method of reconstructing the tree structure from the action on the boundary of the tree.

Theorem 2.4.47. *Let $G_i \curvearrowright T_i$, for $i = 1, 2$, be weakly branch actions of groups on rooted trees. Suppose that $\phi : G_1 \longrightarrow G_2$ is an isomorphism (of abstract groups), and there exist sequences $H_{1,i} \geqslant H_{2,i} \geqslant \dots$ of subgroups of G_i such that for every n we have $H_{n,i} \leqslant \mathsf{Stab}_n(G_i)$, the group $H_{n,i}$ acts level-transitively on every subtree $T_{i,v}$ for v in the nth level of T_i, and $\phi(H_{n,1}) = H_{n,2}$. Then the isomorphism ϕ is induced by an isomorphism $T_1 \longrightarrow T_2$ of trees.*

Proof. We know that there exists a homeomorphism $f : \partial T_1 \longrightarrow \partial T_2$ inducing the isomorphism ϕ. Since the groups $H_{n,i}$ act level-transitive on the subtrees growth from the nth level, the set of minimal closed $H_{n,i}$-invariant subsets of ∂T_i is the set of boundaries $\partial T_{i,v}$ for v in the nth level of T_i. Since $\phi(H_{n,1}) = H_{n,2}$, and f is induced by ϕ, the homeomorphism ϕ maps $\partial T_{1,v}$ to some $\partial T_{2,u}$, where v, u are vertices of the nth level of the trees T_1 and T_2, respectively. We get a map $v \mapsto u$ from the set of vertices of T_1 to the set of vertices of T_2. It is easy to check that it is an isomorphism inducing ϕ. □

Corollary 2.4.48. *Let $G \curvearrowright T$ be a weakly branch group action on a rooted tree. Suppose that there exists a decreasing sequence of characteristic subgroups $H_n \leqslant G$ such that $H_n \leqslant \mathsf{Stab}_n(G)$ and H_n acts level-transitively on the subtrees T_v for all v in the nth level of T. Then every automorphism of G is induced by an automorphism of T, i.e., the automorphism group of G is its normalizer in* $\mathrm{Aut}\, T$.

Example 2.4.49. Consider the group $\mathrm{IMG}\left(z^2 + i\right)$ generated by

$$a = \sigma, \quad b = (a, c), \quad c = (b, 1),$$

see example 2.4.35. We know that it is branch. Define $H_0 = G$, and define H_n as the group generated by the squares of the elements of H_{n-1}. It is clear that $H_n \leqslant \mathsf{Stab}_n$ and that they are characteristic. The group H_1 contains $(abc)^2 = (cab, abc)$, $(bca)^2 = (abc, cab)$, and $(cab)^2 = (bca, abc)$. It follows by induction that the restriction of H_n to the subtrees of the nth level (after their natural identification with X^*) contain the elements abc, bca, cab. But each of them is level-transitive. It follows that $\mathrm{IMG}\left(z^2 + i\right)$ satisfies the conditions of Corollary 2.4.48

Example 2.4.50. Consider the following two groups $G_i = \langle a_i, b_i, c_i \rangle$, for $i = 1, 2$:

$$a_1 = \sigma(1, b_1), \quad b_1 = (1, c_1), \quad c_1 = (1, a_1),$$

and

$$a_2 = \sigma(1, b_2), \quad b_2 = (1, c_2), \quad c_2 = (a_2, 1).$$

They are iterated monodromy groups of two quadratic polynomials $f(z) = z^2 + c$ such that $f^3(0) = 0$.

It is not hard to check (similarly to Example 2.4.34) that for both of these group the derived subgroup $[G_i, G_i]$ contains $[G_i, G_i]^{\mathsf{X}}$, so that they are weakly branch. Similarly to the previous example, consider the subgroups $H_{i,n}$ defined inductively as the groups generated by the squares of the elements of $H_{i,n-1}$, where $H_{i,0} = G_i$. Then $H_{i,n} \leqslant \mathsf{Stab}_n(G_i)$, and we have that $H_{1,1}$ contains $a_1^2 = (b_1, b_1)$, $(a_1 b_1)^2 = (b_1 c_1, b_1 c_1)$ and $(a_1 c_1)^2 = (b_1 a_1, b_1 a_1)$. It follows that the restriction of H_1 to the trees of the first level contain the group generated by $b_1, b_1 c_1, b_1 a_1$, i.e., the whole group G_1. By induction, the restrictions of H_n to the trees of the nth level are equal to G_1, hence are level transitive. The same is true for $H_{2,n}$, since $a_2^2 = (b_2, b_2)$, $(a_2 b_2)^2 = (b_2 c_2, b_2 c_2)$, and $(a_2 c_2)^2 = (b_2 c_2, c_2 b_2)$.

It is clear that any isomorphism $\phi : G_1 \longrightarrow G_2$ must map $H_{1,n}$ to $H_{2,n}$, hence must be induced by an automorphism of X^*. We will see later that this is impossible, and thus prove that G_1 and G_2 are not isomorphic, see...

Proposition 2.4.51. *Suppose that G_1, G_2 are groups acting faithfully on a tree T. Suppose that the rigid stabilizers $\mathsf{Rist}_n(G_i)$ act level-transitively on all subtrees T_v such that v is in the nth level of T. Then any isomorphism $\phi : G_1 \longrightarrow G_2$ is induced by an automorphism of T.*

Proof. By Theorem 2.4.39, the isomorphism ϕ is induced by a homeomorphism $f : \partial T \longrightarrow \partial T$. It follows from Lemma 2.4.44 that for every vertex v of the nth level of the tree T we have $\phi(G_1[v]) \leqslant \mathsf{Stab}_n(G_2)$. By the conditions of the proposition, $G_1[v]$ acts level-transitively on T_v. It follows that the minimal closed $G_1[v]$ invariant subsets of ∂T are the set ∂T_v and the singletons outside of ∂T_v. The homeomorphism f will map them to minimal closed $\phi(G_1[v])$-invariant subsets. Since $\phi(G_1[v]) \leqslant \mathsf{Stab}_n(G_2)$, the set ∂T_v must be mapped into a set ∂T_u for some vertex u of the nth level. We have shown that for every vertex v of T there exists a vertex u of the same level as v such that $f(\partial T_v) \subset \partial T_u$. Since f is a homeomorphism, and the sets ∂T_v for v in the nth level of T form a partition of ∂T, it follows that there is a permutation f_n of the nth level of T such that $f(\partial T_v) = \partial T_{f_n(v)}$. It is easy to see that the sequence f_n defines an automorphism of T inducing f on the boundary and the isomorphism $\phi : G_1 \longrightarrow G_2$. $\square$

Example 2.4.52. The full automorphism group $\mathrm{Aut}\, T$ of a spherically homogeneous tree T satisfies the conditions of Proposition 2.4.51, hence every automorphism of $\mathrm{Aut}\, T$ is inner.

Example 2.4.53. Similarly, the P. Neumann's groups 2.4.33 satisfy the conditions of Proposition 2.4.51.

2.4.12. Free subgroups of groups acting on rooted trees. Everywhere in this subsection "free group" is "free non-abelian group".

Since free groups are residually finite (see, for example Exercise 2.23), we can use the construction of Theorem 2.4.4 to find a faithful action of the free groups on rooted trees, namely the action on the coset tree of a sequence $G_0 > G_1 > G_2 > \dots$ of subgroups of finite index with trivial intersection. For such an action the stabilizer of the point of the boundary of the tree corresponding to the sequence of cosets $1G_n$ has trivial stabilizer. Note that the points with trivial stabilizers are regular (see Definition 2.1.7). Therefore, if the action on the tree is level-transitive, then the set of points with trivial stabilizers is co-meager, see Proposition 2.1.18.

On the other hand, it is possible to construct a faithful action of the free group without free orbits on the boundary. Take an arbitrary faithful action $\tau : F \curvearrowright \mathsf{X}^*$ of the free group F on a regular rooted tree X^* for some alphabet X. (For example, the action from Example 2.4.19.) Consider the action $\tau_n : F \curvearrowright \mathsf{X}^*$ given by the rule

$$\tau_n(g)(vw) = g(v)w,$$

for every word v of length n. In other words, the action τ_n copies the original action on the first n levels, and then extends them "rigidly", by acting identically on all letters beyond the n first ones. The image of F under the action τ_n is finite. Choose two letters $x, y \in \mathsf{X}$, and define a new action $\psi : G \curvearrowright \mathsf{X}^*$ of F on X^* by the rules

$$\psi(g)(y^n xw) = y^n x\tau_n(g)(w),$$

and identically everywhere else. In other words, we "hang" the actions τ_n along the path y^ω, as it is shown on Figure...

Then the F-orbit of every point of the boundary X^ω of X^* for the new action is finite. But the action is faithful, since every element of F acts non-trivially on some vertex of X^* for the original action τ, hence it will act non-trivially on points arbitrarily close to y^ω. In particular, the stabilizer F_{y^ω} is the whole group F, whereas the germ stabilizer $F_{(y^\omega)}$ (see...) is trivial.

The next theorem shows that every faithful action of the free group on a rooted trees contains actions similar to the above two types of actions.

Theorem 2.4.54. *Let $G \leqslant \mathrm{Aut}\, T$. Then one of the following cases takes place.*

(1) *The group G does not contain non-abelian free subgroups.*

(2) *There exists a non-abelian free subgroup $F \leqslant G$ and a point $\xi \in \partial T$ such that F_ξ is trivial.*

(3) *There exists a point $\xi \in \partial T$ and a non-abelian free subgroup F of the group of germs $G_x/G_{(x)}$.*

Note that the cases (2) and (3) are not mutually exclusive.

Proof. Suppose that the theorem is not true. Then there exists a group G acting faithfully on a locally finite rooted tree T, containing free subgroups, and such that the groups $G_x/G_{(x)}$ do not contain free subgroups, and for every free subgroup $F \leqslant G$ and every $\xi \in \partial T$ the stabilizer F_ξ is non-trivial.

Choose a free subgroup $F \leqslant G$. For every $\xi \in \partial T$ the stabilizer F_ξ is non-trivial. It is not cyclic, since otherwise we can find a free subgroup $F_1 < F$ having trivial intersection with F_ξ, which is impossible by the choice of G (as the stabilizer of ξ in F_1 will be trivial).

It follows that the homomorphism $F_\xi \longrightarrow G_x/G_{(x)}$ has a non-trivial kernel, i.e., there exists $g \in F \setminus \{1\}$ such that g acts trivially on a neighborhood U_ξ of ξ. We get a covering $\{U_\xi\}$ of a compact space ∂T, hence we can find a finite cover $U_1, U_2, \ldots, U_n$ of ∂T by open sets such that for every U_i there exists $g_i \in F \setminus \{1\}$ acting trivially on U_i. Note that the set of all elements of F acting trivially on U is a subgroup of F.

The F-orbits of U_i is finite, hence there exists a finite-index subgroup $\tilde{F}$ of F such that $\tilde{F}$ leaves each of the sets U_i invariant. Since intersection of any subgroup of finite index of F with any non-trivial subgroup of F is non-trivial, for every U_i there exists $g \in \tilde{F}$ acting trivially on U_i.

Let us prove the following classical fact...

Lemma 2.4.55. *Let $\phi : F \longrightarrow G_1 \times G_2 \times \cdots \times G_n$ be a homomorphism of a free group to a direct product of groups. If composition of ϕ with every projection $P_i : G_1 \times G_2 \times \cdots \times G_n \longrightarrow G_i$ has a non-trivial kernel, then ϕ has a non-trivial kernel.*

Proof. It is clear that it is enough to prove the lemma for $n = 2$. The general case will follow by induction. Suppose that g_1, g_2 are non-trivial elements of F such that $\phi(g_1) = (1, h_1)$ and $\phi(g_2) = (h_2, 1)$. If one of h_i is trivial, then we are done. Otherwise, consider $[g_1, g_2]$. We have $\phi([g_1, g_2]) = [(1, h_1), (h_2, 1)] = 1$. If $[g_1, g_2] \neq 1$, then we are done. Otherwise, there exists $g \in F$ such that $g_1 = g^{n_1}$ and $g_2 = g^{n_2}$ for some non-zero integers n_1, n_2, see... Then $g_1^{n_2} = g_2^{n_1}$, hence $(1, h_1^{n_2}) = (h_2^{n_1}, 1)$. But this implies $h_1^{n_2} = 1$ and $h_2^{n_1} = 1$, hence $\phi(g_1^{n_2}) = 1$. $\square$

Consider now the homomorphism $\phi : g \mapsto (g|_{U_1}, g|_{U_2}, \ldots, g|_{U_n})$ from $\tilde{F}$ to the direct product of homeomorphism groups of the spaces U_i. By the above, each coordinate of this homomorphism has a non-trivial kernel. The homomorphism ϕ is injective, since the sets U_i cover ∂T. But this is a contradiction with the lemma above. □

2.4.13. Example: almost finitary groups. Let X^* be the tree of words defined by the sequence $\mathsf{X} = (X_1, X_2, \ldots)$, see 2.4.1. Let μ be the $\mathrm{Aut}\,\mathsf{X}^*$-invariant measure on the boundary X^ω of the tree. It is defined by the condition that the measure of the set of sequences with a given beginning of length n is equal to $|\mathsf{X}^n|^{-1} = |X_1|^{-1}|X_2|^{-1} \cdots |X_n|^{-1}$.

Definition 2.4.56. Let $g \in \mathrm{Aut}\,\mathsf{X}^*$. Denote by Σ_g the set of points $w = x_1x_2\ldots \in \mathsf{X}^\omega$ such that for every $n \geqslant 1$ the automorphism $g|_{x_1x_2\ldots x_n}$ of the tree X_n^* is non-trivial. We say that g is *almost finitary* if Σ_g has measure zero.

As a corollary of (2.4) in 2.4.1, we get that the set of all almost finitary automorphisms of X^* is a group.

If the set Σ_g is empty, then we say that g is *finitary*. If g is finitary, then there exists n such that $g|_v = 1$ for all $v \in \mathsf{X}^n$, by compactness of X^ω. Then the element g is uniquely determined by its action on the nth level X^n of the tree X^*. For a given n the set of such automorphisms is a group isomorphic to the automorphism group of the finite subtree of X^* consisting of the levels X^k for $k = 0, 1, \ldots, n$. The set of all finitary automorphism is an increasing union of these finite groups.

Denote, for $g \in \mathrm{Aut}\,\mathsf{X}^*$ and $n \geqslant 0$, by $\theta_g(n)$ the number of vertices $v \in \mathsf{X}^n$ of the nth level of T such that $g|_v$ is non-trivial. More generally, if T is a rooted subtree of X^* (i.e., a subtree containing the root of X^*), then we denote by $\theta_{g,T}(n)$ the number of vertices v of the nth level of T such that $g|_v$ is non-trivial.

If T is a spherically-homogeneous tree, then we denote by m_T the unique $\mathrm{Aut}\,T$-invriant probability measure on the boundary ∂T of the tree. It is defined by the condition that $m_T(\partial T_v)$ is equal to the inverse of the cardinality of the level of v.

Proposition 2.4.57. *Let T be a spherically homogeneous rooted subtree of X^*, and let $g \in \mathrm{Aut}\,\mathsf{X}^*$. Then*

$$m_T(\Sigma_g \cap \partial T) = \lim_{n\to\infty} \frac{\theta_{g,T}(n)}{|L_n|},$$

where $L_n = T \cap \mathsf{X}^n$ is the nth level of the tree T.

Proof. The set Σ_g is closed in X^ω, hence $\Sigma_g \cap \partial T$ is closed both in ∂T and X^ω. A point $\xi \in \partial T$ belongs to Σ_g if and only if for every beginning v of ξ we have $g|_v \neq 1$. It follows that $\Sigma_g \cap \partial T$ is the intersection of the decreasing set of clopen subsets $S_n = \bigcup_{v \in L_n \cap T, g|_v \neq 1} \partial T_v$. We have $m_T(S_n) = \frac{\theta_{g,T}(n)}{|L_n|}$, by definition of the measure m_T. The statement of the proposition follows from continuity of measures. □

Definition 2.4.58. Let Γ be a graph of bounded valency. We say that Γ is *amenable* if there exists a sequence of finite subsets A_n of the set of vertices of Γ such that

$$\lim_{n\to\infty} \frac{|\partial A_n|}{|A_n|} = 0,$$

where ∂A denotes the set of edges connecting a vertex in A to a vertex in the complement of A.

Alternatively, we may define ∂A as the vertices of A adjacent to the vertices in the complement of A. The definition will be equivalent to the given above, since the valency of the vertices of Γ is assumed to be uniformly bounded.

The next proposition is a particular case of a more general statement, see [**Kai01, GN05**].

Proposition 2.4.59. *Suppose that G is a finitely generated group, and let $T \subset \mathsf{X}^*$ be a G-invariant subtree such that the action of G on T is level-transitive. Suppose that for every $g \in G$ we have $m_T(\Sigma_g \cap \partial T) = 0$, where m_T is the G-invariant probability measure on ∂T. Then all orbital graphs of the action of G on ∂T are amenable.*

Proof. Let S be a finite generating set of G such that $S = S^{-1}$. Let Γ_n be the graph of the action of G on the nth level L_n of T (with respect to the generating set S). Let Γ'_n be the subgraph consisting of all edges $(s, v) \in S \times L_n$ such that $s|_v = 1$. Note that if $(s, v) \in \Gamma'_n$, then the inverse arrow $(s^{-1}, s(v))$ is also in Γ'_n. The number of edges of Γ_n not included into Γ'_n is equal to $\sum_{s\in S} \theta_{s,T}(n)$.

Let $\Phi_1, \Phi_2, \ldots, \Phi_k$ be the sets of vertices of the connected components of Γ'_n. Then

$$|\partial\Phi_1| + |\partial\Phi_2| + \cdots + |\partial\Phi_k| = \sum_{s\in S} \theta_{s,T}(n)$$

and

$$|\Phi_1| + |\Phi_2| + \cdots + |\Phi_k| = |L_n|,$$

hence there exists i_n such that

$$\frac{|\partial\Phi_{i_n}|}{|\Phi_{i_n}|} \leqslant \frac{\sum_{s\in S} \theta_{s,T}(n)}{|L_n|} \to 0$$

as $n \to \infty$.

Consider an arbitrary orbital graph Γ_ξ for the action $G \curvearrowright \partial T$. Since the action $G \curvearrowright \partial T$ is minimal, there exists a vertex $v_n\xi \in \partial T$ of Γ_ξ passing through a vertex v_n of Φ_{i_n}. Every vertex u of Φ_{i_n} can be reached from v_n by a path inside Γ'_n. Taking a product of the generators along such a path, we find an element $g \in G$ such that $g(v_n) = u$ and $g|_{v_n} = 1$. It follows that $g(v_n\xi) = u\xi$. It also follows that the map $u \mapsto u\xi$ is an isomorphic embedding of Γ'_n into Γ_ξ. Let A_n be the image of Φ_{i_n} under this embedding. Then $|\partial A_n| \leqslant |\partial\Phi_{i_n}|$, $|A_n| = |\Phi_{i_n}|$, hence $\frac{|\partial A_n|}{|A_n|} \to 0$ as $n \to \infty$. □

Proposition 2.4.60. *Suppose that $G \leqslant \operatorname{Aut} \mathsf{X}^*$ is such that Σ_g is countable for every $g \in G$. Then G has a free non-abelian subgroup if and only if there exists a point $w \in \partial T$ such that the group of germs $G_w/G_{(w)}$ has a free non-abelian subgroup.*

Proof. If Σ_g is countable, then $m_T(T \cap \Sigma_g) = 0$ for every subtree of X^*. It is well known and easy to check that the Cayley graph of a free group is non-amenable. Therefore, Proposition 2.4.59 eliminates the possibility of a free subgroup of G with a free orbit on the boundary. Theorem 2.4.54 finishes the proof. □

Example 2.4.61. Consider the group $\mathrm{IMG}\left(z^2 - 1\right)$ from Example 2.4.34. It is easy to see that $\Sigma_a = \Sigma_b$ is the singleton $\{1^\omega\}$, which implies that Σ_g is finite for every $g \in G$. One can also show that for every $g \in G$ there exists n such that $g|_v \in \{1, a, b, a^{-1}, b^{-1}, ab^{-1}, ba^{-1}\}$ for all $v \in \mathsf{X}^n$. (It is enough to check this for all elements of $\{1, a, b, a^{-1}, b^{-1}, ab^{-1}, ba^{-1}\} \cdot \{a, b, a^{-1}, b^{-1}\}$.) It follows that the groups $G_w/G_{(w)}$ are trivial. Consequently, $\mathrm{IMG}\left(z^2 - 1\right)$ has no free subgroups.

Example 2.4.62. The group $\mathrm{IMG}\left(z^2 + i\right)$ from Example 2.4.35 also can be analyzed in a similar way. We have $\Sigma_a = \varnothing$, $\Sigma_b = \{(10)^\infty\}$, and $\Sigma_c = \{(01)^\infty\}$. We also have that for every $g \in G$ the sections $g|_v$ belong to the set $\{1, a, b, c\}$ for all v long enough. This can be used to show that $G_w/G_{(w)}$ is a group of order at most two.

Example 2.4.63. The following group from ... models the *Hanoi tower game*. The game...

If $a_{i,j}$ is the move involving the pegs number i and j, then we have

$$a_{i,j}(xv) = \begin{cases} jv & \text{if } x = i, \\ iv & \text{if } x = j, \\ xa_{i,j}(v) & \text{otherwise.} \end{cases}$$

Denote by H_n the group generated by the transformations $a_{i,j}$ for all $1 \leqslant i < j \leqslant n$. It is known that the orbital graphs of the action of H_n on X^ω

have sub-exponential growth (see...). In particular, H_n can not have a free subgroup with a free orbit.

It follows from the definition of the generators that if $a_{i,j}(w) \neq w$, then for some long enough beginning v of w we have $a_{i,j}|_v = 1$ (namely v is the beginning such that its last letter is the first occurrence of i or j in w). We have $a_{i,j}(w) = w$ if and only if w does not have neither i nor j as its letters. In this case $a_{i,j}|_v = a_{i,j}$ for all beginnings v of w.

Consequently, if $g \in H_n$ and $g(w) = w$, then for all long enough beginnings v of w the section $g|_v$ belongs to a group generated by elements $a_{i,j}$ such that neither i nor j appears infinitely many times in w. Consequently, for any subgroup G of H_n and any sequence $w \in \mathsf{X}^\omega$ the group of germs $G_w/G_{(w)}$ is a quotient of a subgroup of H_m for $m < n$. Since H_2 is a group of order two, this gives us an inductive proof that H_n have no free subgroups.

2.4.14. Activity growth. Let $g \in \operatorname{Aut}\mathsf{X}^*$, and denote $\theta_g(n) = |\{v \in \mathsf{X}^n : g|_v \neq 1\}|$, see 2.4.13.

Definition 2.4.64. We say that g is of *polynomial activity growth* of degree d if the sequence $\theta_g(n)$ is bounded from above by a degree d polynomial in n.

Note that $\theta_{g_1 g_2}(n) \leqslant \theta_{g_1}(n) + \theta_{g_2}(n)$ and $\theta_{g^{-1}}(n) = \theta_g(n)$. It follows that the group of all automorphisms of g of degree d polynomial activity growth is a subgroup of $\operatorname{Aut}\mathsf{X}^*$.

Denote by $\mathcal{P}_d(\mathsf{X})$ the group of *finite state* automorphisms of X^* with degree d polynomial activity growth. For example $\mathcal{P}_0(\mathsf{X})$ is the group of *bounded automata*, i.e., finite state automorphisms $g \in \operatorname{Aut}\mathsf{X}^*$ such that $\theta_g(n)$ is a bounded sequence. The groups $\mathcal{P}_d(\mathsf{X})$ were defined and studied for the first time by S. Sidki in [**Sid00**].

Let $\mathcal{A}_g$ be the automaton defining g, i.e., the automaton with the set of states $\{g|_v : v \in \mathsf{X}^*\}$, initial state g, and the transition and output functions $\pi(h, x) = h|_x$, $\lambda(h, x) = h(x)$. Then $\theta_g(n)$ is equal to the number of paths of length n in the Moore diagram of $\mathcal{A}$ starting in the initial state of $\mathcal{A}$ and not ending in the trivial state. If A is the adjacency matrix of the Moore diagram, then $\theta_g(n)$ is equal to the sum of all but one entries in a column of A^n. It follows that

$$\theta_g(n) = v_1 A^n v_2$$

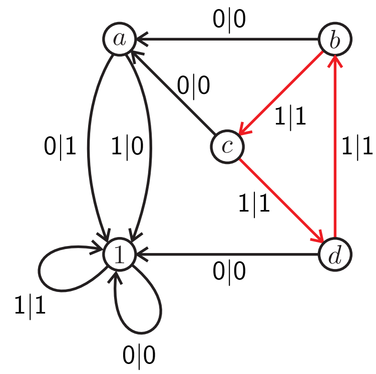

Figure 2.21. The automaton generating the Grigorchuk group

for a column vector v_2 and a vector v_1. Namely, assuming that the first coordinate corresponds to the initial state, and the last coordinate corresponds to the trivial state, then $v_2 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$, and $v_1 = (1, 1, \ldots, 1, 0)$. As a corollary of the Jordan normal form theorem, we get the following.

Proposition 2.4.65. *If g is a finite state automorphism of a rooted tree, then $\theta_g(n)$ is equal to a finite sum of complex functions of the form $p(n)a^n$, where p is a polynomial and a is a complex number. In particular, if $\theta_g(n)$ grows sub-exponentially, then it is bounded from above by a polynomial.*

In fact, we have the following description of the elements of $\mathcal{P}_d(\mathsf{X})$ in terms of the structure of their Moore diagrams, see....

Proposition 2.4.66. *Let Γ be graph obtained from the Moore diagram of the automaton $\mathcal{A}_g$ by removing the trivial state and all arrows adjacent to it. Then $g \in \mathcal{P}_d(n)$ if and only if the oriented cycles of Γ are disjoint. The number $d+1$ is equal to the maximal length of a sequence of oriented cycles $C_1, C_2, \ldots, C_{d+1}$ of Γ such that C_i is connected by an oriented path to C_{i+1} for all $i = 1, 2, \ldots, d$.*

Example 2.4.67. The *Grigorchuk group* is generated by the automaton shown on Figure 2.21. The only non-trivial cycle is highlighted.

It follows that the Grigorchuk group is a subgroup of $\mathcal{P}_0(\{0, 1\})$.

Example 2.4.68. Another classical example of a group generated by bounded automata is the group $\mathrm{IMG}\left(z^2 - 1\right)$ generated by all states of the automaton shown on Figure 2.22. We have introduced it already in 2.4.34. Compare

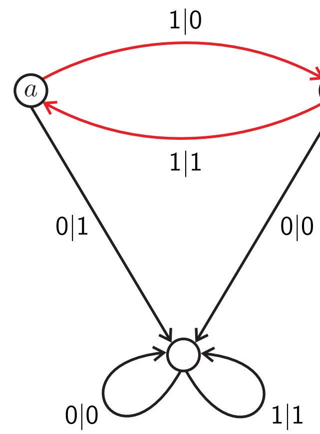

Figure 2.22. The Basilica group

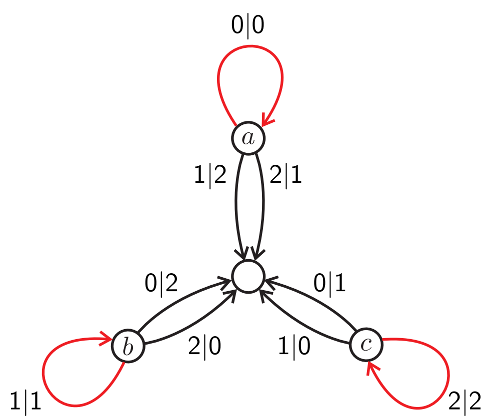

Figure 2.23. Hanoi tower group

the wreath recursion with the automaton.

Example 2.4.69. The Hanoi tower group H_3, see 2.4.63, is generated by the automaton shown on Figure 2.23. We did not show the loops at the trivial state, which is in the center.

We also see that it is a subgroup of $\mathcal{P}_0(\{0,1\})$. The groups H_n for $n \geqslant 4$ are not generated by bounded automata.

Example 2.4.70. Consider the group generated by the wreath recursion $a = \sigma(1, a), b = (a, b)$. We leave it to the readers as an exercise to show that the orbital graphs of the action of this group on the boundary of the binary tree are the graphs Λ_w described in 2.1.1.5.

The wreath recursion defining the generators of this group correspond to the automaton shown on Figure 2.24. We see that the automaton has

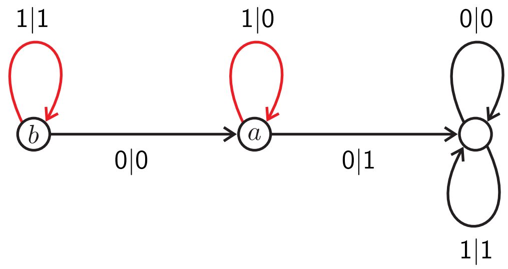

Figure 2.24.

two non-trivial cycles connected by an edge. It follows that this group is a subgroup of $\mathcal{P}_1(\{0,1\})$.

The following theorem was proved by S. Sidki in [**Sid00**]. We give here a shorter proof based on Theorem 2.4.54. (Note that [**Sid00**] is more general as it also considers the case of an infinite alphabet.)

Theorem 2.4.71. *The groups $\mathcal{P}_d(\mathsf{X})$ have no free subgroups.*

Proof. It follows from Proposition 2.4.66 that a finite automaton belongs to $\mathcal{P}_d(\mathsf{X})$ for some d if and only if the number of infinite paths in its Moore diagram that does not pass through the trivial state is countable. In particular, if $g \in \mathcal{P}_d(\mathsf{X})$, then the set Σ_g of sequences $x_1x_2\ldots \in \mathsf{X}^\omega$ such that $g|_{x_1x_2\ldots x_n} \neq 1$ for all n is countable. Consequently, it follows from Proposition 2.4.60 that if there exists a free subgroup in $\mathcal{P}_d(\mathsf{X})$, then there exists a finitely generated group $G \leqslant \mathcal{P}_d(\mathsf{X})$ and a point $w \in \mathsf{X}^\omega$ such that $G_w/G_{(w)}$ has a free subgroup. .. $\square$

More examples of subgroups of $\mathcal{P}_d(\mathsf{X})$ and their relation to dynamics are discussed in 6.6.8.

Exercises

2.1. Describe all possible Schreier graphs of the infinite dihedral group D_∞ with the usual generating set.

2.2. Describe, using Schreier graphs, all subgroups of index 4 in the free group F_2.

2.3. Let Γ be an unlabeled oriented graph such that for every vertex v the number of incoming arrows and the number of outgoing arrows are both equal to some fixed number d. Prove that Γ can be perfectly labeled by a set S such that $|S| = d$.

2.4. Let $\mathsf{X} = A \sqcup B$ be a finite set partitioned into two non-empty subsets. Let $w = \ldots A_{-1}B_{-1}A_0B_0A_1B_1\ldots$ be a random sequence such that $A_n \subset A$ and $B_n \subset B$ are independent and uniformly distributed in the set of all subsets of A and B, respectively. Prove that, with probability one, the group G_w (defined in 2.1.1.1) is isomorphic to the free product $(\mathbb{Z}/2\mathbb{Z})^{|A|} * (\mathbb{Z}/2\mathbb{Z})^{|B|}$.

2.5. Let $G = \langle a, b\rangle$ be the group defined in 2.1.1.3. Prove that the map $a \mapsto a^2, b \mapsto b^2$ extends to an endomorphism of G.

2.6. Transform the substitution σ given in 2.1.1.2 into a graph substitution generating the orbital graphs of the Grigorchuk group G, and prove that the map $a \mapsto aca, b \mapsto d, c \mapsto b, d \mapsto c$ extends to an endomorphism of G.

2.7. The following example of a group is from [Kotowski,Virág] ... Let $\alpha_0, \alpha_1, \ldots$ be a sequence of positive integers. Consider a binary rooted tree T. Replace each vertex by a cycle of length three with edges labeled by b, and replace each edge connecting a vertex of level $n-1$ to a vertex of level n by a cycle of length $2\alpha_n$ labeled by letters a, so that two opposite (i.e., on distance α_n) vertices of the cycle also belong to the three-cycles corresponding to the vertices. Also add a cycle of length $2\alpha_0$ attached to the root, and add loops so that the obtained graph is perfectly labeled, see Figure 2.25, where the graph is shown for $\alpha_0 = 2, \alpha_1 = 3, \alpha_2 = 4, \ldots$.

Prove that if $\alpha_n \to \infty$, then the group G defined by the constructed graph has a locally finite normal subgroup N such that $G/N \cong \mathbb{Z}$.

2.8. Let g_0, g_1 be the homeomorphisms of $\mathbb{R}$ defined in 2.1.1.6. a) Prove that $g_{a_k}g_{a_{k-1}}\cdots g_{a_0}(0) = a_0 + \frac{a_1}{2} + \cdots + \frac{a_k}{2^k}$ for every sequence $a_0, a_1, \ldots, a_k \in \{0, 1\}$. b) Prove that $g_0^{-1}g_1g_{a_k}g_{a_{k-1}}\cdots g_{a_1}(0) \geqslant 2$ and $g_1^{-1}g_0g_{a_k}g_{a_{k-1}}\cdots g_{a_1}(0) \leqslant 0$. c) Prove that the orbital graph Γ_0 of the group $\langle g_0, g_1\rangle$ is isomorphic to the graph shown on Figure 2.6. d) Prove that the group $\langle g_0, g_1\rangle$ is isomorphic to the group defined by the graph. (Use the fact that $\mathbb{Z}\left[\frac{1}{2}\right]$ is dense in $\mathbb{R}$.)

2.9. Note that if we switch the labels in the graph on Figure 2.6, then we get an isomorphic graph. It follow that the transposition of the generators of the group defined by it extends to an automorphism of the group.

Let g_0, g_1 be the homeomorphism of $\mathbb{R}$ given in **??**. Find an order two homeomorphism of $\mathbb{R}$ conjugating g_0 to g_1 and g_1 to g_0.

2.10. Consider an tree T for which every vertex has one incoming arrow labeled by g_0 or g_1 and two outgoing arrows labeled by g_0 and g_1. Add infinite paths with loops, as in 2.1.1.6, and let Γ_T be the obtained perfectly labeled graph. a) Prove that any two graphs Γ_{T_1} and Γ_{T_2} constructed in this way are locally isomorphic and locally contained in the

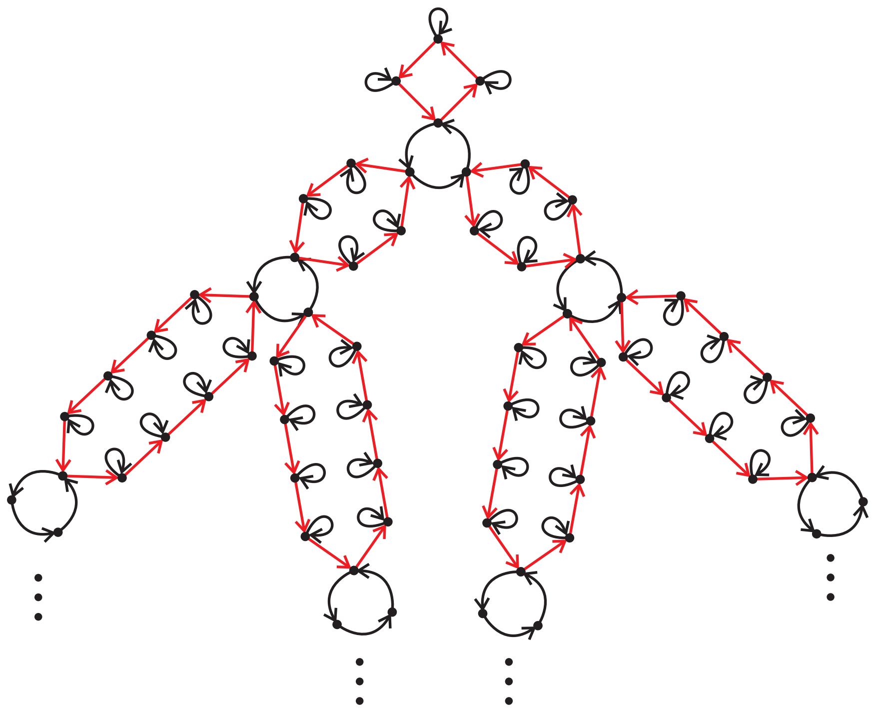

Figure 2.25. Kotowski-Virág groups

graph Γ from 2.1.1.6. b) Prove that the rooted graph (Γ_T, v), where v is a vertex of $T \subset \Gamma_T$, is uniquely determined, up to isomorphism, by the sequence of the labels along the unique infinite path in T going against the arrows and starting in v.

2.11. Let (T, v) be the tree from the previous problem such that the labels of the unique path against the arrows starting in a vertex v are all equal to g_0, and let Γ_0 be the corresponding graph Γ_T, see Figure 2.26.

Let Γ be the graph from 2.1.1.6, see Figure 2.26. Prove that Γ_0 both covers Γ and is locally contained in Γ. Conclude that the groups defined by Γ_0 and Γ are isomorphic.

2.12. Let a and b be permutations of $\mathbb{Z}$ defined by the graph Λ_w, see 2.1.1.5. Prove that a^2b^{-1} and $ab^{-1}a$ commute.

2.13. Let us identify a sequence $x_0x_1\ldots \in \{0,1\}^\omega$ with the diadic number

$$x_0 + 2x_1 + 2^2x_2 + \cdots .$$

Show that Λ_{w_1} and Λ_{w_2} are isomorphic as non-rooted trees if and only if $w_1 - w_2 \in \mathbb{Z}$.

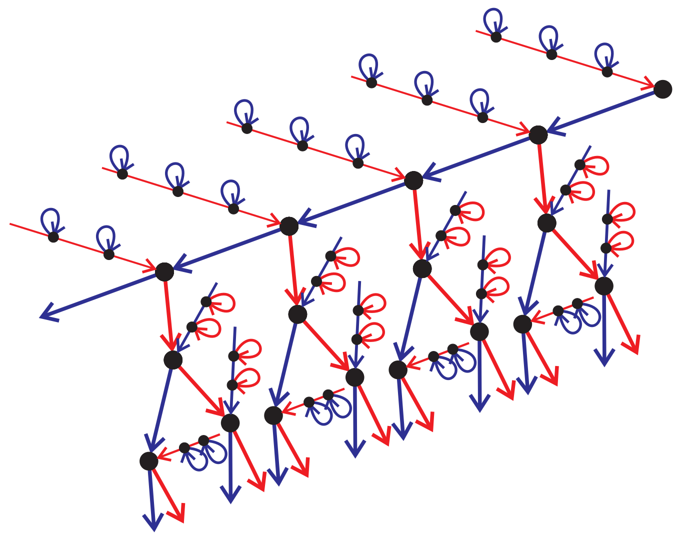

Figure 2.26.

2.14. Show that the realization of the topological graph of the action of a finitely generated group G on a topological space $\mathcal{X}$ is connected if and only if the action is topologically transitive.

2.15. Let $|A| = 1$. Prove that the space $\mathcal{S}_A$ is homeomorphic to the set $\{0\} \cup \{n^{-1} : n \in \mathcal{N}\}$.

2.16. Prove that (Γ, v) is an isolated point of $\mathcal{S}_S$ if and only if Γ is finite.

2.17. Find a subset of $\mathbb{R}$ homeomorphic to the space $\mathcal{S}_{\mathbb{Z}^n}$.

2.18. Prove that if a minimal action $G \curvearrowright \mathcal{X}$ on a compact space has a free orbit, then the orbit of every G-generic point is free.

2.19. Prove that Λ_{w_1} and Λ_{w_2} (see 2.1.1.5) are locally isomorphic if $w_1, w_2 \notin \mathbb{Z}$. Conclude that the group defined by the graph Λ_w does not depend on w.

2.20. Prove the statements of Example 2.1.25.

2.21. Consider the realization of the hull $\overline{\Gamma_0}$ as the direct product $\{g_0, g_1\}^\omega \times \{0, 1, 2, \ldots\}$ with two added points L_{g_0}, L_{g_1}, where the second coordianate $n \in \{0, 1, 2, \ldots\}$ is the distance from the root to the closest vertex of the tree T.... Show that g_i, for $i = 0, 1$, acts on $\overline{\Gamma_0}$ according to the following rules:

$$((g_{i_1}, g_{i_2}, \ldots), 0) \mapsto ((g_i, g_{i_1}, g_{i_2}, \ldots), 0),$$
$$((g_i, g_{i_2}, \ldots), n) \mapsto ((g_i, g_{i_2}, \ldots), n),$$
$$((g_{1-i}, g_{i_2}, \ldots), n) \mapsto ((g_{1-i}, g_{i_2}, \ldots), n-1).$$

2.22. In the conditions of the previous problem, consider the map

$$\lambda((g_{i_0}, g_{i_1}, \ldots), n) = \begin{cases} \sum_{k=0}^{\infty} \frac{i_k}{2^k} + n & \text{if } g_{i_1} = 1, \\ \sum_{k=0}^{\infty} \frac{i_k}{2^k} - n & \text{if } g_{i_1} = 0. \end{cases}$$

a) Prove that $\lambda : \overline{\{\Gamma_0\}} \backslash \{L_{g_0}, L_{g_1}\} \longrightarrow \mathbb{R}$ is continuous, surjective, and $|\lambda^{-1}(x)| = 1$ for all x except for $x \in \mathbb{Z}\left[\frac{1}{2}\right]$, when $|\lambda^{-1}(x)| = 2$. b) Prove that

$$\lambda(g_i(\xi)) = g_i(\lambda(\xi)),$$

where $g_i : \mathbb{R} \longrightarrow \mathbb{R}$ on the right-hand side is the function defined in **??**.

2.23. Consider the set P of all non-empty subsets of the set $\mathsf{X} = \{a, b, c\}$. Let $\mathcal{F} \subset P^{\mathbb{Z}}$ be the subshift consisting of all sequences $(x_n)_{n \in \mathbb{Z}}$ such that $x_n \cap x_{n+1} = \emptyset$ for every $n \in \mathbb{Z}$. It is a subshift of finite type. For every sequence $w \in \mathcal{F}$ consider the graph Γ_w as defined in 2.1.1.1. a) Show that for a co-meager set of sequences $u \in \mathcal{F}$ the graph Γ_u locally contains all graphs Γ_w, $w \in \mathcal{F}$ and the group G_u defined by the graph Γ_u is the free product $\langle a \rangle * \langle b \rangle * \langle c \rangle$ of groups of order 2. b) Show that the set of periodic sequences is dense in $\mathcal{F}$. c) Use this to prove that the free product of three groups of order two is residually finite. (Remark: this is very close to the first proof of residual finiteness of a free group, see....)

2.24. Consider the action of the Thompson group F on the interval and on the Cantor set. Show that the first action is locally minimal, while the second action is only locally transitive. Deduce that local minimality condition in Theorem 2.2.25 can not be replaced by local transitivity.

2.25. Prove that two manifolds are homeomorphic if and only if their homeomorphism groups are isomorphic as abstract groups. Prove that every automorphism of the homeomorphism group of a manifold is inner. See ...

2.26. Let T be a spherically homogeneous tree. Prove that there exists only one level-transitive action of the infinite dihedral group on T, up to conjugacy in $\mathrm{Aut}\, T$.

2.27. Find an embedding of the additive groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ into the group $\mathcal{Q}$ of rational homeomorphisms of the Cantor set.

2.28. Prove that a transformation defined by a finite ω-deterministic automaton can be defined by a finite deterministic automaton.

2.29. Find the distance in the graph of the action of the Hanoi towers game (see...) from the vertex 1^n to the vertex 2^n.

2.30. Show that the set of almost finitary automorphisms of X^* is a group.

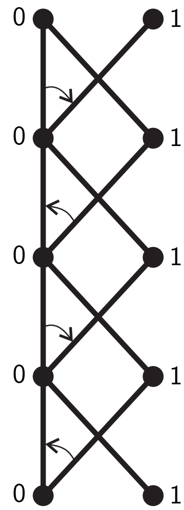

Figure 2.27. Golden mean rotation

2.31. Show that the group of almost finitary automorhpisms of a regular rooted tree X^* contains an isomorphic copy of the group of all automorphisms of X^*.

2.32. Let $G \curvearrowright \mathcal{X}$ be a minimal action on a compact space. Prove that if the orbital graph Γ_x of the action is amenable for a G-regular point $x \in \mathcal{X}$, then all orbital graphs of $G \curvearrowright \mathcal{X}$ are amenable.

2.33. Prove that $\mathrm{GL}_n(\mathbb{Z})$ can be embedded into the group of finite-state automorphisms of X^* (a) for some X, (b) for X^* consisting of two letters. (Hint: use the action of $\mathrm{GL}_n(\mathbb{Z})$ on the set of n-dimensional dyadic vectors.)

2.34. Find an embedding of $\mathbb{Q}$ into the group of rational homeomorphisms of the Cantor set.

2.35. The diagram on Figure 2.14 is equivalent to the Vershik-Bratteli diagram shown on Figure 2.27.

Label the paths in this diagram by sequences of 0 and 1 according to the vertices it passes, as labeled on the figure. Show that in this encoding the adic transformation is given by the automaton shown on Figure 2.28.

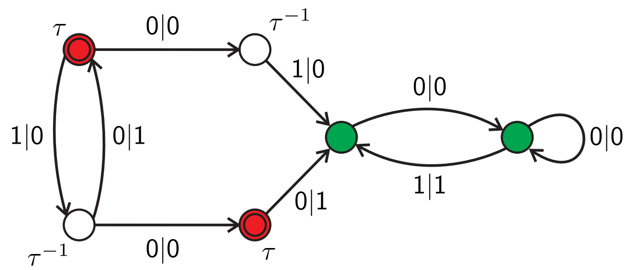

Figure 2.28.

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